REMARK ON THE DUAL EHP SEQUENCE

YASUTOSHI NOMURA

Dedicated to Professor K. Noshiro for his 60th birthday

In this note we will improve the dual EHP sequence which has been constructed in [6] by showing that that can be extended by one term. We then observe that this can be used to deduce a result which has been announced by T. Ganea in [4]. As another application we will establish a theorem which asserts that, under certain conditions, a principal fibration with a loop-space as fibre is principally equivalent to the one induced by some map.

Throughout this note, we make use of the notations and results described in [5] and [6] without specific reference. In particular, $E_{f,g}$ and E_g denote the mapping track of a triad $A \xrightarrow{f} Y \xleftarrow{g} B$ and the fibre of g respectively. Dually, $C_{f,g}$ and C_g denote the mapping cylinder of a cotriad $A \xleftarrow{f} X \xrightarrow{g} B$ and the cofibre of g respectively. We denote the loop and (reduced) suspension functor by Ω and S respectively. We use $\pi(X, Y)$ to denote the set of based homotopy classes of based maps $X \to Y$, but we will permit ourselves not to distinguish between a map and the homotopy class it represents.

1. The dual EHP sequence

For a triad $A \xrightarrow{f} Y \xleftarrow{g} B$, we introduce in [6] the maps

$$\xi' : C_{P_1, P_2} \to Y \text{ and } \eta' : SE_{f,g} \to C_{f \nabla g}$$

which make the following diagram homotopy-commutative:

$$\begin{tikzcd}
 & & & E_{\bar{f}} * E_g \arrow[r] \arrow[dd, "j"] & E_{\eta'} \arrow[dd] & \\
 & A \arrow[drr, "i_1"] & & & & SA \\
E_{f,g} \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & & C_{P_1, P_2} \arrow[r, "Q"] \arrow[dd, "\xi'"] & SE_{f,g} \arrow[ur, "SP_1"] \arrow[dr, "SP_2"'] \arrow[dd, "\eta'"] & \\
 & B \arrow[urr, "i_2"'] & & & & SB \\
 & & A \vee B \arrow[r, "f \nabla g"] & Y \arrow[r, "k"] & C_{f \nabla g} \arrow[r] & SA \vee SB,
\end{tikzcd}$$

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in which the columns are fibre triples, the middle row is the sequence associated with the cotriad $A \xleftarrow{P_1} E_{f,g} \xrightarrow{P_2} B$ consisting of projections, and i_1, i_2, k are appropriate injections. The map η', which is defined by

$$\eta'(a, \gamma, b\,;\, s) = \begin{cases} (a, 4s) & 0 \leqq 4s \leqq 1 \\ \gamma\left(\frac{4s-1}{2}\right) & 1 \leqq 4s \leqq 3 \\ (b, 4-4s) & 3 \leqq 4s \leqq 4 \end{cases}$$

for $a \in A$, $b \in B$, $\gamma \in Y^I$ with $f(a) = \gamma(0)$, $g(b) = \gamma(1)$, induces the "suspension"

$$\mathscr{E}^* : \pi(C_{f\nabla g}, V) \to \pi(SE_{f,g}, V).$$

The composite $\mathscr{H} = Q \circ j$, which is given by

$$\mathscr{H}((1-t)(a, \alpha) \oplus t(\beta, b)) = (a, \alpha + \beta, b\,;\, t)$$

for $a \in A$, $b \in B$, $\alpha, \beta \in Y^I$ with $f(a) = \alpha(0)$, $g(b) = \beta(1)$, $\alpha(1) = \beta(0) = *$, induces the dual Hopf invariant

$$\mathscr{H}^* : \pi(SE_{f,g}, V) \to \pi(E_{\bar{f}} * E_g, V).$$

Now, the cooperation of $SA \vee SB$ on $C_{f\nabla g}$ in the Puppe sequence for $f \nabla g$, defines an action of $\pi(SA \vee SB, V)$ on $\pi(C_{f\nabla g}, V)$. We denote the result of the action of $(\alpha, \beta) \in \pi(SA, V) \oplus \pi(SB, V)$ on $v \in \pi(C_{f\nabla g}, V)$ by $(\alpha, \beta) \top v$. Then we can easily verify the following

LEMMA 1.1. $\mathscr{E}^*((\alpha, \beta) \top v) = (SP_1)^* \alpha + \mathscr{E}^*(v) - (SP_2)^* \beta$.

Next, given $v : C_{f\nabla g} \to V$, let u denote the composite $Y \xrightarrow{k} C_{f\nabla g} \xrightarrow{v} V$. v determines the liftings $\tilde{f} : A \to E_u$ and $\tilde{g} : B \to E_u$ of f and g with respect to the projection $E_u \to Y$. We denote the adjoint of $\mathscr{E}^*(v)$ by $\theta : E_{f,g} \to \Omega V$. Let $j_1 : E_{\bar{f}} \to E_{f,g}$ and $j_2 : E_g \to E_{f,g}$ denote the obvious injections. Then we have

LEMMA 1.2. *The following diagram is homotopy-commutative;*

$$\begin{array}{ccc} E_{\bar{f}} & \xrightarrow{P_1 \circ j_1} & A \\ {\scriptstyle \theta \circ j_1}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ \Omega V & \xrightarrow{i} & E_u \\ {\scriptstyle -\theta \circ j_2}\big\uparrow & & \big\uparrow{\scriptstyle \tilde{g}} \\ E_g & \xrightarrow{P_2 \circ j_2} & B, \end{array}$$

where i is the inclusion of the fibre.

Proof: According to Proposition 5. 14 of [6], we have

$$m_*\{\theta,\ P_2^*(\tilde{g})\} = P_1^*(\tilde{f}),$$

where $m: \Omega V \times E_u \to E_u$ is the action of ΩV on E_u. Note that $P_1 \circ j_2$ and $P_2 \circ j_1$ are trivial maps. Consequently, by composing with j_1, we see that $i_*(\theta \circ j_1) = \tilde{f} \circ P_1 \circ j_1$. Similarly for homotopy-commutativity of the lower square.

The main purpose of this section is to improve Theorem 5. 8 of [6] as follows;

THEOREM 1. 3. *Suppose that f, g and Y are p-, q- and r-connected respectively, and that $\pi_i(V) = 0$ for $i \geqq p+q+r+2$. If A, B and Y have the homotopy type of CW-complexes, then the sequence*

$$\pi(E_{\tilde{f}} * E_g,\ V) \xleftarrow{\mathscr{A}^*} \pi(SE_{f,g},\ V) \xleftarrow{\mathscr{E}^*} \pi(C_{f\nabla g},\ V)$$

is exact.

Proof. Since $E_{\tilde{f}} * E_g$ is $(p+q)$-connected, it follows from a theorem of Sugawara [9, Theorem 6. 5] that the sequence

$$\pi(E_{\tilde{f}} * E_g,\ V) \xleftarrow{j^*} \pi(C_{P_1,P_2},\ V) \xleftarrow{\xi'^*} \pi(Y,\ V)$$

is exact. Consequently, given $\rho \in \pi(SE_{f,g},\ V)$ with $\mathscr{A}^*(\rho) = j^*Q^*(\rho) = 0$, there exists $\tau \in \pi(Y,\ V)$ such that $Q^*(\rho) = \xi'^*(\tau)$. Since

$$\tau \circ (f \nabla g) \circ k_1 = \tau \circ \xi' \circ i_1 \simeq \rho \circ Q \circ i_1 = *$$

for the injection $k_1: A \to A \vee B$, we see that $(f \nabla g)^* \tau = 0$, so that $\tau = k^* v$ for some $v \in \pi(C_{f\nabla g},\ V)$. Thus,

$$Q^* \mathscr{E}^*(v) = \xi'^* k^*(v) = Q^*(\rho).$$

Now, by Lemma 1. 1′ in [6], we can find $\alpha \in \pi(SA,\ V)$, $\beta \in \pi(SB,\ V)$ such that $\rho = (SP_1)^*\alpha + \mathscr{E}^*(v) - (SP_2)^*\beta$, whence, by Lemma 1. 1, we have

$$\mathscr{E}^*((\alpha,\ \beta) \top v) = \rho,$$

which completes the proof of our theorem.

COROLLARY 1. 4. (Sugawara [9, Lemma 7. 4]). *Let Y be a r-connected space which has the homotopy type of a CW complex and let V be such that $\pi_i(V) = 0$*

for $i \geqq 3r+2$. *Then an element of* $\pi(\Omega Y, \Omega V)$ *is primitive if, and only if, it is a suspension element, i.e., lies in the image of* $\pi(Y, V) \to \pi(\Omega Y, \Omega V)$.

This follows by considering a triad $* \to Y \leftarrow *$ and applying Lemma 5.1 in [6].

Now consider a triad $A \xrightarrow{f} Y \xleftarrow{g} B$ in which f and g are fibrations with fibres F_1, F_2 respectively. Let $\mathrm{Ker}(f : g)$ be the pull-back, i.e., $\mathrm{Ker}(f : g) = \{(a, b) | f(a) = g(b)\}$. Let $\pi_1 : \mathrm{Ker}(f : g) \to A$, $\pi_2 : \mathrm{Ker}(f : g) \to B$ denote the projections. Then the map $C_{\pi_1, \pi_2} \to Y$ corresponding to ξ', is essentially the same as the *Whitney sum* of f and g (as defined by I. M. Hall [3]). It is also known as the *fibre-join* of f and g (see [1]). To η' corresponds the map

$$\bar{\mathscr{E}} : S\,\mathrm{Ker}(f ; g) \to C_{f \triangledown g}$$

which is given by

$$\bar{\mathscr{E}}(a, b : s) = \begin{cases} (a, 2s) & \text{if } 2s \leqq 1 \\ (b, 2-2s) & \text{if } 2s \geqq 1. \end{cases}$$

Also, in this case, to the dual Hopf invariant $\mathscr{H}$ corresponds

$$\bar{\mathscr{H}} : F_1 * F_2 \to S\,\mathrm{Ker}(f : g)$$

which is defined by setting

$$\bar{\mathscr{H}}((1-t)a \oplus tb) = (a, b ; t).$$

With these notations we have

COROLLARY 1.5. *Suppose* F_1, F_2 *and* Y *are* $(p-1)$-, $(q-1)$- *and* r-*connected respectively and let* V *be such that* $\pi_i(V) = 0$ *for* $i \geqq p+q+r+2$. *If* A, B *and* Y *have the homotopy type of CW-complexes, then the sequence*

$$\pi(F_1 * F_2, V) \xleftarrow{\bar{\mathscr{H}}^*} \pi(S\,\mathrm{Ker}(f : g), V) \xleftarrow{\bar{\mathscr{E}}^*} \pi(C_{f \triangledown g}, V)$$

is exact.

Finally we observe that the following result announced in [4] can be derived from Lemma 1.2 and Theorem 1.3 by considering a triad $* \longrightarrow Y \xleftarrow{g} B$.

THEOREM OF GANEA. *Let* $F \longrightarrow B \xrightarrow{g} Y$ *be a fibration in which* Y *is* $(m-1)$-*connected and suppose* $\pi_i(F) \neq 0$ *only if* $n \leqq i \leqq n+2m-2$, $m \geqq 1$, $n \geqq 1$. *Let*

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$\theta : F \to \Omega V$ be a homotopy equivalence such that the composite

$$\Omega Y * F \longrightarrow SF \xrightarrow{\bar{\theta}} V$$

is nullhomotopic, where the first is obtained by Hopf construction associated with the action $\Omega Y \times F \to F$ and $\bar{\theta}$ is adjoint to θ. Then there exists a map $u : Y \to V$ and a fibre homotopy equivalence $B \to E_u$ with induced fibre equivalence in $\theta \in \pi(F, \Omega V)$.

Moreover, it follows from Theorem 5.12 in [6] that, if V is an H-space with $\pi_i(V) = 0$ for $i \geqq m+n+\min(m, n+1)$, maps u in the above forms a coset of the image of

$$\mathscr{P}^* : \pi(SC_g \hat{*} SY,\ V) \to \pi(Y,\ V),$$

where $\mathscr{P} = \langle \overline{Sk},\ \overline{1_{SY}} \rangle$ is the cojoin product of the adjoints of $Sk : SY \to SC_g$ and the identity 1_{SY} of SY.

2. An application to principal fibrations in the restricted sense

In [7] we strengthened the notion of principal fibrations in the sense of Peterson-Thomas [8] as follows. A fibration $F \xrightarrow{i} E \xrightarrow{p} B$ is said to be *principal in the restricted sense*, if F is a homotopy-associative H-space (with inversion) and if there exist maps

$$\mu : F \times E \to E \text{ and } h : \operatorname{Ker}(p : p) \to F$$

subject to the following conditions:

(i) $\mu(1_F \times i) = i\mu_0$ where $\mu_0 : F \times F \to F$ is the multiplication of F,

(ii) $p\mu = pq_2$, $h\{q_2, \mu\} \simeq q_1$ where $q_1 : F \times E \to F$ and $q_2 : F \times E \to E$ are the projections,

(iii) $\mu(\mu_0 \times 1_E) \simeq_B \mu(1_F \times \mu)$ where $\simeq_B$ indicates "is vertically homotopic to",

(iv) $\mu\{h, p_1\} \simeq_B p_2$ where $p_1, p_2 : \operatorname{Ker}(p : p) \to E$ are the projections,

(v) $\mu\{0, 1_E\} \simeq_B 1_E$ where 1_E is the identity map of E.

For example, a principal fibre bundle and $E_f \to X$ induced by $f : X \to Y$ from the contractible path space over Y are principal fibrations in the restricted sense. Note that, from (iv), $h\{p_2, p_1\} \simeq -h$ where $\{p_2, p_1\} : \operatorname{Ker}(p : p) \to \operatorname{Ker}(p : p)$ is the permutation.

LEMMA 2.1. *$\{h, p_1\} : \operatorname{Ker}(p : p) \to F \times E$ and $\{q_2, \mu\} : F \times E \to \operatorname{Ker}(p : p)$ are mutually inverse homotopy equivalences.*

Proof. This follows from the following:

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$$\{q_2, \mu\}\{h, p_1\} \simeq \{p_1, p_2\} \qquad \text{by (iv)},$$
$$\{h, p_1\}\{q_2, \mu\} \simeq \{q_1, q_2\} \qquad \text{by (ii)}.$$

LEMMA 2.2. *The composite* $E \xrightarrow{\{1,1\}} \mathrm{Ker}(p : p) \xrightarrow{h} F$ *is nullhomotopic, where* $1 = 1_E$.

Proof. By (v) and (ii) we have

$$h\{1, 1\} \simeq h\{q_2, \mu\}\{0, 1\} \simeq q_1\{0, 1\} = 0.$$

LEMMA 2.3. *Suppose* F *has the inversion* $\omega : F \to F$. *Then the composite*

$$F \times F \xrightarrow{l} \mathrm{Ker}(p : p) \xrightarrow{h} F$$

is homotopic to the composite

$$F \times F \xrightarrow{\tau} F \times F \xrightarrow{1_F \times \omega} F \times F \xrightarrow{\mu_0} F,$$

where l *is the inclusion and* τ *is the switching map.*

Proof. We define $n : F \times F \to F \times F$ by setting $n(x, x') = (x', \mu_0(x, x'))$. Since F has an inversion, n is a homotopy equivalence. We see at once that $\mu_0(1_F \times \omega)\tau n$ is homotopic to the projection $F \times F \to F$ on the first factor. Now, since the diagram

$$\begin{tikzcd}
F \times F \arrow[r, "n"] \arrow[d, "1_F \times i"'] & F \times F \arrow[d, "l"] \\
F \times E \arrow[r, "\{q_2, \mu\}"] \arrow[dr, "q_1"'] & \mathrm{Ker}(p : p) \arrow[d, "h"] \\
& F
\end{tikzcd}$$

is homotopy commutative, it follows that $hln \simeq \mu_0(1_F \times \omega)\tau n$, whence the desired conclusion.

The goal in this section is to prove the following

THEOREM 2.4. *Let* $F \xrightarrow{i} E \xrightarrow{p} B$ *be a principal fibration in the restricted sense such that* B *is* m*-connected and* $\pi_j(F) \neq 0$ *only if* $n + 1 \leqq j \leqq 2n + m + 2$. *Suppose there is given an* H*-homotopy equivalence* $\theta_0 : F \to \Omega V$. *If* E *and* B *have the* *homotopy type of CW-complexes, then there exist a map* $u: B \to V$ *and a fibre homotopy equivalence* $\tilde{p}: E \to E_u$ *with induced fibre equivalence in* $\theta_0 \in \pi(F, \Omega V)$, *so that the diagram*

$$\begin{tikzcd} F \times E \arrow[r, "\mu"] \arrow[d, "\theta_0 \times \tilde{p}"'] & E \arrow[d, "\tilde{p}"] \\ \Omega V \times E_u \arrow[r, "m"] & E_u \end{tikzcd}$$

is homotopy commutative, where m is the action of ΩV *on* E_u.

Proof. We apply Corollary 1.5 to the triad $E \xrightarrow{p} B \xleftarrow{p} E$ and use Lemma 1.2 for $\theta = (-\theta_0) \circ h : \mathrm{Ker}(p : p) \to \Omega V$.

First we show that $\bar{\mathscr{H}}^*(\bar{\theta}) = 0$ for the adjoint $\bar{\theta} : S\mathrm{Ker}(p : p) \to V$ of θ. Consider the diagram

$$\begin{tikzcd} F * F \arrow[rr, "\bar{\mathscr{H}}"] \arrow[dr] \arrow[dd, "\theta_0 * (-\theta_0)"'] & & S\ \mathrm{Ker}(p : p) \arrow[d, "Sh"] & \\ & S(F \times F) \arrow[ur, "Sl"] \arrow[r, "S[\mu_0(1_F \times \omega)\tau]"] & SF \arrow[d, "S(-\theta_0)"] & \\ \Omega V * \Omega V \arrow[rr, "\mathscr{H}"] & & S\Omega V \arrow[r, "\xi'"] & V, \end{tikzcd}$$

in which the row in the bottom is the fibre sequence constructed for the triad $* \to V \leftarrow *$. By Lemma 2.3, we see that the above diagram is homotopy-commutative. Since $\xi' \circ \mathscr{H} \simeq 0$, it follows that

$$\bar{\theta} \circ \bar{\mathscr{H}} = \xi' S(-\theta_0)(Sh)\bar{\mathscr{H}} \simeq 0,$$

as required.

By the assumption on connectedness, Corollary 1.5 now implies that $\bar{\mathscr{C}}^*(v) = \bar{\theta}$ for some $v : C_{p \nabla p} \to V$. Let $u : B \to V$ denote the composite $B \xrightarrow{k} C_{p\nabla p} \xrightarrow{v} V$. Then, by Lemma 1.2, we obtain the homotopy commutative diagram

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d, "\theta\{i,\ 0\}"'] & E \arrow[d, "\tilde{f}"] \\
\Omega V \arrow[r] & E_u \\
F \arrow[r, "i"] \arrow[u, "-\theta\{0,\ i\}"] & E, \arrow[u, "\tilde{g}"']
\end{tikzcd}$$

where $\tilde{f}$, $\tilde{g}$ are liftings of p. Using Lemma 2.3, we see that $\theta\{i,\ 0\} \simeq \theta_0$, $-\theta\{0,\ i\} \simeq \theta_0$. By Proposition 5.14 of [6], $m_*\{-\theta,\ \tilde{f}p_1\} = \tilde{g}p_2$ and, in turn, $m_*(\theta_0 \times \tilde{f})\{h,\ p_1\} = \tilde{g}\mu\{h,\ p_1\}$ by (iv). This, together with Lemma 2.1, yields $m(\theta_0 \times \tilde{f}) \simeq \tilde{g}\mu$.

But $\tilde{f} \simeq \tilde{g}$, because $\tilde{f}$ and $\tilde{g}$ define the separation element in $\pi(E,\ \Omega V)$, the adjoint of which is the composite

$$SE \xrightarrow{S\{1_E,\ 1_E\}} S\mathrm{Ker}(p : p) \xrightarrow{\overline{\mathscr{C}}} C_{p\Delta p} \xrightarrow{v} V.$$

This composite is nullhomotopic by Lemma 2.2. This shows that $m\{\theta_0 \times \tilde{p}\} \simeq \tilde{p}\mu$ for $\tilde{p} = \tilde{f}$, which completes the proof of the theorem.

3. The dual situations

In this section we briefly state the results which are dual to the previous sections. With a cotriad

$$A \xleftarrow{f} X \xrightarrow{g} B$$

we associate in [6] the following homotopy commutative diagram

$$\begin{tikzcd}
\Omega(A \times B) \arrow[r] & E_{f\Delta g} \arrow[r] \arrow[dd, "\eta"] & X \arrow[rr, "f\Delta g"] \arrow[dd, "\xi"] & & A \times B \\
\Omega A \arrow[dr, "\Omega I_1"] & & & A \arrow[dr, "I_1"] & \\
& C_{f,g} \arrow[r, "I"] \arrow[dd, "H"] & E_{l_1, l_2} \arrow[ur] \arrow[dr] \arrow[dd, "k"] & & C_{f,g} \\
\Omega B \arrow[ur, "\Omega I_2"'] & & & B \arrow[ur, "I_2"'] & \\
& C_f \hat{*} C_g & C_\xi \arrow[l, "F'"'] & &
\end{tikzcd}$$

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in which η, a generalization of the Freudenthal suspension, is defined in § 4 of [6], the Hopf invariant H is defined in § 6 of [6], F' is the map defined in § 6 of [6], and I_1, I_2, k are the appropriate injections.

LEMMA 4.1. *For $\alpha \in \pi(V, \Omega A)$ and $\beta \in \pi(V, \Omega B)$ we denote the result of the action of $\{\alpha, \beta\} \in \pi(V, \Omega(A \times B))$ on $v \in \pi(V, E_{f \Delta g})$ by $\{\alpha, \beta\} \top v$. Then we have*

$$\eta_*(\{\alpha, \beta\} \top v) = (\Omega I_1)_* \alpha + \eta^*(v) - (\Omega I_2)_* \beta.$$

Now, given $v \in \pi(V, E_{f \Delta g})$, we denote the composite $V \xrightarrow{v} E_{f \Delta g} \longrightarrow X$ by u; then v determines the extensions

$$\tilde{f} : C_u \to A, \ \tilde{g} : C_u \to B$$

of f, g. Let $\theta \in \pi(SV, C_{f,g})$ denote the adjoint of $\eta_*(v)$, and let $n : C_u \to SV \vee C_u$ be the cooperation. Then we have

LEMMA 4.2. $n^*\{\theta, I_2 \tilde{g}\} = I_1 \tilde{f}$.

LEMMA 4.3. *The following diagram is homotopy-commutative:*

$$\begin{array}{ccc} A & \xrightarrow{p_1 I_1} & C_f \\ \uparrow{\scriptstyle \tilde{f}} & & \uparrow{\scriptstyle p_1 \theta} \\ C_u & \longrightarrow & SV \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle p_2(-\theta)} \\ B & \xrightarrow{p_2 I_2} & C_g \end{array}$$

in which $p_1 : C_{f,g} \to C_f$, $p_2 : C_{f,g} \to C_g$ are the quotient maps which identify B, A with basepoint respectively.

In the sequel we assume that f, g and X are p-, q- and r-connected respectively, and that V is a CW-complex. Assume further A and B are a-, b-connected respectively.

LEMMA 4.4. *The sequence*

$$\pi(V, X) \xrightarrow{\xi_*} \pi(V, E_{I_1, I_2}) \xrightarrow{k_*} \pi(V, C_\xi)$$

is exact for V such that $\dim V \leqq p + q + r - 1$ (cf. Theorem 4.3 or Corollary 4.5

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in [6]).

The proof of the following theorem is similar to that of Theorem 1.3, except that we use the fact that F' is $[p+q+\min(r+1, p, q, \max(a, b))-1]$-connected by Lemma 6.6 in [6].

THEOREM 4.5. *The sequence*

$$\pi(V, E_{f\Delta g}) \xrightarrow{\eta_*} \pi(V, \Omega C_{f,g}) \xrightarrow{H_*} \pi(V, C_f \hat{*} C_g)$$

is exact for V *with* $\dim V \leqq p+q+\min(r+1, p, q, \max(a, b))-2$.

COROLLARY 4.6 (Theorem 5.2 in [2]). *If X is r-connected, then the sequence*

$$\pi(V, X) \xrightarrow{\eta_*} \pi(V, \Omega SX) \xrightarrow{H_*} \pi(V, SX \hat{*} SX)$$

is exact for V *with* $\dim V \leqq 3r+1$.

COROLLARY 4.7. *Assume f and g are cofibrations. Then the sequence*

$$\pi(V, E_{f\Delta g}) \xrightarrow{\overline{E}_*} \pi(V, \Omega \operatorname{Coker}\langle f : g\rangle) \xrightarrow{\overline{H}_*} \pi(V, C \hat{*} D)$$

is exact for V *with* $\dim V \leqq p+q+\min(r+1, p, q, \max(a, b))-2$, *where* C, D *are cofibres of* f, g *respectively*, $\operatorname{Coker}\langle f : g\rangle$ *is the quotient space obtained from* $A \vee B$ *by the identifications* $f(x)=g(x)$, $x \in X$ *and* $\overline{E}$, $\overline{H}$ *are defined as follows*:

$$\overline{E}(x, \alpha\times\beta)(t) = \begin{cases} \alpha(2t) & 0 \leqq 2t \leqq 1, \\ \beta(2-2t) & 1 \leqq 2t \leqq 2, \end{cases}$$

$$\overline{H} = i(\Omega q),\ q : \operatorname{Coker}\langle f : g\rangle \to \operatorname{Coker}\langle f : g\rangle / X = C \vee D,\ i : \Omega(C \vee D) \to C \hat{*} D.$$

It follows from Lemma 4.3 and Theorem 4.5 that

THEOREM OF GANEA. *Let $g : X \to B$ be a cofibration with m-connected cofibre D and let X be $(n-1)$-connected. If there is a homotopy equivalence $\theta : SV \to D$ such that the composite*

$$V \xrightarrow{\overline{\theta}} \Omega D \equiv \Omega C_g \xrightarrow{\overline{H}} SX \hat{*} D$$

is null-homotopic, where $\overline{\theta}$ is adjoint to θ, and if $\dim V \leqq n+m+\min(m, n)-2$, *then g is induced by some map $u : V \to X$.*

Now we strengthen the notion of principal cofibrations introduced in [10] as follows. Let $A \xrightarrow{i} B \xrightarrow{q} C$ be a cofibration with cofibre $C \simeq B/A$ and let $\mathcal{C}$

be an H'-space which is homotopy associative. We say that i is a *principal cofibration in the restricted sense*, if there exist maps

$$\mu' : B \to C \vee B, \quad h : C \to \mathrm{Coker}\langle i\ ;\ i\rangle$$

subject to the following conditions:

(i) $(1_C \vee q)\mu' = \mu_0' q$, where $\mu_0' : C \to C \vee C$ is the comultiplication,

(ii) $\mu' i = i_2 i$, $\{i_2, \mu'\} h \simeq i_1$ where $i_1 : C \to C \vee B$, $i_2 : B \to C \vee B$ are the injections and $\{i_2, \mu'\} : \mathrm{Coker}\langle i : i\rangle \to C \vee B$ is the map determined by i_2 and μ',

(iii) $(\mu_0' \vee 1_B)\mu' \simeq^A (1_C \vee \mu')\mu'$ where $\simeq^A$ indicates "is homotopic rel. A to",

(iv) $\{h, j_1\}\mu' \simeq^A j_2$ where $j_1, j_2 : B \to \mathrm{Coker}\langle i : i\rangle$ denote the injections,

(v) $\{0, 1_B\}\mu' \simeq^A 1_B$.

Then we can readily verify the following properties:

(vi) $\{h, j_1\} : C \vee B \to \mathrm{Coker}\langle i : i\rangle$ and $\{i_2, \mu'\} : \mathrm{Coker}\langle i\ ;\ i\rangle \to C \vee B$ are mutually inverse homotopy equivalences.

(vii) $C \xrightarrow{h} \mathrm{Coker}\langle i : i\rangle \xrightarrow{\{1_B, 1_B\}} B$ is null-homotopic.

(viii) The composite $C \xrightarrow{h} \mathrm{Coker}\langle i : i\rangle / A = C \vee C$ is homotopic to $C \xrightarrow{\mu_0'} C \vee C \xrightarrow{1_C \vee \omega} C \vee C \xrightarrow{\tau} C \vee C$ where ω is the inversion and τ is switching map.

With these preliminaries we can prove

THEOREM 4.10. Let $A \xrightarrow{i} B \xrightarrow{q} C$ *be a principal cofibration in the restricted sense such that* A *is* m-*connected and* C *is an* n-*connected* CW-*complex with* dim $C \leqq 2n + \min(m, n) - 1$. *Suppose given an* H' *homotopy equivalence* $\theta_0 : SV \to C$. *If* V *has the homotopy type of a* CW-*complex, then there exist a map* $u : V \to A$ *and a homotopy equivalence* $\tilde{i} : C_u \to B$ *with induced cofibre equivalence* θ_0 *so that the diagram*

$$\begin{tikzcd} C_u \arrow[r, "m'"] \arrow[d, "\tilde{i}"'] & SV \vee C_u \arrow[d, "\theta_0 \vee \tilde{i}"] \\ B \arrow[r, "\mu'"] & C \vee B \end{tikzcd}$$

is homotopy commutative, where m' *is the coaction of* SV *on* C_u.

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Added in proof. There is an error in computing the connectedness of $C_{f,g}$ in §6 of [6]. Theorem 6.2 of [6] should be stated as follows: Let f, g be p-, q-connected respectively and let X, A, B be r-, a-, b-connected respectively. Then ρ is $[p+q+\min(p, q, \max(a, b))-1]$-connected and ν is $[p+q+\min(p, q, r+1, \max(a, b))-2]$-connected. The word "$\min(p, q, r+1)$" in Lemma 6.6 and Theorem 6.8 of [6] should be replaced by the one "$\min(p, q, r+1, \max(a, b))$".

Department of Mathematics,
Nagoya Institute of Technology,
Nagoya, Japan