# LONG PERIODS IN THE THREE-DIMENSIONAL MOTION OF TROJAN ASTEROIDS

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ABSTRACT. The long periods caused by Jupiter, or Jupiter and Saturn in the motion of a Trojan are studied by numerical integration. Orbital inclinations between 0° and 50° are considered. Results on the length of the periods and on their main effects in the osculating elements are presented. It turns out that secular resonances can be important for the evolution of Trojan orbits.

## 1. INTRODUCTION

E.W. Brown's theory of the Trojan group of asteroids (Brown and Shook, 1933) describes an analytical way of approximation of three-dimensional motion of Trojans. Recently Érdi (1981) has applied a different theoretical method. Numerical results by Chebotarev et al. (1974) refer to real asteroids and cover an interval of 400 yr. One of us (Bien, 1978, 1980a) has studied by numerical integration the motion of real and fictitious Trojans on the basis of a simplified version of the planar elliptic three-body problem, and then (Bien, 1980b) on the basis of the rigorous problem Sun-Jupiter-Saturn-asteroid in three dimensions. Here we present a continuation of this work that refers to orbits of both small and high inclination, and consists in a study of the long periods by means of the main effects caused by them. Especially, we are interested in the long-period effects shown by eccentricity and inclination of an orbit and by the longitudes of perihelion and node. The following is a preliminary discussion of these effects. Our basic numerical integrations cover intervals between $1.5 \times 10^3$ and $2 \times 10^5$ yr.

## 2. MODELS AND EXAMPLES

For our studies of the three-dimensional motion of massless Trojan asteroids we have used four models. The following abbreviations will refer to them in the text:
"circ.3b." = circular restricted three-body problem with sun and Jupiter.
"ell.3b." = elliptic restricted three-body problem.



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"av.ell.3b." = the same problem, but the short-period terms are eliminated in the equations of motion by a procedure of numerical averaging (Schubart, 1978).
"simp.4b." = four-body problem with sun, Jupiter and Saturn. The two major planets move on elliptic orbits, but in the same plane as a simplification (compare Schubart, 1979, model 3).
With the exception of "av.ell.3b." we realize these models by numerical integration of the rigorous equations of motion, using the n-body program by Schubart and Stumpff (1966). The masses of Jupiter and Saturn correspond to the IAU (1976) System of Constants. All other perturbing masses are entirely neglected.

We use the following symbols for the osculating orbital elements of a Trojan: a, e, i, $\Omega$, $\tilde{\omega} = \omega + \Omega$, $\ell$ = mean longitude. The subscript J relates a symbol to Jupiter, so that $\mu = \ell - \ell_J$ represents the mean heliocentric angular distance between asteroid and Jupiter. In the four models mentioned above the orbital plane of Jupiter is the plane of reference for $\omega$, $\Omega$, and i ($i_J = 0°$). At the start of an integration we put $a_J = 1$, $e_J = 0.048$ (except in "circ.3b."), and $\tilde{\omega}_J = 0°$. The time T is counted from a moment that is comparatively close to the present. The longitudes count from the fixed direction or starting value of $\tilde{\omega}_J$. We prefer to replace e, $\tilde{\omega}$ by

$$\psi_1 = e\cos(\tilde{\omega} - \tilde{\omega}_J)\ , \qquad \psi_2 = e\sin(\tilde{\omega} - \tilde{\omega}_J)\ .$$

Variations of e, $\tilde{\omega}$ are studied by curves in rectangular coordinates $\psi_1$, $\psi_2$ (compare Bien, 1978). If the attraction of Saturn is neglected, the simpler relations $\psi_1 = e\cos\tilde{\omega}$, $\psi_2 = e\sin\tilde{\omega}$ hold. In studies of the planar subcase of "av.ell.3b." Bien (1978, 1980a) has found the two long periods $P_L$ (period of libration) $\approx$ 150 yr and $P_{\tilde{\omega}}$ (period of perihelion) $\approx$ 3 600 yr. $\mu$ librates around +60° or -60° for preceding and following Trojans, respectively, while a oscillates around 1, and both variations correspond to $P_L$ as the main period. In this case the "$\psi_1$, $\psi_2$ curve" is approximately a circle around a point $\psi_1 = e_J\cos(\pm 60°)$, $\psi_2 = e_J\sin(\pm 60°)$, where the sign of 60° corresponds to that of $\mu$. For nonplanar cases with large i the $\psi_1$, $\psi_2$ curve becomes an ellipse, and the nodal rate can be positive or negative (Bien, 1980b). In general, the $\psi_1$, $\psi_2$ curve does not always surround the point $\psi_1 = \psi_2 = 0$. Thus $P_{\tilde{\omega}}$ is given by the mean period of revolution along this curve.

Starting values will be designated by the subscript 0 and similarly extremes by max or min. We have chosen a standard set of starting values: $a_0 = 1$, $\mu_0 = 53°$ (then $\mu_0 \approx \mu_{min}$ because $\mu$ reaches extremes near a = 1), $e_0 = 0.11$, $\tilde{\omega}_0 = 60°$ (then e is not very far from its maximum), $\Omega_0 = 70°$. i is varied from 2° to 50°. The choice of $\mu_0 > 0$ is not essential, because we expect similar periods and effects for $\mu_0 < 0$ according to a property of symmetry of "av.ell.3b.". For the three values $i_0 = 4°$, 30°, 40° we have integrated forward over 150 000 yr in both "ell.3b." and "simp.4b.". The two following Trojans ($\mu < 0$), (1208) Troilus and (1873) Agenor have been integrated in "av.ell.3b." over more than $10^5$ yr.

## 3. RESULTS FROM THE THREE-BODY CASES

We have integrated the standard set of starting values with $i_0 = 2°$, $4°$, .... , $50°$ over 1 500 yr in "av.ell.3b.", and derived from this approximate values for $P_L$ and $\mu_{max}$ (with respect to $P_L$), see Fig.1. We find comparatively large $P_L$ values for high-inclination orbits, whereas $\mu_{max}$ decreases with increasing $i_0$ such that $\mu$ oscillates around mean values of less than 60° for large $i_0$. We can approximately identify i with $i_0$ in case of Trojans, because the variations of i are small. Besides the dominant influence of $P_L$ on $\mu$ and a, our more extended calculations mentioned above reveal additional periodicities. An integration of high-inclination orbits in "circ.3b." shows already a superposition of effects by a period of $\frac{1}{2}$ $P_\omega$, where in this model $P_\omega$ can be defined as the period of revolution of $\omega$. Using "ell.3b.", an additional superposition of effects due to $P_{\tilde{\omega}}$ appears. The superimposed effects cause oscillations of about $0\overset{\circ}{.}5$ in $\mu_{max}$ and $\mu_{min}$ with respect to mean values.

The influence of $\frac{1}{2}$ $P_\omega$ appears clearly in all the curves of Fig.2, which are smoothed with respect to $P_L$ and shorter periods, and which correspond to "circ.3b.". Strong effects appear in e and $\dot{\tilde{\omega}}$, and smaller effects in i and $\dot{\Omega}$. In this standard example with $i_0 = 30°$ the four curves coincide roughly in the position of the extremes and can be

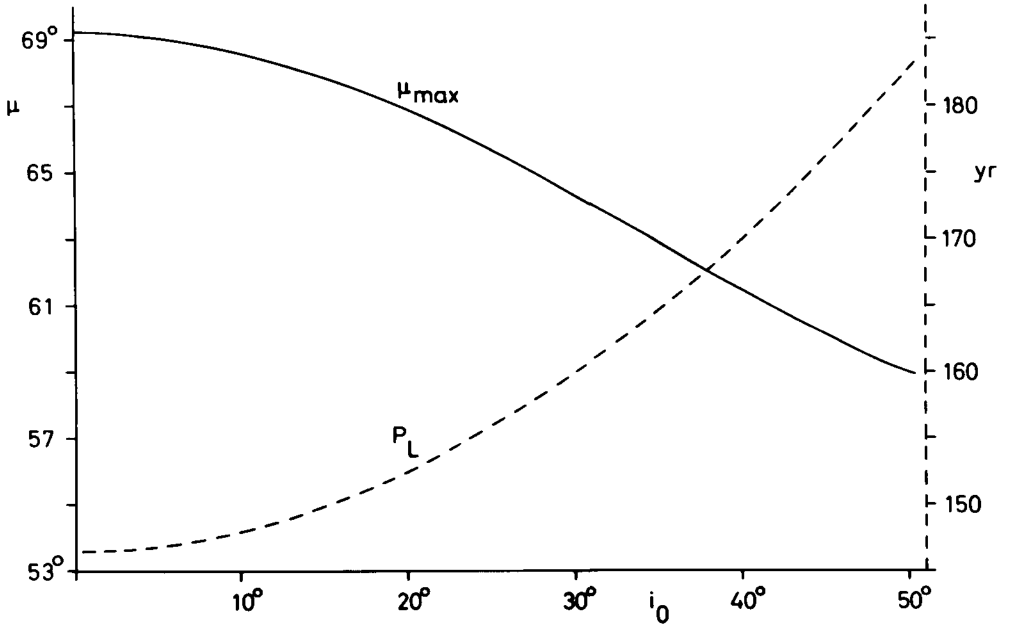


Fig. 1. $\mu_{max}$ (left scale) and $P_L$ (dashed line, right scale) are plotted against the starting value of inclination. $\mu_{min} \approx 53°$ coincides approximately with the horizontal axis. Results from "av.ell.3b.".

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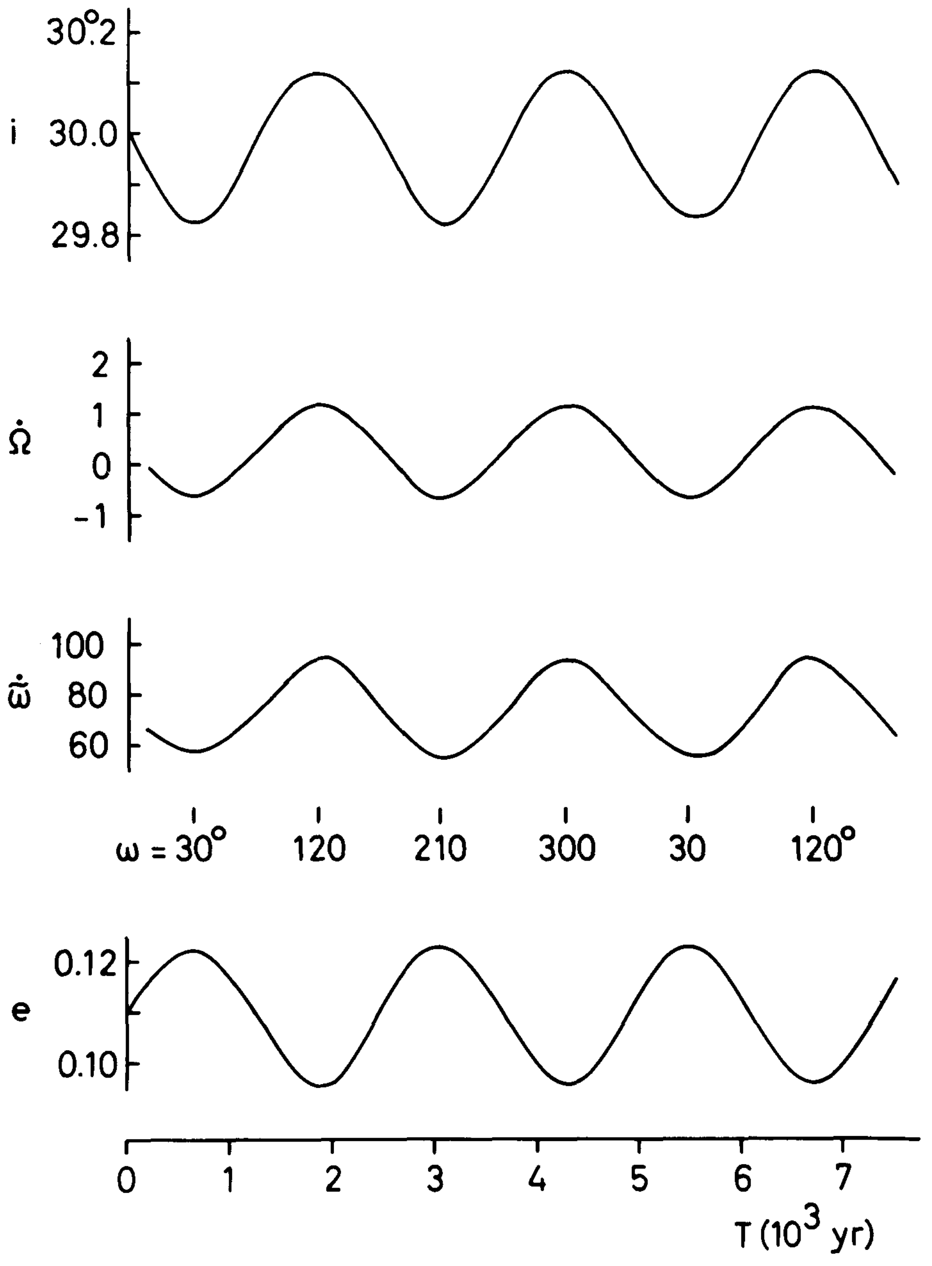


Fig. 2. i, $\dot{\Omega}$, $\dot{\tilde{\omega}}$, e versus T, smoothed curves from "circ.3b.". The unit of $\dot{\Omega}$ and $\dot{\tilde{\omega}}$ is degree/$10^3$ yr. Special values of ω are marked additionally. The curves correspond to the standard example with $i_0 = 30°$.

described qualitatively by functions $\pm\cos(2\omega-\gamma)$ with $\gamma \approx 60°$. For other examples one finds $\gamma \approx \pm 60°$, where the sign corresponds to that of $\dot{\mu}$. The mean value of $\Omega$ is so close to zero, that the sign of $\dot{\Omega}$ can change. An extremely slow motion of $\Omega$ is typical for Trojan orbits.

Fig. 3 is obtained from "av.ell.3b." and shows two stages of the evolution of the $\psi_1$, $\psi_2$ curve of the highly inclined following Trojan (1208) Troilus. In order to describe the quality of such a curve we

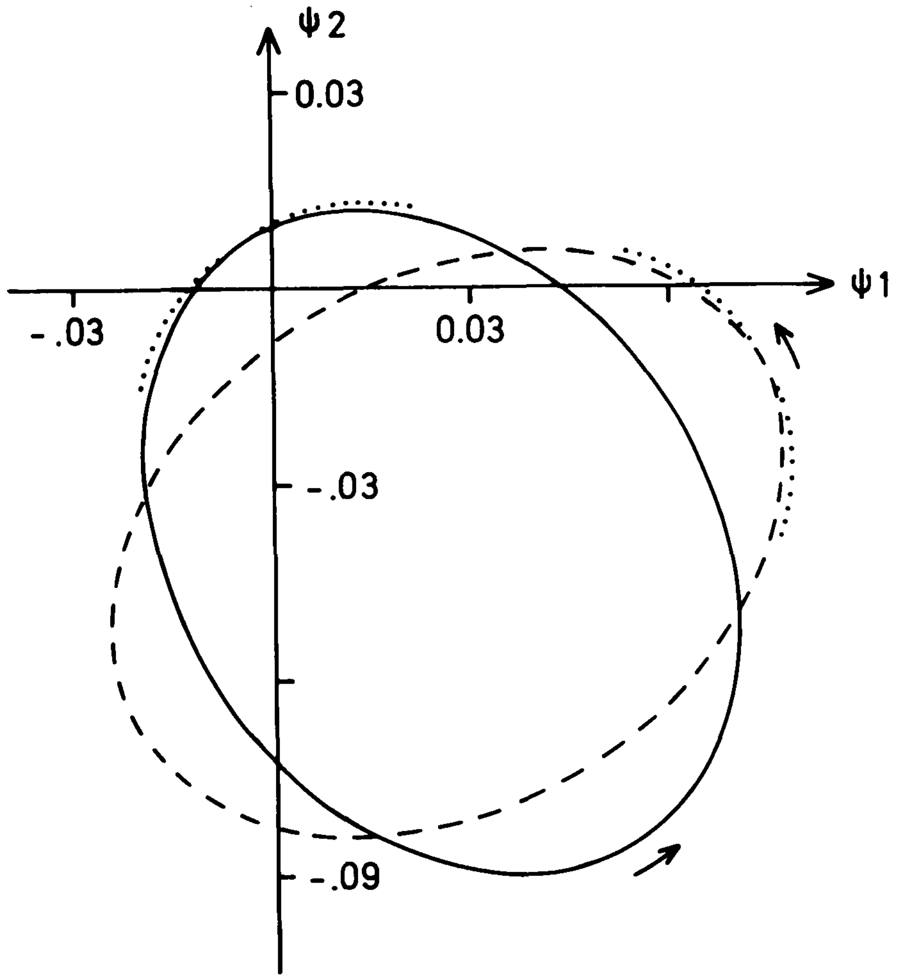


Fig. 3. Two parts of the $\psi_1$, $\psi_2$ curve of (1208) Troilus, smoothed results from "av.ell.3b.". The dotted beginning and end of each part indicates that the curve corresponds to a slowly rotating ellipse. The arrows demonstrate the direction of motion. The dashed part is later by 90 000 yr. $P_{\tilde{\omega}} \approx$ 5 000 yr corresponds to the period of circulation around the center of the rotating ellipse.

introduce the proper eccentricity $e_p$ and the proper perihelion $\tilde{\omega}_p$ by the relations

$$\psi_1 - \psi_1^* = e_p \cos(\tilde{\omega}_p - \tilde{\omega}_J) , \quad \psi_2 - \psi_2^* = e_p \sin(\tilde{\omega}_p - \tilde{\omega}_J) ,$$

where $\psi_1^*$, $\psi_2^*$ is the position of the center of the rotating curve (note: $\tilde{\omega}_J = 0°$ in 3-body cases). This is a generalization of relations used by Bien (1978, 1980a). $e_p$ represents the distance from the center of the curve to the point $\psi_1$, $\psi_2$. In analogy to the variations of e shown in Fig. 2, $e_p$ varies qualitatively like the function $\cos(2\omega_p - \gamma)$, where $\omega_p$ is introduced by $\omega_p = \tilde{\omega}_p - \Omega$, and $\gamma \approx -60°$ refers to the case of Fig. 3. $P_{\tilde{\omega}}$ is now given by the mean period of revolution of $\tilde{\omega}_p - \tilde{\omega}_J$, and a general definition of $P_\omega$ is possible: $P_\omega$ equals the mean period of revolution of $\omega_p$. For the three standard examples treated by "ell.3b." we find $P_{\tilde{\omega}} \approx$ 3 700 yr, 4 800 yr, 6 000 yr for $i_0 = 4°$, 30°, 40°, respectively. According to the slow motion of $\Omega$, $P_\omega \approx P_{\tilde{\omega}}$ holds for all Trojans. Since $\frac{1}{2} P_\omega$ rules the oscillation of $e_p$, two maxima of $e_p$ correspond to one revolution on the $\psi_1$, $\psi_2$ curve. In case of Fig. 3 the mean of $\dot{\Omega}$ equals 1° per $10^3$ yr. Therefore the values of $\tilde{\omega}_p$ that correspond to maxima of $e_p$ will increase according to this rate. Thus the rotating ellipse of Fig. 3 is qualitatively explained by a combination of effects of $P_{\tilde{\omega}}$ and $\frac{1}{2} P_\omega$.

A study of the variations of i shows an influence of $\frac{1}{2} P_\omega$ and $P_{\tilde{\omega}}$, and also of the period corresponding to the argument $\tilde{\omega}_p + \tilde{\omega}_J - 2\Omega$. An influence of the very long period corresponding to $2\Omega - 2\tilde{\omega}_J$ is indicated. Fig. 4 shows mean values of $\dot{\Omega}$. The curve refers to the examples considered in Fig. 1 and shows an increase of $\dot{\Omega}$ with $i_0$, but the crosses indicate a dependence on $\mu_0$. Especially a decrease in $\mu_0$, i.e. an increase in the amplitude of libration of $\mu$, causes a considerable decrease in $\dot{\Omega}$. This decrease is smaller for large $i_0$. The two dots in Fig. 4 refer to following Trojans, but also to fictitious preceding objects according to a symmetry of "av.ell.3b.".

## 4. SPECIAL RESULTS FROM THE FOUR-BODY MODEL

The addition of Saturn to the system of bodies causes important effects which appear in our studies by "simp.4b.". A secular period of about 54 000 yr causes an oscillation of $e_J$ between 0.03 and 0.06 and a temporary retrograde shift of $\tilde{\omega}_J$, although the mean of $\dot{\tilde{\omega}}_J$ is positive. The approximate 5/2 resonance of Jupiter and Saturn refers also to Trojan and Saturn. In case of Jupiter it causes the well-known perturbation in longitude and smaller effects with periods of about 900 yr in the other elements. The graphs by Cohen et al. (1973) demonstrate all variations in the orbit of Jupiter.

The extended integrations of our standard examples with $i_0 = 4°$, 30°, and 40° by "simp.4b." reveal effects of periods near 900 yr in a, $\mu$, i, and in the $\psi_1$, $\psi_2$ curves. Such an effect has temporarily an amplitude of $\approx 0\overset{\circ}{.}02$ in i of the case with $i_0 = 30°$. The earlier results

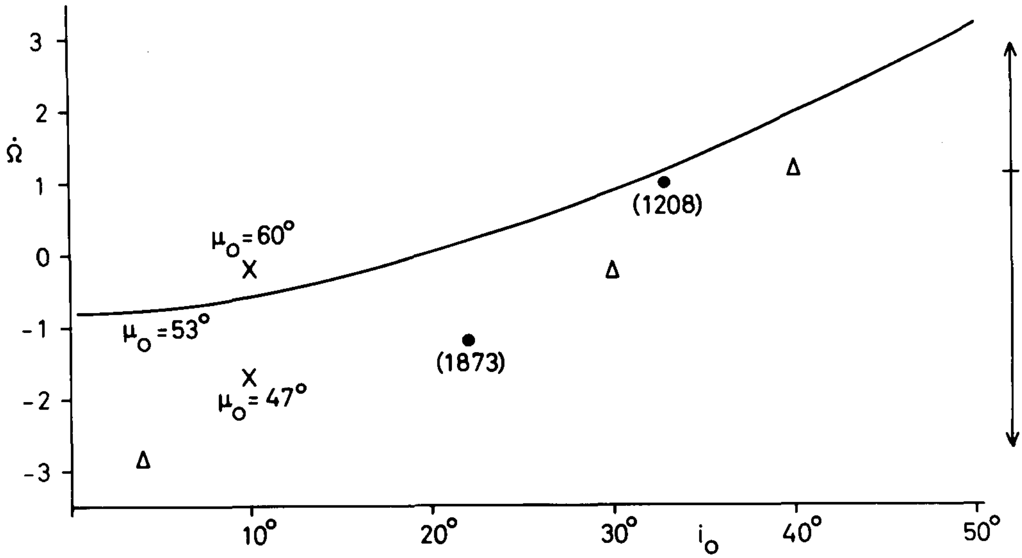


Fig. 4. $\dot{\Omega}$ in degree/$10^3$ yr is plotted against $i_0$. The curve corresponds to those of Fig. 1. The crosses demonstrate effects by a variation of $\mu_0$, the dots refer to real objects. Results from "av.ell.3b.", but the triangles and the vertical line demonstrate effects by Saturn (see text).

indicate a dependence of the center of a $\psi_1$, $\psi_2$ curve on $e_J$. Therefore, it is no surprise that the center $(\psi_1^*, \psi_2^*)$ changes with $e_J$. Now we have to replace $\Omega$ by $\Omega - \tilde{\omega}_J$, and this angle causes the forward or backward rotation of the elliptic curves,as in Fig. 3 the angle $\Omega$. An alternation of forward and backward rotation of such an ellipse can occur in special examples, but the shape of the ellipse does not vary. Saturn changes the mean of $\dot{\Omega}$, as demonstrated by triangles for the three standard examples in Fig. 4. Therefore a positive $\dot{\Omega}$ in "ell.3b." can become negative in "simp.4b." (compare Bien, 1980b). The range of the variation of $\dot{\tilde{\omega}}_J$ which is caused by the secular period of 54 000 yr is demonstrated in Fig. 4 by the vertical straight line that refers to the scale of $\dot{\Omega}$. A bar on this line represents the mean of $\dot{\tilde{\omega}}_J$. It turns out that for many high-inclination orbits an alternation between a forward and a backward shift of $\Omega - \tilde{\omega}_J$ can occur. Our example with $i_0 = 40°$ gives a value $\dot{\Omega}$ which agrees nearly with the mean of $\dot{\tilde{\omega}}_J$. If effects due to periods near 900 yr are neglected, the variations in i resemble those found in "ell.3b.". However, the amplitudes of effects depending on $e_J$ show an oscillation that follows the 54 000 yr period of $e_J$, and this period can also cause variations of the mean of i in case of Trojans. A special feature occurs in the example with $i_0 = 40°$: The slow mean motion of the argument $\Omega - \tilde{\omega}_J$ corresponds to an increase of $\approx 5°$ in $10^5$ yr during the computation, and the mean of i with respect to the visible periodicities decreases monotonously by 0°.2 in $10^5$ yr. We cannot predict

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the variations of $\Omega - \tilde{\omega}_J$ and i of this example during more extended periods, and we do not know, whether there is a relation between the variations of these two quantities.

## 5. DISCUSSION AND OUTLOOK

In the motion of Trojan asteroids the following long periods appear: the libration period $P_L$, the period of perihelion $P_{\tilde{\omega}}$, one half of the period of the argument of perihelion $\frac{1}{2}$ $P_\omega$ and periods that correspond to the angular arguments $\tilde{\omega}_p + \tilde{\omega}_J - 2\Omega$ and $2\Omega - 2\tilde{\omega}_J$. The influence of four of these periods is also suggested by terms with long-period arguments listed by Brown and Shook (1933, p.259). Further, periods due to the 5/2 resonance of Jupiter and Saturn, or asteroid and Saturn play an important role, and the period of 54 000 yr in the oscillation of $e_J$ is even more important. We point out that in case of special Trojan orbits the very slow motion of $\Omega$ allows a secular resonance with the mean motion of $\tilde{\omega}_J$ which is very small as well. Obviously, our standard example with $i_0 = 40°$ is very close to such a resonance. According to a former computation, the mean of $\dot{\Omega}$ of the real object (2146) Stentor ($i \approx 38°$) is close to +1°.0 per $10^3$ yr, which is just 0°.1 smaller than the value corresponding to the secular resonance. However, the attraction of additional major planets can change this difference.

Our results demonstrate that a study of Trojan motion over very long intervals of time must include the effects of Saturn. Fundamentally, the slow nodal motion of a Trojan can make possible a secular resonance with respect to a suitable secular frequency of the major planets, and especially to the mean of $\dot{\tilde{\omega}}_J$ in case of high-inclination orbits. This can be important for the evolution of Trojan orbits (compare Yoder, 1979).

The authors plan a more detailed numerical analysis of the periods and effects in the motion of Trojans. It is planned to extend the number of examples, and to take into account the mutual orbital inclination of Jupiter and Saturn. We thank Mrs.E.Miltenberger and Mrs.I.Seckel for aid in typing and drawing. We used the IBM 370-168 computer at the University of Heidelberg's Rechenzentrum.

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