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Cuspidal quintics and surfaces with $p_g = 0$, $K^2 = 3$ and 5-torsion

Carlos Rito

Abstract

If S is a quintic surface in $\mathbb{P}^3$ with singular set 15 3-divisible ordinary cusps, then there is a Galois triple cover $\phi : X \to S$ branched only at the cusps such that $p_g(X) = 4$, $q(X) = 0$, $K_X^2 = 15$ and ϕ is the canonical map of X. We use computer algebra to search for such quintics having a free action of $\mathbb{Z}_5$, so that $X/\mathbb{Z}_5$ is a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 3$. We find two different quintics, one of which is the van der Geer–Zagier quintic; the other is new.

We also construct a quintic threefold passing through the 15 singular lines of the Igusa quartic, with 15 cuspidal lines there. By taking tangent hyperplane sections, we compute quintic surfaces with singular sets $17\mathsf{A}_2$, $16\mathsf{A}_2$, $15\mathsf{A}_2 + \mathsf{A}_3$ and $15\mathsf{A}_2 + \mathsf{D}_4$.

1. *Introduction*

In the context of Hilbert modular surfaces, van der Geer and Zagier [6] have constructed a smooth minimal surface of general type X such that $K_X^2 = 10$, $p_g(X) = 4$, $q(X) = 0$ and the canonical map ϕ of X is of degree 2 onto a quintic surface S with 20 ordinary double points (nodes), ramified only over these points. These two surfaces have a free action of $\mathbb{Z}_5$; thus, by taking quotients, one gets a numerical Campedelli surface X' ($K^2 = 2$, $p_g = q = 0$), which is a double cover of a Godeaux surface S' ($K^2 = 1$, $p_g = q = 0$, $\pi_1 = \mathbb{Z}_5$), ramified over four nodes. Moreover, as noted by Catanese [5, § 5], X is simply connected, so also $\pi_1(X') = \mathbb{Z}_5$. This construction is a particular case of the Campedelli surfaces constructed in [5].

There is a similar construction involving a triple cover ramified over ordinary cusps (singularities of type A_2) instead of a double cover ramified over nodes. The same paper [6] contains the construction of a quintic surface S in $\mathbb{P}^3$ with 15 cusps. Later, Barth [1, unpublished] has shown that this set of cusps is 3-divisible, that is, there is a labelling A_i, A_i' for the (-2)-curves corresponding to the resolution $A_i + A_i'$, $i = 1, \ldots, 15$, of the cusps such that

$$\sum_1^{15}(2A_i + A_i') \equiv 3L$$

for some divisor L. Following Tan [8, Theorem 4.3.1] (see also [7, § 1.3]), this implies the existence of a Galois triple cover $X \to S$, ramified only over the cusps, such that X is a smooth minimal surface of general type with $K^2 = 15$, $p_g = 4$, $q = 0$ and this triple cover is the canonical map of X.

The quintic S has a free action of $\mathbb{Z}_5$, which extends to a free action on X. So, as noticed by Tan [8, Theorem I], taking quotients we get surfaces X', S' such that $K_{X'}^2 = 3$, $p_g(X') = 0$,

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$K^2_{S'} = 1$, $p_g(S') = 0$ and there is a triple cover $X' \to S'$ ramified only over the three cusps of S'.

The van der Geer–Zagier quintic S seems very special; it is invariant for the action of the symmetric group in five elements. In this paper we construct another quintic surface in $\mathbb{P}^3$ with a 3-divisible set of 15 cusps and with a free action of $\mathbb{Z}_5$, and show that it is not isomorphic to S.

The observation which allowed us to have (computational) success with the construction is the following. We considered some random quartic surfaces in $\mathbb{P}^3$ with (at least) 15 nodes at points $p_1, \ldots, p_{15}$. For each case, the linear system of quintic surfaces with double points at $p_1, \ldots, p_{15}$ is of dimension 4, and we were able to find an element with 15 cusps. So, our strategy is to first compute a $\mathbb{Z}_5$-invariant quartic with 15 nodes and then try to find the quintic.

Consider the $\mathbb{Z}_5$ action

$$(x : y : z : w) \mapsto (ex : e^2y : e^3z : e^4w),$$

where e is a fifth root of unity. The quartic monomials invariant under this action are

$$x^3y,\ y^3w,\ xz^3,\ zw^3,\ x^2w^2,\ y^2z^2,\ xyzw.$$

Thus, a general invariant quartic passes with multiplicity 1 through the four fixed points $(1 : 0 : 0 : 0), \ldots, (0 : 0 : 0 : 1)$ of the action. Semi-invariant quartics contain exactly three of these points and are singular at one of them. For instance, the quartic monomials corresponding to the eigenvalue 4 are

$$x^4,\ y^3z,\ yw^3,\ x^2zw,\ xy^2w,\ xyz^2,\ z^2w^2.$$

A general quartic in this space passes through $(0 : 1 : 0 : 0), (0 : 0 : 0 : 1)$ and is singular at $(0 : 0 : 1 : 0)$. So, the semi-invariant quartic that we are going to compute has in fact 16 nodes, while the invariant quartic corresponding to the van der Geer–Zagier quintic has exactly 15 nodes. We use this to show that our quintic surface is not isomorphic to the van der Geer–Zagier quintic.

Now recall that the Igusa quartic threefold is singular at 15 lines, so, by analogy with the quintic surface and its associated quartic, we might expect the existence of a quintic threefold in $\mathbb{P}^4$ with 15 *cuspidal lines*, that is, such that a general hyperplane section is a quintic surface with 15 cusps. This is in fact the case; we compute an equation for such threefold (in particular with S_5 symmetry). We think that this quintic is interesting on its own.

By taking hyperplanes tangent to this threefold, we found equations for quintic surfaces with singular sets $17\mathsf{A}_2$, $16\mathsf{A}_2$, $15\mathsf{A}_2 + \mathsf{A}_3$ and $15\mathsf{A}_2 + \mathsf{D}_4$. To our knowledge only examples with at most 15 cusps (A_2 singularities) were known.

This suggests that quintic surfaces with more than 17 cusps could exist. The problem of constructing surfaces with many cusps has been addressed by Barth and Rams [**2**, **3**]. The difficulty in finding new examples increases with the number of cusps. Quintic surfaces with $n \geqslant 18$ cusps could provide interesting examples of surfaces with non-birational canonical map (see, for example, [**8**]).

The paper is organized as follows. In § 2, we review the van der Geer–Zagier quintic surface. Then we explain the main steps of the construction of our quintic. In § 4, we prove the 3-divisibility of the 15 cusps of our surface. Then we compute the equation of the quintic threefold and finally § 6 explains the computation of the quintic surfaces with more than 15 cusps.

We use the computational algebra system Magma [**4**] to perform the computations. The corresponding input code lines are given in the appendix (a file containing the output lines is available on the author's web page).

Notation. We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in algebraic geometry.

2. *Van der Geer–Zagier's quintic*

Van der Geer and Zagier [6] have shown the existence of a quintic surface V in $\mathbb{P}^3$ with 15 ordinary cusps (singularities of type A_2) and no other singularities. It is given in $\mathbb{P}^4(x, y, z, w, t)$ by

$$12s_5 - 5s_2s_3 = 0, \quad s_1 = 0,$$

where $s_i := x^i + y^i + z^i + w^i + t^i$. The $\mathbb{Z}_5$ action $(x, y, z, w, t) \mapsto (y, z, w, t, x)$ acts freely on V. The quotient of V by this action is a Godeaux surface ($p_g = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_5$) with three cusps.

Later, Barth [1] showed that these cusps are 3-divisible. He found a relation between the (-2)-curves contained in the resolution of the cusps and some divisors corresponding to 15 lines contained in V. We use an analogous argument below in § 4.

Here we note that there exists exactly one quartic surface Q with singular set 15 ordinary double points at the points $p_1, \ldots, p_{15}$ where V has cusps. Due to the symmetry, we expect it to be in the pencil generated by s_4, s_2^2 and in fact it is given in $\mathbb{P}^4$ by

$$4s_4 - s_2^2 = 0, \quad s_1 = 0.$$

In general, it can be shown that 15 nodes determine a quartic surface.

This surface contains 10 plane conics such that each conic contains six of the points p_i. We show in the appendix that V and Q meet exactly at these conics and confirm the uniqueness of Q.

3. *The $\mathbb{Z}_5$-invariant surface*

THEOREM 1. *Let $q := e^3 + e^2$, with e a fifth root of unity.*
The $\mathbb{Z}_5$-invariant quintic surface S with equation

$$\begin{aligned}
3x^5 - 4y^5 &+ (60q + 120)xy^3z + (-90q - 150)x^2yz^2 + (-220q - 356)z^5 \\
&+ (30q + 30)x^2y^2w + (20q + 30)x^3zw + (390q + 630)yz^3w + (-210q - 350)y^2zw^2 \\
&+ (120q + 195)xz^2w^2 + (-120q - 180)xyw^3 + (20q + 32)w^5 = 0
\end{aligned}$$

has 15 *cusps and no other singularities.*
The quartic surface Q with equation

$$\begin{aligned}
x^4 &+ (4q + 8)y^3z + (-12q - 20)xyz^2 + (4q + 4)xy^2w \\
&+ (4q + 6)x^2zw + (8q + 13)z^2w^2 + (-8q - 12)yw^3 = 0
\end{aligned}$$

has 16 *nodes,* 15 *of which coincide with the cusps of S.*
Moreover, S is not isomorphic to the van der Geer–Zagier quintic.

Proof. The first two sentences are easily verified using computer algebra. This is done in the appendix. Suppose that S is isomorphic to the van der Geer–Zagier quintic V. Then this isomorphism preserves the canonical systems of S and V, that is, hyperplanes of S are mapped to hyperplanes of V, which implies that S and V are projectively equivalent. Hence, the quartic Q is mapped to a 16-nodal surface Q' singular at the 15 cusps of V. But, as noted in § 2, the surface Q' has exactly 15 nodes. □

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The surface S is invariant for the action

$$(x : y : z : w) \mapsto (x : ey : e^2 z : e^3 w)$$

and, since its equation contains the monomials x^5, y^5, z^5, w^5, the action is base-point free.

Now we explain the steps taken to find the equation of S. The corresponding Magma code is given in the appendix.

We start by searching for the equation of a quartic surface Q in $\mathbb{P}^3$ with an action of $\mathbb{Z}_5$ and with 15 ordinary double points.

Fix the $\mathbb{Z}_5$ action as above. The system of quartic polynomials which are invariant under this action is generated by

$$x^4,\ y^3 z,\ yw^3,\ x^2 zw,\ xy^2 w,\ xyz^2,\ z^2 w^2.$$

We compute the monomials s_1, s_2, s_3 which generate the subsystem of the elements singular at the point $(1 : 1 : 1 : 1)$.

We want to find coefficients U, V such that the polynomial $F := s_1 + U s_2 + V s_3$ defines a normal surface with 15 nodes at the $\mathbb{Z}_5$ orbits of points

$$p_1 := (1 : 1 : 1 : 1), \quad p_2 := (x : y : z : w), \quad p_3 := (a : b : c : d).$$

These are given by points in the scheme S_C defined by $F = \partial F/\partial x = \ldots = \partial F/\partial w = 0$.

At this stage a generic point in S_C corresponds to a non-normal quartic surface. To overcome this, we impose the condition that the double point at $(1 : 1 : 1 : 1)$ is ordinary (not all order-3 minors of the Hessian matrix of F vanish).

Still this does not give 15 singular points. We need to add conditions to ensure that the points p_1, p_2, p_3 are in different $\mathbb{Z}_5$ orbits:

$$(xd)^5 - (aw)^5 \neq 0, \quad x^5 - w^5 \neq 0, \quad a^5 - d^5 \neq 0.$$

Now the scheme S_C is zero dimensional. Computing a point in S_C, we get a $\mathbb{Z}_5$-invariant quartic surface Q with 15 double points $p_1, \ldots, p_{15}$. As explained in the Introduction, there is an extra double point at $(0 : 0 : 1 : 0)$, a fixed point for the $\mathbb{Z}_5$ action.

The linear system of $\mathbb{Z}_5$-invariant polynomials of degree 5 with double points at $p_1, \ldots, p_{15}$ has two generators L_1, L_2. We want to compute b such that the quintic surface with equation $F := L_1 + bL_2 = 0$ has cusps at the points $p_1, \ldots, p_{15}$. This is done by imposing the vanishing of derivatives of F and of all order-3 minors of the Hessian matrix of F at these points.

Finally we confirm, using Magma, that all singularities are of type A_2.

4. 3-*divisibility of the cusps*

Let S be the $\mathbb{Z}_5$-invariant quintic computed above and G' be the quotient of S by the $\mathbb{Z}_5$ action. Denote by G the Godeaux surface obtained by resolving the three cusps of G'.

PROPOSITION 2. *The cusps of G' are 3-divisible, that is, there exists a divisor L such that $\sum_1^3 (2A_i + A_i') \equiv 3L$ for some labelling A_i, A_i' for the (-2)-curves in the resolution $A_i + A_i'$, $i = 1, \ldots, 3$, of the cusps.*

Notice that this implies the 3-divisibility of the 15 cusps of S.

Proof. Recall that a quartic Kummer surface has 16 *tropes*, which are plane conics through six nodes. The quintic S meets the Kummer surface Q at the 10 tropes of Q not containing

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the 16th node $(0:0:1:0)$. These conics are divided into two $\mathbb{Z}_5$ orbits $\overline{T}_1$, $\overline{T}_2$. One of the tropes of Q ($\{y=0\}$) through $(0:0:1:0)$ is fixed by the $\mathbb{Z}_5$ action and the remaining five tropes through this point correspond to an orbit $\overline{T}_3$ of plane quintics in S with five double points each.

Let T_1, T_2, T_3 be the strict transforms in G of the quotients of $\overline{T}_1$, $\overline{T}_2$, $\overline{T}_3$, respectively. We compute the intersection matrix of A_i, A_i', $i=1,2,3$, and T_1, T_2, T_3. This is achieved using the Magma function `Blowup` to resolve the singularities of S at a representative of each of the three orbits of five cusps. The matrix is

$$\begin{array}{c} \\ A_1 \\ A_1' \\ A_2 \\ A_2' \\ A_3 \\ A_3' \\ T_1 \\ T_2 \\ T_3 \end{array}
\begin{array}{c} \begin{array}{ccccccccc} A_1 & A_1' & A_2 & A_2' & A_3 & A_3' & T_1 & T_2 & T_3 \end{array} \\
\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & -2 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & -2 & 2 & 0 & 2 \\
1 & 0 & 2 & 0 & 1 & 2 & -4 & 0 & 0 \\
1 & 2 & 0 & 2 & 1 & 0 & 0 & -4 & 0 \\
2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & -1
\end{pmatrix} \end{array}.$$

Its determinant is zero. Since $b_2(G)=9$, these curves are dependent in $\mathrm{Num}(G)$ and this relation must be expressed in the nullspace of the matrix. As the basis for this nullspace is

$$\begin{pmatrix} 2 & 4 & 2 & -2 & -2 & -4 & -3 & 3 & 0 \end{pmatrix}$$

and $q(G)=0$ implies $\mathrm{NS}(G)=\mathrm{Pic}(G)$ (Castelnuovo), then there is a 5-torsion element t such that

$$2A_1+4A_1'+2A_2-2A_2'-2A_3-4A_3'-3T_1+3T_2\equiv t.$$

But $t\equiv t+5t\equiv 6t$; therefore, there exists L such that

$$2A_1+A_1'+2A_2+A_2'+A_3+2A_3'\equiv 3L.$$

The Magma code for the above is in the appendix. □

5. *The quintic threefold*

It is well known that the maximum number of nodes that a cubic threefold can have is 10 and there is exactly one such threefold, the *Segre cubic*, which can be given in $\mathbb{P}^4$ by the equation

$$x^3+y^3+z^3+w^3+t^3+h^3=0,\quad h:=-x-y-z-w-t.$$

The dual of the Segre cubic is the so-called *Igusa quartic*. Its singular set is an union of 15 lines.

Let the *monomial symmetric polynomial* $m_{(\alpha_1,\ldots,\alpha_5)}(x,y,z,w,t)$ be defined as the sum of all monomials $x^{\alpha_i}\ldots t^{\alpha_j}$ over all distinct permutations of $(\alpha_1,\ldots,\alpha_5)$.

PROPOSITION 3. *Let X be the S_5-invariant quintic threefold with equation*

$$\begin{aligned} 7m_{(5,0,0,0,0)}-5m_{(4,1,0,0,0)}-2m_{(3,2,0,0,0)}+4m_{(3,1,1,0,0)}\\ +2m_{(2,2,1,0,0)}-4m_{(2,1,1,1,0)}+8m_{(1,1,1,1,1)}=0. \end{aligned}$$

Then X intersects the Igusa quartic along its singular lines and has cusps along each of these lines.

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This result is proved easily using computer algebra. Such an equation is computed as follows.

Fix 15 points in the 15 singular lines of the Igusa quartic and compute the linear system of S_5-invariant quintics in $\mathbb{P}^4$ singular at these points.

We get a pencil of quintics which are singular at the 15 lines.

Let F, G be the generators of this pencil. Then compute b such that the threefold $\{F + bG = 0\}$ has cusps at these 15 lines. This is done by imposing the vanishing of the order-3 minors of the Hessian matrix of $F + bG$ at the above 15 points.

The corresponding Magma code is in the appendix.

6. *Quintics with* $17\mathsf{A}_2$, $16\mathsf{A}_2$, $15\mathsf{A}_2 + \mathsf{A}_3$ *and* $15\mathsf{A}_2 + \mathsf{D}_4$

A general hyperplane section of the quintic threefold computed above is a surface with 15 ordinary cusps. A tangent hyperplane section gives a surface $\{f = 0\}$ with an extra double point, which is ordinary only if one of the order-3 minors of the Hessian matrix of f is non-zero. By imposing the vanishing of all order-3 minors, we found quintic surfaces with singular sets $17\mathsf{A}_2$, $16\mathsf{A}_2$, $15\mathsf{A}_2 + \mathsf{A}_3$ and $15\mathsf{A}_2 + \mathsf{D}_4$.

The appendix contains the computer code for the case $17\mathsf{A}_2$. This surface is the hyperplane section

$$3x - 13y - 13z + 7w + 7t = 0$$

of the quintic computed in the previous section.

For the remaining cases, the hyperplane is not defined over the rationals. The details can be found on the author's web page.

Appendix. *Magma code*

The following code is implemented on the computational algebra system Magma, version V2.21-8.

Van der Geer–Zagier's quintic

Here we verify the assertions made in § 2 about the existence of a quartic surface Q which is singular at the 15 singular points of the van der Geer–Zagier quintic V.

```
P3<x,y,z,w>:=ProjectiveSpace(Rationals(),3);
t:=-(x+y+z+w);
s:=[x^i+y^i+z^i+w^i+t^i:i in [1..5]];
V:=Surface(P3,12*s[5]-5*s[2]*s[3]);
pts:=SingularPoints(V);#pts eq 15;
HasSingularPointsOverExtension(V) eq false;
L4:=LinearSystem(P3,4);
L:=LinearSystem(L4,[P3!x:x in pts],[2:x in pts]);
#Sections(L) eq 1;
Q:=Surface(P3,4*s[4]-s[2]^2);
Q eq Surface(P3,Sections(L)[1]);
#SingularPoints(Q) eq 15;
HasSingularPointsOverExtension(Q) eq false;
pc:=PrimeComponents(V meet Q);pc;
[#Points(pc[i] meet SingularSubscheme(V)) eq 6:i in [1..10]];
```

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The $\mathbb{Z}_5$-invariant surface

```
K<e>:=CyclotomicField(5);
P<x,y,z,w>:=ProjectiveSpace(K,3);
```

The following sequences of monomials s_4 and s_5 are invariant under the action $(x:y:z:w) \mapsto (x : ey : e^2z : e^3w)$. We compute the elements of s_4 which are singular at the point $(1:1:1:1)$.

```
s4:=[x^4,y^3*z,x*y*z^2,x*y^2*w,x^2*z*w,z^2*w^2,y*w^3];
s5:=[x^5,x^3*z*w,x^2*y^2*w,x^2*y*z^2,x*y^3*z,x*y*w^3,x*z^2*w^2,
     y^5,y^2*z*w^2,y*z^3*w,z^5,w^5];
LinearSystem(LinearSystem(P,s4),P![1,1,1,1],2);

A12<x,y,z,w,a,b,c,d,U,V,N1,N2>:=AffineSpace(K,12);
s:=[
    x^4 - 2*x^2*z*w + z^2*w^2,
    y^3*z - 2*x*y^2*w + x^2*z*w - z^2*w^2 + y*w^3,
    x*y*z^2 - x*y^2*w - z^2*w^2 + y*w^3
];
F:=s[1]+U*s[2]+V*s[3];
```

We want to compute U, V such that $\{F = 0\}$ has double points at (the orbits of) $(1:1:1:1)$, $(x:y:z:w)$ and $(a:b:c:d)$.

```
M:=Submatrix(HessianMatrix(Scheme(A12,F)),1,1,4,4);
m:=Minors(Evaluate(M,[1,1,1,1,a,b,c,d,U,V,N1,N2]),3)[1];
```

If there is a number N_1 such that $1+mN_1 = 0$ at $(1:1:1:1)$, then the minor $m \neq 0$ and the double point at $(1:1:1:1)$ is ordinary. This reduces the probability of getting a non-normal surface.

```
Hx:=[F] cat [Derivative(F,i):i in [1..4]];
Ha:=[Evaluate(Hx[i],[a,b,c,d,a,b,c,d,U,V,N1,N2]):i in [1..#Hx]];
```

To search for the 15 double points, we define the scheme S_C of the points such that $H_x = H_a = 0$ and $m \neq 0$.

```
SC:=Scheme(A12,Hx cat Ha cat [w-1,d-1,1+N1*m]);
```

We impose extra conditions to ensure that the points $(1:1:1:1)$, $(x:y:z:w)$ and $(a:b:c:d)$ are in different $\mathbb{Z}_5$ orbits:

```
SC:=Scheme(SC,[1+N2*(x^5-1)*(x^5-a^5)*(a^5-1)]);
Dimension(SC) eq 0;
```

We compute the points in $\mathbb{P}^3$ corresponding to one of the points of S_C:

```
p:=Points(SC)[1];
p:=[P![1,1,1,1],P![p[1],p[2],p[3],p[4]],P![p[5],p[6],p[7],p[8]]];
```

The unique $\mathbb{Z}_5$-invariant quartic with nodes at (the orbits of) the above three points:

```
L:=LinearSystem(LinearSystem(P,s4),p,[2,2,2]);
Q:=Scheme(P,Sections(L)[1]);
r:=SingularPoints(Q);
#r eq 16;
```

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The quartic Q has 16 nodes. One of these points is $(0:0:1:0)$, a fixed point for the action of $\mathbb{Z}_5$. Now we compute the linear system of invariant quintics with nodes at the 15 points different from $(0:0:1:0)$:

```
L:=LinearSystem(LinearSystem(P,s5),p,[2,2,2]);
#Sections(L) eq 2;
```

It remains to find the element of this pencil with 15 cusps.

```
A5<X,Y,Z,W,b>:=AffineSpace(K,5);
h:=hom<CoordinateRing(P)->CoordinateRing(A5)|[X,Y,Z,W]>;
F:=h(Sections(L)[1])+b*h(Sections(L)[2]);
M:=HessianMatrix(Scheme(A5,F));RemoveColumn(~M,5);RemoveRow(~M,5);
H:=[Evaluate(Minors(M,3)[i],[1,1,1,1,b]):i in [1..#Minors(M,3)]];
```

If all these minors vanish, the surface $\{F=0\}$ has a non-ordinary double point at $(1:1:1:1)$.

```
SC:=Scheme(A5,H cat [X,Y,Z,W]);
Dimension(SC) eq 0;
```

The points in S_C give three possibilities for b. One of these corresponds to a quintic surface with 15 ordinary cusps (A_2 singularities):

```
b:=Points(SC)[1][5];
F:=Sections(L)[1]+b*Sections(L)[2];
S:=Surface(P,F);
r:=SingularPoints(S);
```

There are exactly 15 singular points:

```
#r eq 15;
HasSingularPointsOverExtension(S) eq false;
```

We confirm that these singularities are of type A_2:

```
for x in r do IsSimpleSurfaceSingularity(S!x);end for;
```

3-divisibility of the cusps

The surfaces S and Q meet at the 10 tropes of Q not containing the point $(0:0:1:0)$:

```
pp:=PrimeComponents(Q meet S);
[#Points(pp[i] meet SingularSubscheme(S)):i in [1..10]];

P<x,y,z,w>:=P;
psi:=map<P->P|[x,y*e,z*e^2,w*e^3]>;
```

These conics are divided into two $\mathbb{Z}_5$ orbits $\overline{T}_1$, $\overline{T}_2$:

```
T1:=&join[pp[i]:i in [1,2,5,6,9]];
T1 eq psi(T1);
T2:=&join[pp[i]:i in [3,4,7,8,10]];
T2 eq psi(T2);
[Multiplicity(T1,r[i]):i in [4,5,6]] eq [1,2,3];
[Multiplicity(T2,r[i]):i in [4,5,6]] eq [3,2,1];
```

We define $\overline{T}_3$ as the $\mathbb{Z}_5$ orbit of plane quintics corresponding to five of the six tropes of Q through the fixed point $(0:0:1:0)$:

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```
L1:=LinearSystem(LinearSystem(P,1),[P!r[i]:i in [1,3,11,12,13]]);
t:=[Scheme(P,Sections(L1)[1])];
#Points(t[1] meet SingularSubscheme(S)) eq 5;
for i in [1..4] do t:=t cat [psi(t[#t])];end for;
T3:=(&join t) meet S;
T3 eq psi(T3);
[Multiplicity(T3,r[i]):i in [4,5,6]] eq [4,2,4];
```

The strict transform of T_3 does not intersect the strict transform of $T_1 + T_2$:

```
Points(T3 meet (T1 join T2)) eq r;
```

The resolution of each cusp is a union of two (-2)-curves A_i, A_i'. We compute the intersection number of T_1, T_2 and T_3 with these curves. First we blow up the quintic at a cusp:

```
T:=[T1,T2,T3];
for j in [4,5,6] do
 X,mp:=Blowup(S,P!r[j]);
```

We define the exceptional divisor E and the (-2)-curves A_i, A_i':

```
 E:=(P!r[j]) @@ mp;
 A:=PrimeComponents(E meet Complement(X,E));
```

and compute the intersection number of the strict transform of T_i with the (-2)-curves:

```
 for i in [1,2,3] do
  t:=Complement(Scheme(S,DefiningEquations(T[i])) @@ mp,E);
  #Points(t meet A[1]),#Points(t meet A[2]);
 end for;
end for;
```

This shows that the intersection matrix of the curves, in the Godeaux surface, corresponding to the A_2 configurations of the three cusps and the orbits T_1, T_2 and T_3, is

```
M:=SymmetricMatrix([-2,1,-2,0,0,-2,0,0,1,-2,0,0,0,0,-2,0,0,
0,0,1,-2,1,0,0,2,1,2,-4,1,2,2,0,1,0,0,-4,2,2,1,1,2,2,0,0,-1
]);M;
Determinant(M) eq 0;
Nullspace(M);
```

The quintic threefold

We define the Segre cubic S_3, the Igusa quartic I_4 and compute the singular set of I_4:

```
P4<x,y,z,w,t>:=ProjectiveSpace(Rationals(),4);
S3:=Scheme(P4,x^3+y^3+z^3+w^3+t^3+(-x-y-z-w-t)^3);
rho:=map<P4->P4|Basis(JacobianIdeal(DefiningEquation(S3)))>;
I4:=rho(S3);
SI4:=SingularSubscheme(I4);
```

The S_5 symmetric quintics are generated by

```
e:=[ElementarySymmetricPolynomial(CoordinateRing(P4),i):i in [1..5]];
s5:=[e[1]^5, e[1]^3*e[2], e[1]^2*e[3], e[1]*e[4],
e[1]*e[2]^2, e[2]*e[3], e[5]];
```

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We fix 15 singular points of I_4 and compute the symmetric quintics singular at these points:

```
r:=Points(Scheme(SI4,x+2*y+3*z+4*w+5*t));
L5:=LinearSystem(P4,s5);
L:=LinearSystem(L5,[P4!x:x in r],[2:x in r]);
```

A generic element of the pencil L is singular exactly at the 15 singular lines of the Igusa quartic:

```
X1:=SingularSubscheme(Scheme(P4,Sections(L)[1]));
X2:=SingularSubscheme(Scheme(P4,Sections(L)[2]));
SI4 eq ReducedSubscheme(X1);
SI4 eq ReducedSubscheme(X2);
```

We compute the element of L which contains 15 cuspidal lines:

```
A<X,Y,Z,W,T,b>:=AffineSpace(Rationals(),6);
h:=hom<CoordinateRing(P4)->CoordinateRing(A)|[X,Y,Z,W,T]>;
F:=h(Sections(L)[1])+b*h(Sections(L)[2]);
```

We compute b such that the quintic $\{F = 0\}$ has cusps at the 15 points above:

```
H:=HessianMatrix(Scheme(A,F));RemoveColumn(~H,6);RemoveRow(~H,6);
G:=[F] cat [Derivative(F,i):i in [1..5]] cat Minors(H,3);
```

The vanishing of these minors implies non-ordinary double points.

```
G:=[Evaluate(G[j],Coordinates(r[i]) cat [b]):
j in [1..#G],i in [1..#r]];
S:=Scheme(A,G cat [X,Y,Z,W,T]);
Dimension(S) eq 0;
PointsOverSplittingField(S);
```

This gives $b = -5/7$.

The quintic threefold with 15 cuspidal lines:

```
F:=Sections(L)[1]-5/7*Sections(L)[2];
Q:=Scheme(P4,F);
SI4 eq ReducedSubscheme(SingularSubscheme(Q));
Degree(SingularSubscheme(Q)) eq 30;
```

The coefficients of F in Sections(L_5):

```
CoefficientMap(L5)(F);
```

We verify that a random hyperplane section is a quintic surface with 15 cusps:

```
a:=[Random(1,100):i in [1..5]];
S:=Surface(P4,[F,a[1]*x+a[2]*y+a[3]*z+a[4]*w+a[5]*t]);
#SingularPoints(S) eq 15;
r:=SingularPoints(S);
for x in r do IsSimpleSurfaceSingularity(S!x);end for;
```

The quintic surface with 17 cusps

Now we compute an hyperplane section of the above quintic threefold which is a quintic surface with 17 cusps (A_2 singularities):

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```
A<X,Y,Z,W,T,a,b,c,d>:=AffineSpace(Rationals(),9);
h:=hom<CoordinateRing(P4)->CoordinateRing(A)|[X,Y,Z,W,T]>;
G:=a*X+b*Y+c*Z+d*W+T;
f:=Evaluate(h(F),T,-(a*X+b*Y+c*Z+d*W));
```

So, $\{f = 0\}$ is an hyperplane section of the quintic threefold.

```
H:=HessianMatrix(Scheme(A,f));
H:=Submatrix(H,1,1,5,5);
min1:=Minors(H,3);
```

If all these minors vanish, then the double point is non-ordinary.

```
J:=[JacobianSequence(h(F))[i]:i in [1..5]];
min2:=Minors(Matrix([J,[a,b,c,d,1]]),2);
```

If all these minors vanish, then the hyperplane is tangent to the quintic.

```
H:=[T-1,h(F),G] cat min1 cat min2;
SC:=Scheme(A,H cat [X,Y-2]);
pp:=PrimeComponents(SC);
[Dimension(x):x in pp];
```

Since S_C has zero-dimensional components, we can compute some points. These data allow us to compute the quintic surface with 17 A_2 points:

```
p:=Points(pp[4]);
a:=p[1][6];b:=p[1][7];c:=p[1][8];d:=p[1][9];
Q:=Surface(P4,[F,a*x+b*y+c*z+d*w+t]);
r:=Points(SingularSubscheme(Q));
#r eq 17;
for i in [1..#r] do IsSimpleSurfaceSingularity(Q!r[i]);end for;
HasSingularPointsOverExtension(Q);
```

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Carlos Rito
Universidade de Trás-os-Montes e Alto Douro (UTAD)
Quinta de Prados
5000-801 Vila Real
Portugal

crito@utad.pt
www.utad.pt

Current address:
Departamento de Matemática
Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre 687
4169-007 Porto
Portugal

crito@fc.up.pt
www.fc.up.pt