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Nilpotent measures on compact semigroups

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Let S be a compact semigroup and $P(S)$ the set of probability measures on S. Suppose $P(S)$ has zero θ and define a measure $\mu \in P(S)$ nilpotent if $\mu^n \to \theta$. It is shown that any measure with support containing that of θ is nilpotent, and the set of nilpotent measures is convex and dense in $P(S)$. A measure μ is called mean-nilpotent if $\left(\mu + \mu^2 + \ldots + \mu^n\right)/n \to \theta$, and can be characterized in terms of its support.

Throughout this paper S is a compact topological semigroup such that its minimal ideal $K(S)$ is a (compact) group. As is well-known, the set $P(S)$ of probability measures on S is a compact semigroup under convolution and the weak* topology, [3]. For $\mu, \nu \in P(S)$, we then have [3, Lemma 2.1],

$$\operatorname{supp}\mu\nu = \operatorname{supp}\mu \operatorname{supp}\nu ,$$

where $\operatorname{supp}\mu$ is the support of μ, and so on. It follows that the (normalized) Haar measure θ of $K(S)$ is the zero of $P(S)$. Then a measure $\mu \in P(S)$ is said to be *nilpotent* if $\mu^n \to \theta$ as $n \to \infty$, and we denote by N the set of nilpotent elements in $P(S)$. In [2], the case when $K(S)$ is a singleton has been considered; we established a characterization of nilpotent measures in terms of their supports, and examined the set N. It is the purpose of this note to obtain some possible extensions of those results to the general situation.

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For a subset A of S or $P(S)$, let $S(A)$ denote the closed semigroup generated by A and $K(S(A))$ its minimal ideal. In case $A = \{x\}$, we write $S(x)$ for $S(A)$. Recall that $K(S(x))$ is a group containing exactly the cluster points of the sequence $\{x^n\}$, (see [4, Theorem 3.1.1]). Now if $A \subset P(S)$, we write $\operatorname{supp} A = \overline{\bigcup_{\mu \in A} \operatorname{supp} \mu}$, where the bar denotes closure; it follows that $\operatorname{supp} S(A) = S(\operatorname{supp} A)$, by the same argument given in the proof of [2, Lemma 3]. Moreover, the minimal ideal of $\operatorname{supp} S(A)$ is $K(\operatorname{supp} S(A)) = \operatorname{supp} K(S(A))$; see [1, Theorem 5].

Let $N_0 = \{\mu \in P(S) : \operatorname{supp} \mu \supset K(S)\}$, which is easily seen to be an ideal of $P(S)$.

THEOREM 1. *$N_0 \subset N$; that is, any measure with support containing $K(S)$ is nilpotent.*

Proof. Take $\mu \in N_0$ and let τ be the identity of the group $K(S(\mu))$. Then $\operatorname{supp} \tau \subset \operatorname{supp} K(S(\mu)) = K(\operatorname{supp} S(\mu)) = K(S(\operatorname{supp} \mu)) = K(S)$. Hence $\operatorname{supp} \mu\tau = \operatorname{supp} \mu \operatorname{supp} \tau \supset K(S) \operatorname{supp} \tau = K(S)$. On the other hand, $\operatorname{supp} \mu\tau \subset K(S)$ since $\mu\tau \in K(S(\mu))$. Consequently $\operatorname{supp} \mu\tau = K(S)$. Since $K(S(\mu))$ is a group, there exists $\nu \in K(S(\mu))$ such that $(\mu\tau)\nu = \tau$. It follows, therefore, that $\operatorname{supp} \tau = \operatorname{supp} \mu\tau \operatorname{supp} \nu = K(S) \operatorname{supp} \nu = K(S)$, since $\operatorname{supp} \nu \subset K(S)$. By virtue of Theorem 1 of [6], we see that τ is the Haar measure of $K(S)$; that is, $\tau = \theta$, whence $K(S(\mu)) = \{\theta\}$. This means that the sequence $\{\mu^n\}$ has only one cluster point θ and so $\mu^n \to \theta$, that is $\mu \in N$, completing the proof.

REMARK. Equality need not hold in the theorem above. For instance let S be the multiplicative semigroup of real numbers in the closed unit interval with the usual topology. If $\delta(\frac{1}{2})$ denotes the Dirac measure at $\frac{1}{2} \in S$, clearly $\delta(\frac{1}{2}) \in N \backslash N_0$.

COROLLARY 2. *Let $\mu \in P(S)$ and $\operatorname{supp} \mu^n \supset K(S)$ for some positive integer n; then $\mu \in N$.*

Proof. In view of the fact that $\mu^n \in N_0 \subset N$, we see $\mu \in N$ by Lemma 2.1.4 of [4].

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THEOREM 3. *The set* N_0 *is convex and everywhere dense, and* $\text{supp}N_0 = S$.

Proof. The argument parallels that in [2]. First, taking $\mu, \nu \in N_0$, we let $\tau = t\mu + (1-t)\nu$ for $0 < t < 1$. Then $\text{supp}\tau = \text{supp}\mu \cup \text{supp}\nu \supset K(S)$ implies $\tau \in N_0$; that is, N_0 is convex. To see that N_0 is everywhere dense, let $\lambda \in P(S)$ and consider $\lambda_n = \frac{1}{n}\theta + \frac{n-1}{n}\lambda$ for positive integers n . It is obvious that $\lambda_n \in N_0$ and $\lambda_n \to \lambda$ as $n \to \infty$; that is, N_0 is dense in $P(S)$. Finally, for any $x \in S$, let $\omega = \frac{1}{2}(\theta+\delta(x))$. Since $x \in \text{supp}\omega$ and $\omega \in N_0$, the result is immediate.

THEOREM 4. *The set* N *is connected and everywhere dense, and* $\text{supp}N = S$.

Proof. By Theorem 3, N_0 is convex and so connected. Moreover, N_0 is everywhere dense and $N \supset N_0$, whence N is connected. The rest is clear.

Note that N is convex when $K(S)$ is a singleton, (see [2, Theorem 12]). But, in the present case, we can only establish the connectedness of N .

Given $\mu \in P(S)$, it is known that the sequence $\left(\frac{\mu+\mu^2+\ldots+\mu^n}{n}\right)$ must converge to an idempotent measure $L(\mu)$ with $\text{supp}L(\mu) = K(S(\text{supp}\mu))$, [5]. The measure μ is termed *mean-nilpotent* if $L(\mu) = \theta$. Let M denote the set of mean-nilpotent measures on S ; it is evident that $M \supset N$; that is, a nilpotent measure is mean-nilpotent. That the converse does not hold may be seen from the example below. But first we characterize mean-nilpotency in the next theorem.

THEOREM 5. *Let* $\mu \in P(S)$. *Then* $\mu \in M$ *if and only if* $S(\text{supp}\mu) \supset K(S)$.

Proof. To prove the "if" part, we note that $S(\text{supp}\mu) \supset K(S)$ implies $K(S(\text{supp}\mu)) = K(S)$, whence $\text{supp}L(\mu) = K(S)$. Accordingly, $L(\mu) = \theta$; that is, $\mu \in M$. Conversely, since

$$S(\text{supp}\mu) \supset K(S(\text{supp}\mu)) = \text{supp}L(\mu) = \text{supp}\theta = K(S) ,$$

the "only if" part follows easily.

COROLLARY 6. *If* $\mu^n \in M$ *for some* n *, then* $\mu \in M$.

Proof. Because $S(\text{supp}\mu) \supset S(\text{supp}\mu^n)$, we apply the previous theorem to obtain the result.

COROLLARY 7. *The set* M *is convex and everywhere dense, and* $\text{supp}M = S$.

Proof. We need only to show that M is convex. Take $\mu, \nu \in M$ and let $\tau = t\mu + (1-t)\nu$ for $0 < t < 1$. Since

$$\text{supp}\tau = \text{supp}\mu \cup \text{supp}\nu \supset \text{supp}\mu ,$$

it follows that $S(\text{supp}\tau) \supset S(\text{supp}\mu) \supset K(S)$. This together with Theorem 5 gives $\tau \in M$.

EXAMPLE 8. A mean-nilpotent measure is not nilpotent. Take the group $S = \{a, e\}$, e being the identity, and let $\mu = \delta(a)$. Then $\mu \in M$ since $S(\text{supp}\mu) = S(a) = S = K(S)$, but μ is clearly not nilpotent.

When the group $K(S)$ is a singleton, it can be shown that a measure in $P(S)$ is nilpotent if and only if mean-nilpotent, [2, Theorem 14], so that nilpotency of measures is dependent of supports only. The author has been unable to prove whether this carries over to the general case. That is, given $\mu, \nu \in P(S)$ with $\text{supp}\mu = \text{supp}\nu$, if $\mu \in M$, is it true that $\nu \in N$ also?

References

[1] H.L. Chow, "On supports of semigroups of measures", *Proc. Edinburgh Math. Soc.* (2) **19** (1974), 31-33.

[2] H.L. Chow, "Quasi-nilpotent sets in semigroups", *Proc. Amer. Math. Soc.* (to appear).

[3] Irving Glicksberg, "Convolution semigroups of measures", *Pacific J. Math.* **9** (1959), 51-67.

https://doi.org/10.1017/S000497270002373X Published online by Cambridge University Press

[4] A.B. Paalman-de Miranda, *Topological semigroups*, 2nd ed. (Mathematical Centre Tracts, 11. Mathematisch Centrum, Amsterdam, 1970).

[5] Štefan Schwarz, "Convolution semigroup of measures on compact non-commutative semigroups", *Czechoslovak Math. J.* 14 (89) (1964), 95-115.

[6] J.G. Wendel, "Haar measure and the semigroup of measures on a compact group", *Proc. Amer. Math. Soc.* 5 (1954), 923-929.

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