



Iterated Integrals and Higher Order Invariants

Anton Deitmar and Ivan Horozov

Abstract. We show that higher order invariants of smooth functions can be written as linear combinations of full invariants times iterated integrals. The non-uniqueness of such a presentation is captured in the kernel of the ensuing map from the tensor product. This kernel is computed explicitly. As a consequence, higher order invariants form a free module of the algebra of full invariants.

Introduction

Modular forms of higher order have been studied extensively in recent years [4, 5, 8–14, 18]. To construct them, one often uses iterated integrals. From dimension formulae [13] it is clear that in the case of holomorphic forms, iterated integrals do not give all higher order forms. But there are strong indicators [11, 12, 14, 18] that in the case of smooth functions, all higher order forms are indeed obtainable via iterated integrals. This has been an implicit open question for a while that is answered affirmatively in this paper. It is shown that on any smooth manifold, the smooth module of higher order invariants is generated by the space of homotopy invariant iterated integrals, thus allowing us to deduce structure assertions on higher order invariants from iterated integrals.

In Section 1 we recall the ingredients of the theory of iterated integrals needed in the sequel. In Section 2 we show that the restriction map from free homotopy invariant iterated integrals to loops is a surjective map. This assertion is the key to the subsequent sections. In Section 3 we first show that homotopy invariant iterated integrals always give higher order invariants. For the case of surfaces, this was proven by Sreekantan in [18]. We then use the result from Section 3 to deduce the main result, saying that higher order invariants can always be expressed by iterated integrals. In the final section we consider restricted forms of higher order, which come with triviality assumptions along boundary components. This is typical for automorphic forms, where these restrictions refer to cusp forms or cuspidal cohomology. We formulate the corresponding theorem asserting that the main result is stable under boundary restrictions.

1 Generalities on Iterated Integrals

In this section we fix notations. Let X be a smooth connected manifold and $x_0, x \in X$ points. We write PX for the *path space*, i.e., the set of all smooth maps $p: [0, 1] \rightarrow X$.

Received by the editors July 27, 2011.

Published electronically June 26, 2012.

AMS subject classification: 14F35, 11F12, 55D35, 58A10.

Keywords: higher order forms, iterated integrals.

We also write PX_{x_0} for the subset of all paths that start at x_0 and $PX_{x_0,x}$ for the subset of all smooth paths from x_0 to x . The space $LX_{x_0} = PX_{x_0,x_0}$ is also called the *loop space* at x_0 .

For a path p and 1-forms $\omega_1, \dots, \omega_r$ we define the *iterated integral*:

$$\int_p \omega_1 \cdots \omega_r = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} p^* \omega_1(t_1) p^* \omega_2(t_2) \cdots p^* \omega_r(t_r).$$

For an integer s , let $B_s(X)$ denote the space of all maps $\omega: PX \rightarrow \mathbb{C}$ that are linear combinations of iterated integrals of length $\leq s$. Here we include constants as they may be considered as iterated integrals of length zero. We also write $B(X)$ for the union of all $B_s(X)$ as s varies. Let

$$T(\Omega^1(X)) = \mathbb{C} \oplus \Omega^1(X) \oplus [\Omega^1(X) \otimes \Omega^1(X)] \oplus \cdots$$

be the tensorial algebra over the space $\Omega^1(X)$ of smooth 1-forms. The map assigning $\omega_1 \otimes \cdots \otimes \omega_r$ to the map $p \mapsto \int_p \omega_1 \cdots \omega_r$ is a linear map from $T(\Omega^1(X))$ to $B(X)$. This map has a non-trivial kernel that was determined by Chen in [2].

We denote by $B_s(X)_{x_0}$ the set of restrictions of elements of $B_s(X)$ to PX_{x_0} , and the space $B_s(X)_{x_0,x}$ is defined analogously.

Lemma 1.1 (i) *If φ is an orientation preserving diffeomorphism on $[0, 1]$, then*

$$\int_p \omega_1 \cdots \omega_r = \int_{p \circ \varphi} \omega_1 \cdots \omega_r.$$

(ii) *If F is a diffeomorphism on X , then $\int_{F \circ p} \omega_1 \cdots \omega_r = \int_p (F^* \omega_1) \cdots (F^* \omega_r)$.*

(iii) *If p and q are composable paths, then*

$$\int_{pq} \omega_1 \cdots \omega_r = \sum_{j=0}^r \int_p \omega_1 \cdots \omega_j \int_q \omega_{j+1} \cdots \omega_r.$$

(iv) *One has*

$$\left(\int_p \omega_1 \cdots \omega_r \right) \left(\int_p \omega_{r+1} \cdots \omega_{r+s} \right) = \sum_{\sigma} \int_p \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)},$$

where the sum runs over all (r, s) -shuffles, i.e., permutations σ on $r + s$ letters with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r + 1) < \cdots < \sigma^{-1}(r + s)$.

(v) $\int_{p^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_p \omega_r \cdots \omega_1$, where $p^{-1}(t) = p(1 - t)$.

(vi) For given $\omega \in B(X)$, we extend the map $p \mapsto \int_p \omega$ to the free abelian group $\mathbb{Z}[PX]$ generated by PX . For given $\alpha_1, \dots, \alpha_s \in PX_{x_0,x_0}$ let $\eta = (\alpha_1 - 1)(\alpha_2 - 1) \cdots (\alpha_s - 1) \in \mathbb{Z}[PX]$. For 1-forms $\omega_1, \dots, \omega_r$, $r \leq s$, we have

$$\int_{\eta} \omega_1 \cdots \omega_r = \begin{cases} \prod_{i=1}^s \int_{\alpha_i} \omega_i & r = s, \\ 0 & r < s. \end{cases}$$

Note that (iv) implies that

$$B(X)_{x_0} = \bigcup_{s=0}^{\infty} B_s(X)_{x_0}$$

is a filtered algebra and likewise for $B(X)_{x_0, x}$.

Proof (i)–(v) are easy exercises. (vi) is a result of Hain; see [15, Proposition 2.13]. ■

If we replace the tensor product on $T(X)$ by the *shuffle product* $*$ given by

$$\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma} \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)},$$

where the sum runs over all (r, s) -shuffles, we obtain the *shuffle algebra* $\text{Sh}(X)$. We have shown that the iterated integrals form an algebra homomorphism

$$\text{Sh}(X) \rightarrow B(X),$$

where the latter is an algebra under pointwise multiplication.

Let $B_s(X)^{\text{hom}}$ denote the space of all elements of $B_s(X)$ that are invariant under homotopies with fixed end-points. Similarly define $B_s(X)_{x_0}^{\text{hom}}$ and $B_s(X)_{x_0, x}^{\text{hom}}$.

2 The Restriction Map

Theorem 2.1 *The restriction map $B_s(X)_{x_0}^{\text{hom}} \rightarrow B_s(X)_{x_0, x_0}^{\text{hom}}$ is surjective.*

Proof For a real vector space V , let $T(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots$ be the tensorial algebra. Consider the map $D: T(\Omega^1(X)) \rightarrow T(\Omega(X))$ given by

$$\begin{aligned} D(\omega_1 \otimes \cdots \otimes \omega_n) &= \sum_{j=1}^n \omega_1 \otimes \cdots \otimes (d\omega_j) \otimes \cdots \otimes \omega_n \\ &\quad + \sum_{j=1}^{n-1} \omega_1 \otimes \cdots \otimes (\omega_j \wedge \omega_{j+1}) \otimes \cdots \otimes \omega_n, \end{aligned}$$

and $D(c) = 0$, for $c \in \mathbb{R}$. By [3, Proposition 1.5.2] the map $\int: T(\Omega^1 X) \rightarrow \Omega^0(PX)$ sending $\omega_1 \otimes \cdots \otimes \omega_s$ to the map $p \mapsto \int_p \omega_1 \cdots \omega_s$ extends to a map $T(\Omega X) \rightarrow \Omega(PX)$ such that for $\omega \in T(\Omega^1(C))$ one has

$$d\left(\int \omega\right) = - \int D\omega - p_0^* \omega_1 \int \omega_1 \cdots \omega_s + (-1)^{s-1} p_1^* \omega_s \int \omega_1 \cdots \omega_{s-1},$$

where $p_0, p_1: PX \rightarrow X$ are the evaluation maps that map a path to its start and end point, respectively. Therefore, the kernel of D is mapped to homotopy invariant

iterated integrals on PX_{x_0} . Let M be the kernel of D . Then M has a natural filtration by degrees:

$$\mathbb{R} = M_0 \subset M_1 \subset \dots$$

The map $I_{x_0} : \omega_1 \otimes \dots \otimes \omega_s \mapsto \int \omega_1 \dots \omega_s$ maps M_s to $B_s(X)_{x_0}^{\text{hom}}$. We get a commutative diagram:

$$\begin{array}{ccc} M_s & \xrightarrow{I_{x_0}} & B_s(X)_{x_0}^{\text{hom}} \\ & \searrow I_{x_0, x_0} & \downarrow \text{res} \\ & & B_s(X)_{x_0, x_0}^{\text{hom}} \end{array}$$

The integral $\int \omega_1 \dots \omega_s$ is a function on the path space PX . Its restriction to the loop space $L_{x_0}X$ is homotopy invariant if and only if it is locally constant, which is the case if and only if it is annihilated by the differential of the complex $\Lambda(PM)$ as in [3]. Now the differential D above also coincides with the differential of the bar construction on [3, Section 4.1]. Theorem 4.1.1 of [3] states that the iterated integral map is an isomorphism of graded differential algebras from that bar construction to the iterated integrals on the loop space. Therefore the iterated integral map I_{x_0, x_0} is surjective, hence the restriction map is also surjective. ■

3 The Fundamental Group

For a group Γ we write its group ring as $A = \mathbb{Z}\Gamma$. Let $J \subset \mathbb{Z}\Gamma$ be the augmentation ideal, *i.e.*, the span of all elements of the form $(\gamma - 1)$, where $\gamma \in \Gamma$. For any $\mathbb{Z}\Gamma$ -module V we write $H_s^0(\Gamma, V)$ for the \mathbb{Z} -module of all $v \in V$ with $J^s v = 0$. This space can be identified with $\text{Hom}_{\mathbb{Z}\Gamma}(\mathbb{Z}\Gamma/J^s, V)$. The elements of $H_s^0(\Gamma, V)$ for varying s are called *higher order invariants*. If v is in $H_s^0(\Gamma, V)$, but not in $H_{s-1}^0(\Gamma, V)$, then s is called the *order* of v .

Let X be a connected smooth manifold, $x_0 \in X$ a base-point, and $\Gamma = \pi_1(X, x_0)$ the corresponding fundamental group. We consider Γ as group of deck transformations on the universal covering \tilde{X} of X . We also fix a pre-image x_0 in \tilde{X} , which we will denote by the same symbol x_0 as no confusion can arise.

As \tilde{X} is simply connected, the iterated integral $\int_p \omega$ for $\omega \in B_s(\tilde{X})_{x_0}^{\text{hom}}$ only depends on the endpoints x, y of the path p . We therefore write $\int_x^y \omega = \int_p \omega$.

Every $\gamma \in \Gamma$ can be viewed as a homotopy class of a loop based at $x_0 \in X$. In this way we get a map $B_s(X)_{x_0, x_0}^{\text{hom}} \rightarrow \text{Map}(\Gamma, \mathbb{C})$ that maps $\omega \in B_s(X)_{x_0, x_0}^{\text{hom}}$ to the map $\gamma \mapsto \int_\gamma \omega$. The latter map induces a \mathbb{Z} -linear map from the group ring $\mathbb{Z}\Gamma$ to \mathbb{C} . It is the content of Chen's de Rham Theorem for fundamental groups (see [3, Corollary 1 to Theorem 2.6.1], see also [17]) that this map induces a bijection

$$B_s(X)_{x_0, x_0}^{\text{hom}} \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma/J^{s+1}, \mathbb{C}).$$

Each $\omega \in B_s(X)_{x_0}^{\text{hom}}$ lifts to \tilde{X} and gives an element of $B_s(\tilde{X})_{x_0}^{\text{hom}}$. For $x \in \tilde{X}$ we write $\int_{x_0}^x \omega$ for the iterated integral of this lift over any path joining x_0 and x , or, what amounts to the same, the integral of ω over the projection to X of any such path.

Theorem 3.1 *If $\omega \in B_s(X)_{x_0}^{\text{hom}}$, then the function $\int_{x_0}^x \omega$, $x \in \tilde{X}$, is an invariant of order at most $s + 1$ in the Γ -module $C^\infty(\tilde{X})$. This defines an injective linear map*

$$\Psi : B_s(X)_{x_0}^{\text{hom}} \hookrightarrow H_{s+1}^0(\Gamma, C^\infty(\tilde{X})).$$

The case when X is the hyperbolic plane is in [18].

Proof Let $\omega \in B_s(X)^{\text{hom}}$, and for $x \in \tilde{X}$, set $f_\omega(x) = \int_{x_0}^x \omega$. We have to show that

$$[(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]^* f_\omega = 0$$

for any $\gamma_1, \dots, \gamma_{s+1} \in \Gamma$. For given $x \in X$ and $\gamma \in \Gamma$, we choose a path γ_x from x to γx . The map $\gamma \mapsto \gamma_x$ is extended linearly to a map $\mathbb{Z}\Gamma \rightarrow \mathbb{Z}[PX]$. For every $x \in X$ we also fix a smooth path p_x from x_0 to x . Let $\omega \in B_s(X)^{\text{hom}}$, and let $\eta = \sum_\gamma c_\gamma \gamma$ be an arbitrary element of the group ring $\mathbb{Z}\Gamma$. We have

$$\begin{aligned} \eta^* f_\omega(x) &= \sum_\gamma c_\gamma \gamma^* f_\omega(x) = \sum_\gamma c_\gamma \int_{x_0}^{\gamma x} \omega \\ &= \sum_\gamma c_\gamma \int_{p_x \gamma_x} \omega = \int_{p_x \sum_\gamma c_\gamma \gamma_x} \omega = \int_{p_x \eta_x} \omega. \end{aligned}$$

We apply this to the element $(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)$ of the group ring, and we look at any monomial $\omega_1 \cdots \omega_r$ in ω , where the ω_j are 1-forms on X . We then have

$$\int_{p_x [(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]_x} \omega_1 \cdots \omega_r = \sum_{k=0}^r \int_{p_x} \omega_1 \cdots \omega_k \int_{[(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]_x} \omega_{k+1} \cdots \omega_r.$$

Let $\bar{x} \in X$ be the image of $x \in \tilde{X}$, and let $\gamma_{\bar{x}}$ be the image of γ_x in X . Then $\gamma_{\bar{x}}$ is a loop based at \bar{x} . As the forms ω_j are Γ -invariant, we have

$$\begin{aligned} \int_{[(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]_x} \omega_{k+1} \cdots \omega_r &= \int_{[(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]_{\bar{x}}} \omega_{k+1} \cdots \omega_r \\ &= \int_{(\gamma_{1, \bar{x}} - 1) \cdots (\gamma_{s+1, \bar{x}} - 1)} \omega_{k+1} \cdots \omega_r \\ &= 0 \end{aligned}$$

by Lemma 1.1(v). This proves the first claim. For the injectivity of the induced map, let $\omega \in B_s(X)_{x_0}^{\text{hom}}$ with $\int_{x_0}^x \omega = 0$. This just means that $\omega = 0$ in $B_s(X)_{x_0}^{\text{hom}}$. ■

We formally set $H_0^0 = 0$ and $B_{-1} = 0$.

Theorem 3.2 *Let K be the ideal in the algebra $B(X)_{x_0}$ generated by $d(C^\infty(X))$ and let $K_s^{\text{hom}} = B_s(X)_{x_0}^{\text{hom}} \cap K$. Then K_s^{hom} is the kernel of the restriction map*

$$B_s(X)_{x_0}^{\text{hom}} \xrightarrow{\text{res}} B_s(X)_{x_0, x_0}^{\text{hom}},$$

and Ψ induces an isomorphism of $C^\infty(X)$ -modules

$$C^\infty(X) \otimes (B_s(X)_{x_0}^{\text{hom}}/K_s^{\text{hom}}) \xrightarrow{\cong} H_{s+1}^0(\Gamma, C^\infty(\tilde{X})).$$

It follows that $H_{s+1}^0(\Gamma, C^\infty(\tilde{X}))$ is a free module of the algebra $C^\infty(X)$ of smooth functions.

Proof The fact that K_s^{hom} is the kernel of the restriction map is a consequence of [2, Theorem 4.5]. Write $\bar{B}_s = B_s/B_{s-1}$ and let

$$\bar{K}_s^{\text{hom}} = \ker [B_s(X)_{x_0}^{\text{hom}} \rightarrow \bar{B}_s(X)_{x_0, x_0}^{\text{hom}}].$$

Then \bar{K}_s^{hom} contains $B_{s-1}(X)_{x_0}^{\text{hom}}$, and our assertion is equivalent to

$$C^\infty(X) \otimes (B_s(X)_{x_0}^{\text{hom}}/\bar{K}_s^{\text{hom}}) \xrightarrow{\cong} \bar{H}_{s+1}^0(\Gamma, C^\infty(\tilde{X})),$$

which is what we prove.

By Chen’s de Rham Theorem for fundamental groups (see [3, Corollary 1 to Theorem 2.6.1], see also [17]) the evaluation of iterated integrals gives an isomorphism

$$\bar{B}_s(X)_{x_0, x_0}^{\text{hom}} \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(J^s/J^{s+1}, \mathbb{C}).$$

The right-hand side can also be viewed as $\text{Hom}_{\mathbb{Z}\Gamma}(J^s/J^{s+1}, \mathbb{C})$ and as such be embedded into

$$\text{Hom}_A(J^s/J^{s+1}, C^\infty(X)) \cong \text{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})),$$

where we have written $A = \mathbb{Z}\Gamma$. More precisely, the image in

$$\text{Hom}_A(J^s/J^{s+1}, C^\infty(X)) \cong C^\infty(X) \otimes \text{Hom}_A(J^s/J^{s+1}, \mathbb{C})$$

is a basis of this $C^\infty(X)$ -module, which means that we have an isomorphism of $C^\infty(X)$ -modules,

$$C^\infty(X) \otimes \bar{B}_s(X)_{x_0, x_0}^{\text{hom}} \xrightarrow{\cong} \text{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})). \quad \blacksquare$$

Lemma 3.3 *The cohomology group $H^1(\Gamma, C^\infty(\tilde{X}))$ is trivial.*

Proof A 1-cocycle is a map $\alpha: \Gamma \rightarrow C^\infty(\tilde{X})$ such that $\alpha(\gamma\tau) = \gamma\alpha(\tau) + \alpha(\gamma)$ holds for all $\gamma, \tau \in \Gamma$. We have to show that for any given such map α there exists $f \in C^\infty(\tilde{X})$ such that $\alpha(\tau) = \tau f - f$.

Fix a smooth map $u: \tilde{X} \rightarrow [0, 1]$ such that

$$\sum_{\tau \in \Gamma} u(\tau^{-1}x) \equiv 1,$$

where we can assume that the sum is locally finite. Set

$$f(x) = - \sum_{\tau \in \Gamma} \alpha(\tau)(x) u(\tau^{-1}x).$$

Then the function f lies in the space $C^\infty(\tilde{X})$. We now compute for $\gamma \in \Gamma$,

$$\begin{aligned} \gamma f(x) - f(x) &= f(\gamma^{-1}x) - f(x) \\ &= \sum_{\tau \in \Gamma} \alpha(\tau)(x) u(\tau^{-1}x) - \alpha(\tau)(\gamma^{-1}x) u(\tau^{-1}\gamma^{-1}x) \\ &= \sum_{\tau \in \Gamma} \alpha(\tau)(x) u(\tau^{-1}x) + \alpha(\gamma)(x) \sum_{\tau \in \Gamma} u((\gamma\tau)^{-1}x) \\ &\quad - \sum_{\tau \in \Gamma} \alpha(\gamma\tau)(x) u((\gamma\tau)^{-1}x) \end{aligned}$$

The first and the last sum cancel, and the middle sum is $\alpha(\gamma)(x)$. Therefore, the lemma is proven. ■

Since $H^1(\Gamma, C^\infty(\tilde{X})) = 0$, [6, Lemma 2.1] implies that the exact sequence

$$0 \rightarrow J^s/J^{s+1} \rightarrow A/J^{s+1} \rightarrow A/J^s \rightarrow 0,$$

induces an isomorphism

$$\begin{aligned} \text{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})) &\cong \underbrace{\text{Hom}_A(A/J^{s+1}, C^\infty(\tilde{X})) / \text{Hom}_A(A/J^s, C^\infty(\tilde{X}))}_{\stackrel{\text{def}}{=} \overline{\text{Hom}}_A(A/J^{s+1}, C^\infty(\tilde{X}))} \\ &\cong \check{H}_{s+1}^0(\Gamma, C^\infty(\tilde{X})). \end{aligned}$$

We have to show that the ensuing diagram

$$\begin{array}{ccc} \bar{B}_s(X)_{x_0}^{\text{hom}} & \xrightarrow{\text{res}} & \bar{B}_s(X)_{x_0, x_0}^{\text{hom}} \\ \downarrow & & \downarrow \\ \overline{\text{Hom}}_A(A/J^{s+1}, C^\infty(\tilde{X})) & \xrightarrow{\cong} & \text{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})) \end{array}$$

commutes. This proves the claim, as we already have seen that the right vertical arrow becomes an isomorphism after tensoring with $C^\infty(X)$. This commutativity is a direct consequence of Lemma 1.1(vi). ■

4 Parabolic Restrictions

To satisfy requirements in the theory of automorphic forms, we also introduce further structure: Let $P \subset \Gamma$ be a conjugation-invariant subset and let $\langle P \rangle$ be the normal subgroup generated by P . Then AJ_P is a two-sided ideal of A , where $J_P \subset \mathbb{Z} \langle P \rangle \subset A$ is the augmentation ideal of $\langle P \rangle$. Let $J_s = J^s + AJ_P$, and let

$$H_{P,s}^0(\Gamma, V) = \{v \in V : J_s v = 0\}.$$

In the theory of automorphic forms (see [5, 8]), P will be the set of parabolic elements.

The P -restriction translates on the side of iterated integrals to the following. Recall Chen’s map

$$B_s(X)_{x_0, x_0}^{\text{hom}} \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma / J^{s+1}, \mathbb{C}).$$

We define the space $B_{P,s}(X)_{x_0, x_0}^{\text{hom}}$ to be the inverse image of

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma / J_{s+1}, \mathbb{C}) = \{ \alpha \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma / J^{s+1}, \mathbb{C}) : \alpha(p - 1) = 0 \forall p \in P \}$$

under this map. Finally, we set $B_{P,s}(X)_{x_0}^{\text{hom}}$ to be the inverse image of $B_{P,s}(X)_{x_0, x_0}^{\text{hom}}$ under the restriction map. So then $B_{P,s}(X)_{x_0}^{\text{hom}}$ is the set of all $\omega \in B_s(X)_{x_0}^{\text{hom}}$ with $\int_{p-1} \omega = 0$ for every $p \in P$. Theorems 3.1 and 3.2 generalize to the following.

Theorem 4.1 *If $\omega \in B_{P,s}(X)_{x_0}^{\text{hom}}$, then the function $\int_{x_0}^x \omega$ is an invariant of order at most $s + 1$ in the Γ -module $C^\infty(\tilde{X})$. This defines an injective linear map*

$$\Psi : B_{P,s}(X)_{x_0}^{\text{hom}} \hookrightarrow H_{P,s+1}^0(\Gamma, C^\infty(\tilde{X})).$$

Theorem 4.2 *Let K be the ideal in the algebra $B(X)_{x_0}$ generated by $d(C^\infty(X))$ and let $K_{P,s}^{\text{hom}} = B_{P,s}(X)_{x_0}^{\text{hom}} \cap K$. Then $K_{P,s}^{\text{hom}}$ is the kernel of the restriction map*

$$B_{P,s}(X)_{x_0}^{\text{hom}} \xrightarrow{\text{res}} B_{P,s}(X)_{x_0, x_0}^{\text{hom}},$$

and Ψ induces an isomorphism of $C^\infty(X)$ -modules

$$C^\infty(X) \otimes (B_{P,s}(X)_{x_0}^{\text{hom}} / K_{P,s}^{\text{hom}}) \xrightarrow{\cong} H_{P,s+1}^0(\Gamma, C^\infty(\tilde{X})).$$

Proof The proofs are the same as in the unrestricted case. One only has to check that the restriction conditions match, which is easy to see. ■

References

[1] K.-T. Chen, *Iterated integrals and exponential homomorphisms*. Proc. London Math. Soc. (3) **4**(1954), 502–512. <http://dx.doi.org/10.1112/plms/s3-4.1.502>
 [2] ———, *Algebras of iterated path integrals and fundamental groups*. Trans. Amer. Math. Soc. **156**(1971), 359–379. <http://dx.doi.org/10.1090/S0002-9947-1971-0275312-1>
 [3] ———, *Iterated path integrals*. Bull. Amer. Math. Soc. **83**(1977), no. 5, 831–879. <http://dx.doi.org/10.1090/S0002-9904-1977-14320-6>

- [4] G. Chinta, N. Diamantis, and C. O'Sullivan, *Second order modular forms*. Acta Arith. **103**(2002), no. 3, 209–223. <http://dx.doi.org/10.4064/aa103-3-2>
- [5] A. Deitmar, *Higher order group cohomology and the Eichler-Shimura map*. J. Reine Angew. Math. **629**(2009), 221–235. <http://dx.doi.org/10.1515/CRELLE.2009.032>
- [6] ———, *Higher order invariants in the case of compact quotients*. Cent. Eur. J. Math. **9**(2011), no. 1, 85–101. <http://dx.doi.org/10.2478/s11533-010-0081-9>
- [7] ———, *Lewis-Zagier correspondence for higher order forms*. Pacific J. Math. **249**(2011), no. 1, 11–21. <http://dx.doi.org/10.2140/pjm.2011.249.11>
- [8] A. Deitmar and N. Diamantis, *Automorphic forms of higher order*. J. Lond. Math. Soc. (2) **80**(2009), no. 1, 18–34. <https://dx.doi.org/10.1112/jlms/jdp015>
- [9] ———, *A new multiple Dirichlet series induced by a higher-order form*. Acta Arith. **142**(2010), no. 4, 303–309. <http://dx.doi.org/10.4064/aa142-4-1>
- [10] N. Diamantis and P. Kleban, *New percolation crossing formulas and second-order modular forms*. Commun. Number Theory Phys. **3**(2009), no. 4, 677–696.
- [11] N. Diamantis, M. Knopp, G. Mason, and C. O'Sullivan, *L-functions of second-order cusp forms*. Ramanujan J. **12**(2006), no. 3, 327–347. <http://dx.doi.org/10.1007/s11139-006-0147-2>
- [12] N. Diamantis and C. O'Sullivan, *The dimensions of spaces of holomorphic second-order automorphic forms and their cohomology*. Trans. Amer. Math. Soc. **360**(2008), no. 11, 5629–5666. <http://dx.doi.org/10.1090/S0002-9947-08-04755-7>
- [13] N. Diamantis and D. Sim, *The classification of higher-order cusp forms*. J. Reine Angew. Math. **622**(2008), 121–153. <http://dx.doi.org/10.1515/CRELLE.2008.067>
- [14] N. Diamantis and R. Sreekantan, *Iterated integrals and higher order automorphic forms*. Comment. Math. Helv. **81**(2006), no. 2, 481–494. <http://dx.doi.org/10.4171/CMH/60>
- [15] R. M. Hain, *The geometry of the mixed Hodge structure on the fundamental group*. In: Algebraic geometry, Bowdoin, 1985 (Brunswick, Main, 1985), Proc. Sympos. Pure Math., 46, American Mathematical Society, Providence, RI, 1987, pp. 247–282.
- [16] M. Kontsevich, *Vassiliev's knot invariants*. In: I. M. Gel'fand Seminar, Adv. Soviet Math., 16, American Mathematical Society, Providence, RI, 1993, pp. 137–150.
- [17] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 52, Springer-Verlag, Berlin, 2008.
- [18] R. Sreekantan, *Higher order modular forms and mixed Hodge theory*. Acta Arith. **139**(2009), no. 4, 321–340. <http://dx.doi.org/10.4064/aa139-4-2>

Mathematisches Institut, Auf der Morgenstelle 10, 72076 Tübingen, Germany
 e-mail: deitmar@uni-tuebingen.de ivan.horozov@uni-tuebingen.de