# INJECTIVE ENDOMORPHISMS AND MAXIMAL LEFT IDEALS OF LEFT ARTINIAN RINGS

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**1. Introduction.** Given a ring $R$ and an injective ring endomorphism $\alpha: R \rightarrow R$, not necessarily surjective, it is possible to define a minimal overring $A(R, \alpha)$ of $R$ to which $\alpha$ extends as an automorphism. The ring $A(R, \alpha)$ was first studied by D. A. Jordan in his paper [**5**], where he also introduces the central objects of this paper—the closed left ideals of $R$. As can be seen from Theorem 4.7 of [**5**], the left ideal structure of $A(R, \alpha)$ depends very strongly on the closed left ideals of $R$, and our aim here is to show that each maximal left ideal of a left Artinian ring is closed.

It is known [**5**, Lemma 4.2] that any annihilator left ideal is closed, but it should be noted that maximal left ideals of left Artinian rings need not be annihilator left ideals. Indeed, consider the maximal left ideal $M = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ of the ring $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$. Then $r(M) = 0$, so that $lr(M) \neq M$, and $M$ cannot be a left annihilator.

The method employed to prove the main result involves three steps: first, the result is proved for completely primary rings, then this is extended to primary rings, and finally to left Artinian rings.

The final section of the paper obtains conditions under which the Jacobson radical $J(R)$ of a ring $R$ satisfies $\alpha^{-1}(J(R)) = J(R)$, where $\alpha: R \rightarrow R$ is a monomorphism. Once this is done, the main theorem can be applied to show that the result obtained is a generalization of Lemma 1.1 of [**4**], in which Jategaonkar proves the result for the case where $R$ is left Artinian.

**2. Preliminaries.** In this section, we present the relevant results concerning primary and completely primary rings, together with the important definitions from [**5**].

All rings will be assumed to have unity, and all monomorphisms $\alpha: R \rightarrow R$ are assumed to satisfy $\alpha(1) = 1$. The term "ideal" will refer to a two-sided ideal.

The first definition deals with closed left ideals, and is due to Jordan.

DEFINITION 2.1 [**5**]. Let $\alpha: R \rightarrow R$ be a ring monomorphism. Then a left ideal $I$ of $R$ is said to be *closed* if

$$\bigcup_{n \geqslant 0} \alpha^{-n}(R\alpha^n(I)) \subseteq I.$$

We now turn immediately to primary and completely primary rings.

DEFINITION 2.2. A left Artinian ring $R$ will be called *completely primary* if $R/J(R)$ is a division ring. $R$ will be called *primary* if $R/J(R)$ is simple Artinian.

The following result is standard.

THEOREM 2.3. *A primary ring is isomorphic to a full matrix ring over a completely primary ring.*

*Proof.* See [**3**, p. 55].

DEFINITION 2.4. A subset $\{e_{ij} \mid i,j=1, \ldots, n\}$ of a ring $R$ is called a set of *matrix units* in $R$ if $\sum_{i=1}^{n} e_{ii}=1$ and $e_{ij}e_{kl}=e_{il}\delta_{jk}$, where $\delta_{jk}$ is the Kronecker delta.

DEFINITION 2.5. An idempotent element of a ring $R$ is said to be *primitive* if it cannot be written as the sum of two non-zero orthogonal idempotents. An idempotent element of a semi-simple Artinian ring is called *semiprimitive* if it generates a minimal ideal.

LEMMA 2.6. *Let $R$ be a ring with two sets of primitive orthogonal idempotents* $\{e_i \mid i=1, \ldots, s\}$ and $\{f_i \mid i=1, \ldots, t\}$ *such that* $\sum_{i=1}^{s} e_i = 1 = \sum_{j=1}^{t} f_j$, *and the rings $e_iRe_i$ and $f_jRf_j$ are completely primary, for each $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$. Then $s=t$ and if the $f_i$ are suitably ordered, then there exists a unit $u$ of $R$ such that $u^{-1}e_iu=f_i$ for all $i=1, \ldots, s$.*

*Proof.* See [**3**, Theorem 2, p. 59].

THEOREM 2.7. *Let $R$ be a left Artinian ring with Jacobson radical $J(R)$. If $\phi: R \rightarrow R/J(R)$ denotes the natural surjection, and $e \in R$ is an idempotent, then*:

(i) *if $e$ is primitive then $eRe$ is completely primary*;

(ii) $J(eRe)=eJ(R)e=J(R) \cap eRe$;

(iii) *if $\phi(e)$ is a semiprimitive idempotent of $R/J(R)$, then $eRe$ is primary.*

**3. The primary case.** In this section we show that maximal left ideals of primary rings are closed. As stated above, it is necessary to deal with the completely primary case first: note that if $R$ is completely primary then $J(R)$ is the unique maximal left ideal of $R$.

PROPOSITION 3.1. *If $R$ is a completely primary ring with maximal left ideal $M$, then $M$ is closed.*

*Proof.* Since $M=J(R)$ and $R$ is left Artinian, $M$ is nilpotent. Therefore $\alpha^n(M)$ is a nil subring for any $n \geqslant 0$, and so $\alpha^n(M) \subseteq M$. Thus $R\alpha^n(M) \subseteq M$ and, since $\alpha^{-n}(M)$ is a nilpotent left ideal, $\alpha^{-n}(R\alpha^n(M)) \subseteq M$ and $M$ is closed.

At this point, it is useful to introduce some terminology concerning the maximal left ideals of the ring $M_n(S)$, where $n \in \mathbb{N}$ and $S$ is completely primary. The left ideal of $M_n(S)$ formed by insisting that the entries of the $i$th column $(1 \leqslant i \leqslant n)$ are elements of $J(S)$ is a maximal left ideal, and will be called the *i-th standard maximal left ideal* of $M_n(S)$.

LEMMA 3.2. *Let $S$ be a completely primary ring, and let $M$ be a maximal left ideal of $M_n(S)$. Then there exists a unit $u$ of $M_n(S)$ such that $M=Ku$, where $K$ is the first standard maximal left ideal of $M_n(S)$.*

*Proof.* Let $R = M_n(S)$. Since $R/J(R)$ is simple Artinian, all irreducible left $R/J(R)$-modules are isomorphic. In particular, $R/M$ and $R/K$ are isomorphic as $R/J(R)$-modules, and hence as $R$-modules. Let $\psi: R/M \to R/K$ be an $R$-isomorphism, and let $s \in R$ be such that $\psi(1+M) = s+K$. It can be shown that there exists a unit $v$ of $R$ such that $s+K = v+K$, so that $\psi(1+M) = v+K$. Thus, for any $r \in R$, $\psi(r+M) = rv+K$, and consequently $M = Ku$ where $u = v^{-1}$.

LEMMA 3.3. *Let $u \in R$ be a unit, $M$ any subset of $R$, and assume that $1 \in R\alpha^n(M)$ for some $n \geq 1$. Then $1 \in R(\bar{u}_0\alpha)^n(M)$, where $\bar{u}: R \to R$ is defined by $\bar{u}(r) = u^{-1}ru$.*

*Proof.* We proceed by induction on $n$. The case where $n = 1$ is immediate.

Now assume the conclusion to be true for $n-1$ and assume that $1 \in R\alpha^n(M)$. Then $1 = \sum_i r_i\alpha^n(m_i)$ for $r_i \in R$, $m_i \in M$, and so $\alpha^{n-1}(u) = \sum_i r_i\alpha^n(m_i)\alpha^{n-1}(u)$. Therefore

$$\begin{aligned} 1 &= \sum_i \alpha^{n-1}(u)^{-1}r_i\alpha^{n-1}(u)\alpha^{n-1}(u^{-1})\alpha^n(m_i)\alpha^{n-1}(u) \\ &= \sum_i \alpha^{n-1}(u)^{-1}r_i\alpha^{n-1}(u)\alpha^{n-1}(u^{-1}\alpha(m_i)u) \\ &\in R\alpha^{n-1}(\bar{u}_0\alpha(M)). \end{aligned}$$

By the induction hypothesis,

$$1 \in R(\bar{u}_0\alpha)^{n-1}(\bar{u}_0\alpha(M)) = R(\bar{u}_0\alpha)^n(M).$$

We are now in a position to prove the major theorem of this section.

THEOREM 3.4. *Every maximal left ideal of a primary ring is closed.*

*Proof.* Let $R$ and $S$ be isomorphic rings, $\psi: R \to S$ an isomorphism. If $M$ is a maximal left ideal of $R$ which is not closed under the monomorphism $\alpha: R \to R$, then $\psi(M)$ is not closed under the endomorphism $\psi_0\alpha_0\psi^{-1}$ of $S$. Thus, by Theorem 2.3, it is sufficient to prove the result for rings of the form $M_n(S)$, where $S$ is completely primary.

Now, if $M$ is a maximal left ideal of $M_n(S)$ then, by Lemma 3.2, $M = Ku$, where $K$ is the first maximal left ideal of $M_n(S)$ and $u$ is a unit. If $M$ is not closed, then for some $k \geq 1$, $1 \in M_n(S)\alpha^k(M)$ (since $M$ is maximal and $\alpha(1) = 1$), or $1 \in M_n(S)\alpha^k(Ku)$. Thus, $1 \in M_n(S)\alpha^k(K)$ and $K$ is not closed either. It is therefore sufficient to prove the result for the first standard maximal left ideal of $M_n(S)$.

Let $\{e_{ij} \mid i,j = 1, \dots, n\}$ be the standard set of matrix units for $M_n(S)$, i.e. $e_{ij}$ is the matrix whose $(i,j)$-entry is 1, but all other entries zero. Since $\{\alpha(e_{ij}) \mid i,j = 1, \dots, n\}$ is another set of matrix units for $M_n(S)$, it can be shown that there exists a unit $u$ of $M_n(S)$ such that $e_{ij} = \bar{u}_0\alpha(e_{ij})$ for each $i, j = 1, \dots, n$. (See, for instance, [**1**, Lemma 2.2].)

Define a map $\psi: S \to S$ by setting $\psi(s) = (\bar{u}_0\alpha(se_{11}))_{11}$, where $se_{11}$ is the matrix with $s$ in the $(1,1)$-position and zero elsewhere, and $(m)_{11}$ denotes the $(1,1)$-entry of the matrix $m$. Then, using the fact that $e_{ij} = \bar{u}_0\alpha(e_{ij})$ for each $i,j$, it can be seen that $\psi$ is an injective ring endomorphism of $S$.

Now let $M$ be the first standard maximal left ideal of $M_n(S)$, and assume that $\bar{u}_0\alpha(M) \not\subseteq M$. Then for some $m \in M$, $\bar{u}_0\alpha(m)$ has a unit of $S$ appearing in the first column—say $(\bar{u}_0\alpha(m))_{p1}$ is a unit. But $e_{1p}m \in M$ and $\bar{u}_0\alpha(e_{1p}m) = e_{1p}\bar{u}_0\alpha(m)$, which has a a unit appearing in its $(1, 1)$-position. Therefore, in order to show that $\bar{u}_0\alpha(M) \subseteq M$, it is sufficient to show that, for any $m \in M$, $(\bar{u}_0\alpha(m))_{11}$ cannot be a unit of $S$.

Now, for any $m \in M$,

$$\begin{aligned}(\bar{u}_0\alpha(m))_{11} &= (e_{11}\bar{u}_0\alpha(m)e_{11})_{11}\\ &= \psi(m_{11}).\end{aligned}$$

But $\psi$ is an injective endomorphism of the completely primary ring $S$, and $m_{11} \in J(S)$. The proof of Proposition 3.1 shows that $\psi(m_{11}) \in J(S)$, so that $(\bar{u}_0\alpha(m))_{11}$ cannot be a unit. Consequently $\bar{u}_0\alpha(M) \subseteq M$, and for any $k \geqslant 0$, $1 \notin M_n(S)(\bar{u}_0\alpha)^k(M)$. Lemma 3.3 completes the proof.

**4. The Artinian case.** We now turn our attention to the left Artinian case. One lemma is required before going on to prove the main result.

LEMMA 4.1. *Let $R$ be a semisimple Artinian ring, $M$ a maximal left ideal of $R$, and $e \in R$ an idempotent. Then either $eMe = eRe$, or $eMe$ is a maximal left ideal of $eRe$.*

*Proof.* Assume that $eMe \neq eRe$. Since $R$ is semisimple Artinian, there exists a left ideal $K$ of $R$ such that $R = M \oplus K$, so that

$$eRe = eMe + eKe. \tag{1}$$

It is claimed that $eKe$ is a minimal left ideal of $eRe$. Indeed, let $X$ be a non-zero left ideal of $eRe$ with $X \subseteq eKe$. Then $RX$ is a left ideal of $R$ and $0 \neq RX \subseteq Ke$. Since $K$ is a minimal left ideal, the map $\psi: K \to Ke$ given by $\psi(k) = ke$ has kernel either $K$ or 0. If $\ker \psi = K$ then $Ke = 0$, which, from (1), implies that $eRe = eMe$. Thus, $\ker \psi = 0$, and $Ke$ is a minimal left ideal of $R$. Therefore $RX = Ke$, and $X = eKe$, so that $eKe$ is a minimal left ideal.

This means that either $eKe \cap eMe = 0$ or $eKe \cap eMe = eKe$. The second alternative would imply, from (1), that $eRe = eMe$. Thus $eKe \cap eMe = 0$, and the sum at (1) is direct. Therefore, $eMe$ is a maximal left ideal of $eRe$.

THEOREM 4.2. *Every maximal left ideal of a left Artinian ring is closed.*

*Proof.* Let $R$ be a left Artinian ring, let $M$ be a maximal left ideal of $R$, and let $\{\bar{f}_i \mid i = 1, \ldots, n\}$ be the semiprimitive idempotents of $R/J(R)$. Each $\bar{f}_i$ may be written as the sum of mutually orthogonal primitive idempotents, and these may be arranged so that, for $0 = k_0 < k_1 < \cdots < k_n = m$,

$$\bar{f}_i = \sum_{j=k_{i-1}+1}^{k_i} \bar{e}_j, \tag{1}$$

where each $\bar{e}_j$ is a primitive idempotent, and $\sum_{j=1}^{m} \bar{e}_j = \bar{1}$.

By [**3**, Proposition 5, p. 54], there exist orthogonal, primitive idempotents $\{e_i \mid i=1,\ldots,m\}$ of $R$ such that $\phi(e_i)=\bar{e}_i$ for each $i=1,\ldots,m$, where $\phi: R\to R/J(R)$ is the natural surjection, and $\sum_{i=1}^{m} e_i = 1$.

Putting $f_i = \sum_{j=k_{i-1}+1}^{k_i} e_j$ for $i=1,\ldots,n$ gives a set $\{f_i \mid i=1,\ldots,n\}$ of orthogonal idempotents of $R$ such that $\sum_{i=1}^{n} f_i = 1$ and $\phi(f_i)=\bar{f}_i$ for $i=1,\ldots,n$. It is now claimed that, for some $i$, $1\le i\le n$, $f_iMf_i$ is a maximal left ideal of the subring $f_iRf_i$ of $R$. Indeed, there exists $i$, $1\le i\le n$, such that $\bar{f}_i\bar{M}\bar{f}_i$ is a maximal left ideal of $\bar{f}_i\bar{R}\bar{f}_i$, otherwise by Lemma 4.1, $\bar{f}_j\bar{M}\bar{f}_j=\bar{f}_j\bar{R}\bar{f}_j$, and $\bar{f}_j\in\bar{f}_j\bar{M}\bar{f}_j$ for all $j=1,\ldots,n$. Since each $\bar{f}_j$ is central, this means $\bar{f}_j\in\bar{M}$ for each $1\le j\le n$, and therefore that $\bar{1}\in\bar{M}$.

Since the natural surjection $\phi: R\to R/J(R)$, when restricted to $f_iRf_i$, has image $\bar{f}_i\bar{R}\bar{f}_i$ and kernel $f_iRf_i\cap J(R)$, Theorem 2.7(ii) gives $f_iRf_i/J(f_iRf_i)\cong\bar{f}_i\bar{R}\bar{f}_i$. Theorem 2.7(ii) also shows that $J(f_iRf_i)\subseteq f_iMf_i$, and the image of $f_iMf_i/J(f_iRf_i)$ is the maximal left ideal $\bar{f}_i\bar{M}\bar{f}_i$ of $\bar{f}_i\bar{R}\bar{f}_i$. Therefore, $f_iMf_i$ is a maximal left ideal of $f_iRf_i$, and the claim is proved.

Now, it is clear that $\{\alpha(e_i)\mid i=1,\ldots,m\}$ is a set of mutually orthogonal idempotents of $R$, with $\sum_{i=1}^{m}\alpha(e_i)=1$, and each $\alpha(e_i)$ primitive. Thus, by Lemma 2.6 and Theorem 2.7(i), there exists a unit $u$ of $R$ and a permutation $\pi$ on $\{1,\ldots,m\}$ such that $\bar{u}_0\alpha(e_i)=e_{\pi(i)}$ for each $i=1,\ldots,m$. If $p$ denotes the period of $\pi$, then $(\bar{u}_0\alpha)^p(e_i)=e_i$ for each $i=1,\ldots,m$.

Now assume $M$ is not closed, so that for some $k\ge 0$, $1\in R\alpha^k(M)$. By Lemma 3.3, this means that

$$1\in R(\bar{u}_0\alpha)^k(M). \tag{2}$$

If $q$ is the smallest integer such that $pq\ge k$, then applying $(\bar{u}_0\alpha)^{pq-k}$ to both sides of (2) gives

$$1\in R(\bar{u}_0\alpha)^{pq}(M). \tag{3}$$

The monomorphism $(\bar{u}_0\alpha)^p$ will be denoted by $\beta$. Multiplying both sides of (3) on the right by $f_i$ gives

$$f_i\in R\beta^q(Mf_i),$$

since $\beta(f_i)=f_i$. Therefore,

$$f_i=\sum_j r_j\beta^q(m_jf_i),$$

where $r_j\in R$, $m_j\in M$. Multiplying on the left by $f_i$ gives

$$f_i=\sum_j f_ir_j\beta^q(m_jf_i)=\sum_j f_i^2r_j\beta^q(m_jf_i).$$

Since $\bar{f}_i$ is central in $R/J(R)$,

$$f_i = \sum_j f_i(r_j f_i + a_j)\beta^q(m_j f_i)$$

$$= \sum_j (f_i r_j f_i)\beta^q(f_i m_j f_i) + \sum_j f_i a_j \beta^q(m_j f_i),$$

where $a_j \in J(R)$. Since $\beta(f_i) = f_i$, the second summand becomes

$$\sum_j f_i a_j \beta^q(m_j) f_i \in f_i J(R) f_i.$$

But, by Theorem 2.7(ii), $f_i J(R) f_i = J(f_i R f_i)$, so that

$$f_i - \sum_j f_i a_j \beta^q(m_j) f_i = \sum_j (f_i r_j f_i)\beta^q(f_i m_j f_i)$$

is a unit of $f_i R f_i$. Thus, the maximal left ideal $f_i M f_i$ of $f_i R f_i$ is not closed.

By Theorem 2.7(iii), $f_i R f_i$ is a primary ring, so this contradicts Theorem 3.4, and $M$ must be closed.

**5. Conclusion.** In Lemma 1.1 of his paper [**4**], Jategaonkar proves that if $R$ is a left Artinian ring with Jacobson radical $J(R)$ and $\alpha: R \to R$ is a monomorphism, then $\alpha^{-1}(J(R)) = J(R)$.

Theorem 4.2 may be used to show that the following is a generalization of Jategaonkar's original result.

THEOREM 5.1. *Let $R$ be a ring and $\alpha: R \to R$ a monomorphism such that*

(i) *each maximal left ideal of $R$ is closed under $\alpha$;*

(ii) *the ring $A(R, \alpha)$ is left Noetherian.*

*Then $\alpha^{-1}(J(R)) = J(R)$.*

*Proof.* The following is an indication of the method of proof only. For a full proof, refer to Theorem 3.34 of [**6**].

Let $J$ denote the Jacobson radical of the ring $A(R, \alpha)$, as constructed in [**5**], and for each integer $i \geq 0$, let $J_i = \{r \in R \mid x^{-i} r x^i \in J\}$. It can be shown [**6**, Theorem 3.23] that condition (ii) ensures that $J_i = \bigcap_{M \in B} M$, where $B$ denotes the set of all maximal closed left ideals of $R$. But $B$ consists precisely of the maximal left ideals of $R$, and therefore $J_i = J(R)$ for all $i \geq 0$. It is now only necessary to note that $\alpha^{-1}(J_{i+1}) = J_i$ for all $i \geq 0$ to obtain the result.

To see that Theorem 5.1 is indeed a generalization of Lemma 1.1 of [**4**], note that if $R$ is left Artinian then (i) is satisfied by Theorem 4.2, and (ii) is satisfied by Corollary 5.3 of [**5**].

However, there exist rings $R$ and monomorphisms $\alpha: R \to R$ which satisfy (i) and (ii) without $R$ being left Artinian. Such a pair $(R, \alpha)$ is provided in Example 3.36 of [**6**], where $R$ is constructed as follows.

Let $K$ be a field, $\sigma: K \to K$ a monomorphism which is not surjective, $S = K[y]$, where $y$ is indeterminate, and define $\bar{\alpha}: S \to S$ by $\bar{\alpha}(\sum_i f_i y^i) = \sum_i \sigma(f_i) y^i$. Let $P$ denote the prime ideal of $S$ generated by $y$. Then $\bar{\alpha}$ extends to a monomorphism $\alpha: R \to R$, where $R$ is the localization of $S$ at $P$. It can be shown that $R$ satisfies condition (ii) of Theorem 5.1, and that the unique maximal ideal $PR$ of $R$ is closed. However, $R$ is clearly not Artinian.

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