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Symbolic dynamics for pointwise hyperbolic systems on open regions

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Abstract. Under certain conditions, we construct a countable Markov partition for pointwise hyperbolic diffeomorphism $f : M \to M$ on an open invariant subset $O \subset M$, which allows the Lyapunov exponents to be zero. From this partition, we define a symbolic extension that is finite-to-one and onto a subset of O that carries the same finite f-invariant measures as O. Our method relies upon shadowing theory of a recurrent-pointwise-pseudo-orbit that we introduce. As a canonical application, we estimate the number of closed orbits for f.

Key words: pointwise hyperbolicity, Markov partition, symbolic dynamics
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1. *Introduction*

In this paper, we consider a family of systems which is called *pointwise hyperbolic*, introduced by Chen, Hu, and Zhou in [10], and we construct countable Markov partition with a finite-to-one almost everywhere induced coding for it. A pointwise hyperbolic diffeomorphism $f : M \to M$ is different from the uniformly hyperbolic situation since the expansion and contraction depend on points. If the system is defined on an open invariant subset $O \subset M$, as stated in [10], hyperbolicity may not be uniform near the boundary of O. Under some additional conditions (see [10, Assumptions U and S]), the authors proved that such a system has unstable and stable manifolds over any orbit with initial value $x \in O$. As applications, they give two examples, the almost Anosov diffeomorphisms [15] and gentle perturbations of Katok's map [17], and show that under certain conditions, the requirements given in Assumptions U and S can be verified. For more details, see [10].

In 2013, Sarig constructed countable Markov partitions with full topological entropy for $C^{1+\alpha}(0 < \alpha < 1)$ closed surface diffeomorphisms [22]. His method has been proven to be successful for other classes of systems. Here are some of the recent developments: surface maps with singularities [20], $C^{1+\alpha}$ closed manifolds diffeomorphisms in dimension $d \geq 2$ [6], non-invertible and/or singular maps [4, 18], and injective semi-conjugacies

[9]. A comprehensive survey on the construction of Markov partitions for non-uniformly hyperbolic systems can be found in [19].

As pointed out in [7], the key engine that allows Sarig's methods to be applied to more suitable adaptations and generalizations is the Pesin theory. For a fixed constant $\chi > 0$ and a non-uniformly hyperbolic orbit, Pesin's idea [5] was to construct local charts to represent the action of the differential as a small perturbation of a hyperbolic matrix. After the local change of coordinates, although the domain of the chart is no longer uniform in size, as it is in the uniformly hyperbolic situation, this approach produces uniform estimates in charts, that is, one contracts in rate at most $e^{-\chi}$, and one expands at least e^{χ}. Thanks to these uniform estimates in charts, one can carry out Pesin's unstable and stable manifold theorem over a chain (no longer than an orbit), which is then used in Sarig's and subsequent work mentioned above to construct a coding for big classes of systems.

Unfortunately, similar to 0-summable points considered in [7, Definition 2.1], the orbits in pointwise hyperbolic systems may not be the χ-hyperbolic ones mentioned above since hyperbolicity may become weak near ∂O. An extreme case is an almost Anosov diffeomorphism on the two-dimensional torus studied in [13, 15], where the orbit spends almost all the time near the boundary of O for almost every initial point in the sense of Lebesgue measure. Therefore, the estimates in charts may not be held uniform over an orbit. To construct unstable manifolds, in [10], they require a prior condition that the expansion of the map along unstable directions is stronger than the rates of the point moving away from the boundary [10, Assumption U(ii)].

Since a chain or a pseudo-orbit is an orbit up to small errors at each iteration, a slightly stronger assumption (see Assumption US(ii) and (iii) in §2) than [10, Assumptions U(ii) and S(ii)] is indeed necessary if we construct unstable manifolds over the *recurrent-pointwise-pseudo-orbit* we introduce in Definition 5.1 of §5. However, Assumption US(i) is weaker than [10, Assumptions U(i) and S(i)] since they constructed unstable manifolds along the orbit of f and thus they required the assumption of pointwise dominated splitting. At the same time, the Assumption US(ii) and (iii) we require in this paper were designed to be sufficiently lax that allows us to cover the examples considered in [10].

The paper is organized as follows. In §2, we give definitions and statements of the main results. The results apply to counting the number of closed orbits for pointwise hyperbolic diffeomorphism f under certain pointwise hyperbolic conditions (Assumption US) and regularity assumption (Assumption R). In §3, we represent the dynamics as a small perturbation of a pointwise hyperbolic block-form matrix in a suitable (varying) size of a neighborhood in tangent space (rather than Euclidean space as was done in [22]) and estimate some inequalities for the ratio of two adjacent neighborhoods under the action of f. Since the charts no longer have uniform estimates (the orbits may have 0 Lyapunov exponents), we need to find a suitable definition for admissible manifolds on which the graph transform is operable over some suitable edge, this is the content of §4. Section 5 is devoted to the construction of local unstable and stable manifolds over the recurrent-pointwise-pseudo-orbits, which are the limit points of compositions of a graph transform acting on admissible manifolds. Finally, in §6, we show how to employ the shadowing theory of a recurrent-pointwise-pseudo-orbit to construct a Markov partition which induces a finite-to-one almost everywhere coding for pointwise hyperbolic systems on open regions O satisfying Assumptions US and R.

Although our coding works on some examples, the question of whether or not there is an example to clarify the results of the present paper are not included in the work of [7], which is an interesting and unresolved question so far. In [7, Theorem 8.3 and Corollary 8.4], Ben Ovadia shows that if an almost Anosov diffeomorphisms defined in a two-dimensional torus is topologically transitive and the contraction rates along the stable direction are smaller than 1 even at indifferent fixed point p, then it is 0-summable for every $x \neq p$, and moreover it is codable. Note that the example of almost Anosov diffeomorphisms in this paper is not necessarily topologically transitive and the contraction rates can reach 1 at p. However, this does not mean that our example includes the former, because near p, the rates in the expansion direction are not limited in [7, Definition 8.2(3)], but we require the rates of expansion to be greater than the distance from the point to p (see Assumption US(ii)). We mention that there are non-uniformly contracting maps satisfying Assumption US(iii) in a subset of an open invariant set with every point near p that is not summable. For details, see Example 2.1 in §2.

2. *Definition and statements of results*

Let M be a C^∞ connected compact Riemannian manifold with dimension $d \geq 2$, and $O \subset M$ an open connected subset of M. We will always assume that the diameter of M is smaller than one (just multiply the metric by a small constant).

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism ($\alpha > 0$) such that $f(O) = O$. Denote by $\|D_x f\|$ and $m(D_x f)$ the norm and the minimal norm of $D_x f$, respectively. Note that we have $m(D_x f) = \|D_{f(x)} f^{-1}\|^{-1}$.

Definition 2.1. The $C^{1+\alpha}$ diffeomorphism ($\alpha > 0$) f is called a *pointwise hyperbolic* on an invariant set $O \subset M$ if there exist two continuous functions $\lambda^u, \lambda^s : O \to \mathbb{R}_+$ with $\lambda^s(x) < 1 < \lambda^u(x)$ for any $x \in O$, and an invariant decomposition $T_O M = E^u \oplus E^s$ such that for any $x \in O$,

$$\lambda^u(x) \leqslant m(D_x f|_{E^u(x)}),$$
$$\|D_x f|_{E^s(x)}\| \leqslant \lambda^s(x),$$

where E^u_x and E^s_x are called *unstable* and *stable subspaces*, respectively.

Remark 2.1. Although it is possible that $m(D_x f|_{E^u(x)}) \to 1$ or $\|D_x f|_{E^s(x)}\| \to 1$ as $x \to \partial O$, hyperbolicity is uniform when restricted to any compact set contained in O because of continuity of $\lambda^u(x)$ and $\lambda^s(x)$.

The following Assumption US is motivated by [10, Assumptions U and S].

Assumption US.

(i) The distributions E^u and E^s are continuous.

(ii) (U-boundary control) There exist $r^u_0, \beta^u > 0, \gamma^u > \max\{1, \beta^u/\alpha\}$, $0 < \kappa^u < 1$, and $C^u > 1$ such that for any $x \in O$ satisfying $d(x, \partial O) \leqslant r^u_0$, we have

$$m(Df|_{E^u(x)})^{\kappa^u} - 1 \geqslant C^u \max\left\{d(x, \partial O)^{\beta^u}, \left(\frac{d(f(x), \partial O)}{d(x, \partial O)}\right)^{\gamma^u} - 1\right\}.$$

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(iii) (S-boundary control) There exist $r_0^s, \beta^s > 0$, $\gamma^s > \max\{1, \beta^s/\alpha\}$, $0 < \kappa^s < 1$, and $C^s > 1$ such that for any $x \in O$ satisfying $d(x, \partial O) \leqslant r_0^s$, we have

$$m(Df^{-1}|_{E^s(x)})^{\kappa^s} - 1 \geqslant C^s \max\left\{d(x, \partial O)^{\beta^s}, \left(\frac{d(f^{-1}(x), \partial O)}{d(x, \partial O)}\right)^{\gamma^s} - 1\right\}.$$

Remark 2.2. Assumption US(i) is a corollary of [10, Assumption U(i)], see [10, Lemma 3.1]. It is therefore weaker than the latter.

Remark 2.3. Near the boundary, the forward images of *u-admissible manifolds* defined in Definition 4.2 grow essentially $m(D_x f|_{E^u(x)}) \to 1$ as $x \to \partial O$, that is, the hyperbolicity may become weak. *A priori*, the ratio of size of u-admissible manifolds $(d(f(x), \partial O)/d(x, \partial O))^{\gamma^u}$ may be bigger and far from 1. Fortunately, this is not the case in our setting: Assumption US(ii) means that near the boundary, the expansion rates along the unstable direction are stronger than the ratio of size along the orbit which is near the boundary. Therefore, Assumption US(ii) is required for the unstable manifold theorem (Proposition 5.2(1)) since the size of local unstable manifolds can be recovered under the forward iteration of f when orbits are near the boundary of O.

Assumption US is similar to the boundary condition control in [10, Assumptions U and S]. The additional requirements of κ^u and κ^s were designed to recover the local unstable and stable manifolds, but also sufficiently lax that allow us to cover the examples considered in [10]. It is needed since the local unstable and stable manifold in this paper are constructed over the *recurrent-pointwise-pseudo-orbit* defined in Definition 5.1 rather than an orbit of f as done in [10]. In fact, under certain conditions (see [10, Assumptions A and K]), for the almost Anosov diffeomorphisms and gentle perturbations of Katok's map, we can take $(n+2\delta)/(n+3\delta) < \kappa^t < 1$ and $(2\alpha+1)/(2\alpha+1.5) < \kappa^t < 1$, respectively, where $t = u, s$. For more details, see [10, Theorems D and E].

Given $a, b > 0$, we write $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. The next assumption is on the regularity of f.

Assumption R. $\alpha > (\beta/\gamma)$, where $\gamma = \gamma^u \wedge \gamma^s$, $\beta = \beta^u \vee \beta^s$.

Remark 2.4. This is stronger than $\gamma^u > \max\{1, \beta^u/\alpha\}$ together with $\gamma^s > \max\{1, \beta^s/\alpha\}$ given in Assumption US(ii) and (iii). However, at the end of the proof of [10, Theorems D and E], we can see that for almost Anosov diffeomorphisms and Katok's map with gentle perturbations, the authors took $\alpha = n \geq 2$, $\beta^u = n + \delta$, $\gamma^u = n + 2\delta$ where $\delta \in (0, 1/3)$ and $\alpha \geq 2$, $\beta^u = 2\alpha + 1/2$, $\gamma^u = 2\alpha + 1$, respectively. Because of symmetry between the expression of the mapping and its inverse, we can take $\beta^s = \beta^u$, $\gamma^s = \gamma^u$. Therefore, $\alpha \geq 2$ selected in [10, Theorems D and E] is enough for Assumption R.

We now give a non-uniformly contracting map satisfying Assumption US(iii) in a subset of O which is unsummable in the sense of [7, Definition 2.1]: a point $x \in M$ is called summable if its tangent space decomposes uniquely into $T_x M = H^s(x) \oplus H^u(x)$ where for all $\xi^s \in H^s(x)$, $\xi^u \in H^u(x)$,

$$\sum_{m \geq 0} \|D_x f^m \xi^s\|^2 < \infty, \sum_{m \geq 0} \|D_x f^{-m} \xi^u\|^2 < \infty.$$

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We mention that contrary to the present work, the distribution $H^s(x) \oplus H^u(x)$ is usually not more than just measurable.

Example 2.1. For simplicity, let us consider a $C^{1+\alpha}(\alpha = 2k, k \in \mathbb{N}_+)$ diffeomorphism f of a two-dimensional closed (compact without boundary) connected smooth Riemannian manifold M that has a fixed point p. Suppose f is a non-uniformly contracting map on $O = M/p$ and that there exist a neighborhood U of p and a local coordinate system (x, y) with $p = (0, 0)$. We assume further that on U, f has the form

$$f(x, y) = (x - x^{\alpha+1}, y - \varepsilon_1 y x^{\alpha}),$$

where $\varepsilon_1 > 0$.

Denote $z = (x, y)$. From

$$D_z f = \begin{pmatrix} 1 - (1+\alpha)x^{\alpha} & 0 \\ -\varepsilon_1 \alpha y x^{\alpha-1} & 1 - \varepsilon_1 x^{\alpha} \end{pmatrix},$$

it is easy to verify that if $z \in U$, we have

$$D_z f^{-1} = \begin{pmatrix} 1 + (1+\alpha)x^{\alpha} + O(x^{\alpha+1}) & 0 \\ \varepsilon_1 \alpha y x^{\alpha-1} + yO(x^{2\alpha-1}) & 1 + \varepsilon_1 x^{\alpha} + O(x^{\alpha+1}) \end{pmatrix}.$$

Now let us consider a subset of U:

$$U_1 := U \cap \{z \in U : |y| \le |x|^{\varepsilon_2} \text{ where } \varepsilon_2 > 1\}.$$

CLAIM 1. *If $\varepsilon_1 > 1$, then there exist $r_0^s, \beta^s > 0, \gamma^s > \max\{1, \beta^s/\alpha\}$, $0 < \kappa^s < 1$, and $C^s > 1$ such that for any $z \in U_1$ satisfying $\|z\| \leqslant r_0^s$, we have*

$$m(D_z f^{-1})^{\kappa^s} - 1 \geqslant C^s \max\left\{\|z\|^{\beta^s}, \left(\frac{\|f^{-1}(z)\|}{\|z\|}\right)^{\gamma^s} - 1\right\}.$$

Proof of Claim 1. Note that if $z \in U_1$, we get

$$D_z f^{-1} = \begin{pmatrix} 1 + (1+\alpha)x^{\alpha} + O(x^{\alpha+1}) & 0 \\ O(|x|^{\alpha+\varepsilon_2-1}) & 1 + \varepsilon_1 x^{\alpha} + O(x^{\alpha+1}) \end{pmatrix},$$

and thus $m(D_z f^{-1})^{\kappa^s} - 1 = \varepsilon_1 \kappa^s x^{\alpha} + O(|x|^{\alpha+2(\varepsilon_2-1)})$.

It is easy to see that

$$f^{-1}(z) = (x + x^{\alpha+1} + O(x^{\alpha+2}), y + \varepsilon_1 y x^{\alpha} + O(\|z\|^{\alpha+2})).$$

One can choose r_0^s small enough such that $f^{-1}(z) \in U$ for any $z \in U_1$ satisfying $\|z\| \leqslant r_0^s$, and therefore

$$f^{-1}(z) = (x + x^{\alpha+1} + O(x^{\alpha+2}), y + O(|x|^{\alpha+\min\{\varepsilon_2, 2\}})).$$

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Let us compute

$$C^s\left[\left(\frac{\|f^{-1}(z)\|}{\|z\|}\right)^{\gamma^s}-1\right]=C^s\left[\left(\frac{\|f^{-1}(z)\|}{\|z\|}\right)^{\gamma^s}-1\right]$$
$$=C^s\left[\left(\frac{x^2+2x^{2+\alpha}+y^2+O(|x|^{\alpha+\min\{2\varepsilon_2,3\}})}{x^2+y^2}\right)^{\gamma^s/2}-1\right]$$
$$=C^s\gamma^s\frac{2x^{2+\alpha}+O(|x|^{\alpha+\min\{2\varepsilon_2,3\}})}{2(x^2+y^2)}$$
$$\leq C^s\gamma^s x^\alpha+O(|x|^{\alpha+\min\{2\varepsilon_2-2,1\}}).$$

Hence, we get

$$m(D_zf^{-1})^{\kappa^s}-1\geqslant C^s\max\left\{\|z\|^{\beta^s},\left(\frac{\|f^{-1}(z)\|}{\|z\|}\right)^{\gamma^s}-1\right\},$$

where $1/\sqrt{\varepsilon_1}<\kappa^s<1$, $1<C^s=\gamma^s<\sqrt[4]{\varepsilon_1}$, and $\alpha<\beta^s<\alpha\gamma^s$. □

CLAIM 2. *If* $\varepsilon_1<(\alpha/2)+1$, *then* $\sum_{m\geq 0}\|D_zf^m(\partial/\partial y)\|^2=\infty$ *for all* $z\in U$, *where* $(\partial/\partial y)=(0,1)^T$.

Proof of Claim 2. Denote $z_m=(x_m,y_m):=f^m(z)$. Notice that $z_m\in U$ for all $z\in U$. By using induction, we get for all $m\geq 1$ that

$$D_zf^m\frac{\partial}{\partial y}=(0,g(x_{m-1})x_m)^T,$$

where $g(x_m)=((1-\varepsilon_1x_m^\alpha)/(1-x_m^\alpha))g(x_{m-1})$ and $g(x)=(1-\varepsilon_1x^\alpha)/(x-x^{\alpha+1})$.

Clearly, $x_{m-1}=x_m+x_m^{\alpha+1}+O(x_m^{\alpha+1})$ for all $m\geq 1$ since $z_m\in U$ for all $z\in U$, so by [14, Lemma 3.1], for all large m, we have

$$|x_m|=\left(\frac{1}{\alpha(m+k)}\right)^{1/\alpha}+O\left(\frac{1}{m^\theta}\right)$$

for some integer k, where $\theta>1/\alpha$.

Now it is easy to get

$$\lim_{m\to\infty}m\left(\frac{\|D_zf^m(\partial/\partial y)\|^2}{\|D_zf^{m+1}(\partial/\partial y)\|^2}-1\right)$$
$$=\lim_{m\to\infty}m\left(\frac{g(x_{m-1})^2x_m^2}{g(x_m)^2x_{m+1}^2}-1\right)$$
$$=\frac{2(\varepsilon_1-1)}{\alpha}$$
$$<1.$$

Therefore, $\sum_{m\geq 0}\|D_zf^m(\partial/\partial y)\|^2=\infty$ from Raabe's test. This concludes Claim 2. □

In §6, we show how Assumptions US and R for pointwise hyperbolic diffeomorphism f on O are sufficient for the construction of a Markov partition. The conclusion is the existence of a *symbolic model* for f. Let us give the definitions in the general situation.

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Let $\widehat{\mathscr{G}} = (\widehat{V}, \widehat{E})$ be a directed graph with a countable collection of vertices $\widehat{V}$ such that every vertex has at least one edge coming in and at least one edge coming out, that is, for each $v \in \widehat{V}$, there are $u, w \in \widehat{V}$ and a path $u \to v \to w$ which is made up of two edges. The *topological Markov shift* (TMS) associated to $\widehat{\mathscr{G}}$ is the pair $(\widehat{\Sigma}, \widehat{\sigma})$ which is the set

$$\widehat{\Sigma} = \{\{v_n\}_{n\in\mathbb{Z}} \in \widehat{V}^{\mathbb{Z}} : v_n \to v_{n+1} \text{ for all } n \in \mathbb{Z}\}$$

equipped with the left-shift $\widehat{\sigma} : \widehat{\Sigma} \to \widehat{\Sigma}$, $\widehat{\sigma}(\{v_n\}_{n\in\mathbb{Z}}) = \{v_{n+1}\}_{n\in\mathbb{Z}}$. An element of $\widehat{\Sigma}$ is denoted by $\underline{v} = \{v_n\}_{n\in\mathbb{Z}}$ for the sake of simplicity. We endow $\widehat{\Sigma}$ with the distance $d(\underline{v}, \underline{w}) := \exp[-\min\{|n| : n \in \mathbb{Z} \text{ such that } v_n \neq w_n\}]$. With this metric, $\widehat{\Sigma}$ is a complete separable metric space. Recall that $\widehat{\Sigma}$ is compact if and only if $\widehat{\mathscr{G}}$ is finite. Additionally, $\widehat{\Sigma}$ is locally compact if and only if every vertex of $\widehat{\mathscr{G}}$ has finite ingoing and outgoing degree.

Let $\widehat{\Sigma}^\#$ be the *recurrent set* of $\widehat{\Sigma}$ which is due to Sarig [22]:

$$\widehat{\Sigma}^\# := \{\underline{v} \in \widehat{\Sigma} : \text{there exists } u, w \in \widehat{V}, \text{ there exists } n_k, m_k \uparrow \infty \\ \text{such that } v_{n_k} = u \text{ and } v_{-m_k} = w \text{ for all } k \in \mathbb{Z}\}.$$

By the Poincaré recurrence theorem, every $\widehat{\sigma}$-invariant probability measure is carried by $\widehat{\Sigma}^\#$. Furthermore, every periodic point of $\widehat{\sigma}$ is in $\widehat{\Sigma}^\#$. Notice that when V is finite, $\widehat{\Sigma}^\# = \widehat{\Sigma}$.

Let $\widehat{M}$ be a Riemannian manifold and $\widehat{f} : \widehat{M} \to \widehat{M}$ be a diffeomorphism.

Definition 2.2. A *symbolic model* for $\widehat{f}$ is a triple $(\widehat{\Sigma}^\#, \widehat{\sigma}, \widehat{\pi})$ satisfying:

(1) $(\widehat{\Sigma}^\#, \widehat{\sigma})$ is a recurrent TMS;
(2) $\widehat{\pi} : \widehat{\Sigma}^\# \to \widehat{\pi}(\widehat{\Sigma}^\#) \subset \widehat{M}$ is continuous and finite-to-one;
(3) $\widehat{\pi} \circ \widehat{\sigma} = \widehat{f} \circ \widehat{\pi}$ on $\widehat{\Sigma}^\#$.

In contrast to the notion of a symbolic model of uniform and non-uniform hyperbolic systems, which requires the map $\widehat{\pi}$ to define in the whole of $\widehat{\Sigma}$, the notion here assumes not necessarily in $\widehat{\Sigma}$, but in a recurrent subset $\widehat{\Sigma}^\#$. Intuitively, a diffeomorphism can have many symbolic models. Roughly speaking, an acceptable symbolic model is one for which $\widehat{\pi}(\widehat{\Sigma}^\#)$ (rather than $\widehat{\pi}(\widehat{\Sigma})$) contains a subset where $\widehat{f}$ displays chaotic dynamics. For us, this occurs when $\widehat{f} = f$ is pointwise hyperbolic on an open invariant subset $O \subset M$ satisfying Assumptions US and R.

We assume the metric on M is taken in such a way that for any $x \in O$, $E^u(x)$ and $E^s(x)$ are pairwise orthogonal [10].

THEOREM 2.1. *Let f be a pointwise hyperbolic diffeomorphism on an invariant set $O \subset M$, and suppose it satisfies Assumptions US and R, then f possesses a symbolic model $(\widehat{\Sigma}^\#, \widehat{\sigma}, \widehat{\pi})$. More precisely, there is a set $O^\# \subset O$, a locally compact countable topological Markov shift $(\widehat{\Sigma}, \widehat{\sigma})$, and a continuous map $\widehat{\pi} : \widehat{\Sigma}^\# \to O$ such that:*

(1) $\widehat{\pi} \circ \widehat{\sigma}|_{\widehat{\Sigma}^\#} = f \circ \widehat{\pi}$*;*
(2) $\mu(O^\#) = \mu(O)$ *for every finite f-invariant measure* μ*;*
(3) $\widehat{\pi} : \widehat{\Sigma}^\# \to O^\#$ *is a finite-to-one surjective map.*

In fact, the subset $O^\# \subset O$ in Theorem 2.1 is given by

$$O^\# := \Big\{x \in O : \limsup_{n\to\infty} d(f^n(x), \partial O) > 0 \text{ and } \limsup_{n\to-\infty} d(f^n(x), \partial O) > 0\Big\}. \quad (2.1)$$

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By the Poincaré recurrence theorem, if μ is an f-invariant probability measure supported on O, then it is supported on $O^\#$.

We stress the argument in the proof of Theorem 2.1(3) providing more details on the number of the $\widehat{\pi}$-pre-image, and traditionally refer to the following theorem which precisely characterizes the loss of injectivity of $\widehat{\pi}$.

THEOREM 2.2. *For $(\widehat{\Sigma}^\#, \widehat{\sigma}, \widehat{\pi})$ given by Theorem 2.1, denote the set of vertices of $\widehat{\Sigma}$ by $\widehat{V}$. Then there exists a function $N : \widehat{V} \to \mathbb{N}$ such that for every $x \in O^\#$ which can be written as $x = \widehat{\pi}(\underline{v})$ with $v_n = u$ for infinitely many $n > 0$, and $v_n = w$ for infinitely many $n < 0$, we have*

$$|\widehat{\pi}^{-1}(x)| \le N(u)N(w).$$

Set $P_n(f) := |\{x \in M : f^n(x) = x\}|$ and let $h_{\mathrm{top}}(f, O) := \sup\{h_{\mathrm{top}}(f, \mathscr{O}) : \mathscr{O}$ *finite open cover of* $O\}$ denote the topological entropy of $f|_O$, where the supremum runs over all finite open covers of O. As an application, when $f|_O$ has an ergodic measure of maximal entropy μ satisfying $h_\mu(f|_O) = h_{\mathrm{top}}(f, O)$, where $h_\mu(f|_O)$ is the Kolmogorov–Sinai entropy of $f|_O$-invariant probability measure μ, we have the following consequence.

COROLLARY 2.1. *Under the assumptions in Theorem 2.1, suppose that $f|_O$ has an ergodic measure of maximal entropy, then there exists $C > 0$ and $p \in \mathbb{N}$ such that f has at least $Ce^{pnh_{\mathrm{top}}(f,O)}$ periodic points of period pn for all $n \in \mathbb{N}$, that is,*

$$P_{pn}(f) \ge Ce^{pnh_{\mathrm{top}}(f,O)} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. The proof follows the arguments in [22, Theorem 1.1], we provide some details.

Write $\widehat{\Sigma} = \widehat{\Sigma}(\widehat{\mathscr{G}})$ and let μ be an ergodic measure of maximal entropy for $f|_O$. Proceeding as in [22, §13], we can lift μ to an ergodic measure of maximal entropy $\widehat{\mu}$ for $\widehat{\sigma}$ via

$$\widehat{\mu} = \int_O \frac{1}{|\widehat{\pi}^{-1}(x)|}\bigg(\sum_{\underline{v}\in\widehat{\pi}^{-1}(x)} \delta_{\underline{v}}\bigg)\, d\mu(x), \tag{2.2}$$

such that $h_{\widehat{\mu}}(\widehat{\sigma}) = h_\mu(f|_O)$.

By [11, 12], $\widehat{\mu}$ is supported on a subset $\widehat{\Sigma}(\widehat{\mathscr{G}'}) \subset \widehat{\Sigma}(\widehat{\mathscr{G}})$, where the smaller TMS $(\widehat{\Sigma}(\widehat{\mathscr{G}'}), \widehat{\sigma})$ is topologically transitive. For a topologically transitive TMS with countable many vertexes, Gurevič [11, 12] showed a good estimate on $\mathrm{Per}_n(\widehat{\sigma})$: a topologically transitive TMS $(\widehat{\Sigma}(\widehat{\mathscr{G}'}), \widehat{\sigma})$ admits at most one measure of maximal entropy $h_{\max}(\widehat{\Sigma}(\widehat{\mathscr{G}'}))$, and such a measure exists if and only if there exists $p \in \mathbb{N}$ such that for every vertex $v \in \widehat{\mathscr{G}'}$, there exists $C_v \ge 1$, such that

$$C_v^{-1} \le \frac{\mathrm{Per}_{pn}(\widehat{\sigma}|_{\widehat{\Sigma}(\widehat{\mathscr{G}'})}, v)}{e^{pnh_{\max}(\widehat{\Sigma}(\widehat{\mathscr{G}'}))}} \le C_v$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} h_{\max}(\widehat{\Sigma}(\widehat{\mathscr{G}'})) &:= h_{\max}(\widehat{\sigma}) \\ &= \max\{h_\nu(\widehat{\sigma}) : \nu \text{ is } \widehat{\sigma}\text{-invariant Borel probability measure on } \widehat{\Sigma}(\widehat{\mathscr{G}'})\}, \end{aligned}$$

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and

$$\mathrm{Per}_{pn}(\widehat{\sigma}|_{\widehat{\Sigma}(\widehat{\mathscr{G}}')}, v) := |\{\underline{v} \in \widehat{\Sigma}(\widehat{\mathscr{G}}') : v_0 = v, \widehat{\sigma}^{pn}(\underline{v}) = \underline{v}\}|.$$

From Theorem 2.1(1) and (3), every periodic point in $\{\underline{v} \in \widehat{\Sigma}(\widehat{\mathscr{G}}') : v_0 = v, \widehat{\sigma}^n(\underline{v}) = \underline{v}\} \subset \widehat{\Sigma}^{\#}$ is mapped by $\widehat{\pi}$ to a periodic point of the same period in O. By Theorem 2.2, this mapping is at most $N(v)^2$-to-one. Therefore, for any $v \in \widehat{\mathscr{G}}'$ and all $n \geq 1$,

$$P_{pn}(f) \geq P_{pn}(f|_O) \geq \frac{1}{N(v)^2} \mathrm{Per}_{pn}(\widehat{\sigma}|_{\widehat{\Sigma}(\widehat{\mathscr{G}}')}, v). \tag{2.3}$$

Finally, by assumption and construction, $h_{\max}(\widehat{\Sigma}(\widehat{\mathscr{G}}')) = h_{\widehat{\mu}}(\widehat{\sigma}) = h_\mu(f|_O) = h_{\mathrm{top}}(f, O)$. Therefore, we have

$$P_{pn}(f) \geq \frac{1}{C_v N(v)^2} e^{pnh_{\max}(\widehat{\Sigma}(\widehat{\mathscr{G}}'))} = \frac{1}{C_v N(v)^2} e^{pnh_{\mathrm{top}}(f,O)}. \qquad \square$$

Remark 2.5. It should be noted that the reverse of the second inequality in equation (2.3) above might not hold, because $\widehat{\pi}$ may not code trajectories inside $O/O^{\#}$. A wealth of diffeomorphisms such that the growth rate of $P_n(f)$ is superexponential can be found in [**16**].

Notational convention and standing assumption. From now on, f will be a pointwise hyperbolic on an open invariant subset O of a connected compact smooth Riemannian manifold of dimension greater than 1, and satisfies Assumptions US and R.

Suppose P is a property. The statement 'for all ε, small enough P holds' means 'there exists $\varepsilon_0 > 0$ which only depends on f, M, C_{1-4} (see Lemma 4.1), and the parameters given by Assumption US(ii) and (iii) such that for all $0 < \varepsilon < \varepsilon_0$, P holds'.

3. *Linearization up to small errors*

In this section, we will show that $F_{x,y} := \exp_y^{-1} \circ f \circ \exp_x$ is a small perturbation of a hyperbolic block-form matrix for any $x, y \in O$ such that $F_{x,y}$ is well defined and $d(f(x), y)$ small enough. First, we denote some constants and introduce two main functions, the $\varepsilon(x)$ and $Q(x)$, which will be used many times throughout the paper.

Denote $r_0 = r_0^u \wedge r_0^s$, where r_0^u and r_0^s are given by Assumption US. Let

$$C_0 := C_0(r_0) = \min_{x \in O(r_0)} \{\min\{m(Df|_{E^u(x)}), m(Df^{-1}|_{E^s(x)})\}\}, \tag{3.1}$$

here

$$O(r_0) = \{x \in O : d(x, \partial O) \geq r_0\}$$

is a compact subset of O since M is compact. Observe that $C_0 > 1$ as hyperbolicity is uniform in $O(r_0)$.

Fix a small parameter $\varepsilon > 0$ (how small depends on a finite number of inequalities that ε has to satisfy). For $x \in O$, set

$$\varepsilon(x) = \varepsilon \min\{r_0^\beta, d(x, \partial O)^\beta\}, \tag{3.2}$$

where $\beta = \beta^u \vee \beta^s$ as in Assumption R. We start with a basic lemma.

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LEMMA 3.1. *For all ε small enough and for every $x \in O$,*

$$C\varepsilon(x) \leq \min\{m(Df|_{E^u(x)})^\kappa - 1, m(Df^{-1}|_{E^s(x)})^\kappa - 1\}, \tag{3.3}$$

where $C = C^u \wedge C^s$ and $\kappa = \kappa^u \vee \kappa^s$.

Proof. Our first constraint on ε is that it has to be small so that

$$0 < \varepsilon \leq \min\left\{\frac{C_0^\kappa - 1}{Cr_0^\beta}, 1\right\}.$$

Recall that the diameter of M is smaller than 1. There are the following two cases.

Case 1. $\varepsilon(x) = \varepsilon r_0^\beta$. In this case, we note that $x \in O(r_0)$ and $1 < C_0 \leq \min\{m(Df|_{E^u(x)}), m(Df^{-1}|_{E^s(x)})\}$. Therefore,

$$C\varepsilon(x) \leq C_0^\kappa - 1 \leq \min\{m(Df|_{E^u(x)})^\kappa - 1, m(Df^{-1}|_{E^s(x)})^\kappa - 1\}.$$

Case 2. $\varepsilon(x) = \varepsilon d(x, \partial O)^\beta$. Now we have $d(x, \partial O) \leq r_0$. Since $\beta = \beta^u \vee \beta^s$, $C = C^u \wedge C^s$, and $\kappa = \kappa^u \vee \kappa^s$, the required inequality is an immediate result of Assumption US(ii) and (iii). □

Now we fix another parameter $0 < \delta < \min\{1, \alpha - \beta/\gamma\}$, which is well defined from Assumption R. Setting

$$Q(x) = \varepsilon^{2/(\alpha-\delta)} \min\{r_0^\gamma, d(x, \partial O)^\gamma\}, \tag{3.4}$$

note that $Q(x)^{\alpha-\delta} \leq \varepsilon\varepsilon(x)$ because $(\alpha - \delta)\gamma > \beta$ and $Q(x) \leq d(x, \partial O)$ for $\gamma > 1$ when $0 < \varepsilon < 1$.

LEMMA 3.2. *The following holds for all ε small enough. For every $x \in O$, we have*

$$\frac{C}{\|Df|_{E^s(x)}\|^{-\kappa} + (C-1)} \leq \frac{Q(f(x))}{Q(x)} \leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1 \tag{3.5}$$

and

$$C\varepsilon(x) \leq \min\{m(Df|_{E^u(x)})^\kappa - 1, 1 - \|Df|_{E^s(x)}\|^\kappa\}. \tag{3.6}$$

A similar statement holds for f^{-1}.

Proof. We give the proofs for equations (3.5) and (3.6). The proofs for f^{-1} are almost identical, the major change being the substitution of f^{-1} for f.

For the companion inequality in equation (3.5), there are four cases. It is worth pointing out that Case 2 is the most interesting, since it is the case where both x and $f(x)$ are close to ∂O, and hence there is little hyperbolicity.

Case 1. $Q(x) = Q(f(x)) = \varepsilon^{2/(\alpha-\delta)} r_0^\gamma$. In this case, equation (3.5) is straightforward since for any $x \in O$,

$$\frac{C}{\|Df|_{E^s(x)}\|^{-\kappa} + (C-1)} \leq 1 = \frac{Q(f(x))}{Q(x)} = 1 \leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1.$$

Case 2. $Q(x) = \varepsilon^{2/(\alpha-\delta)} d(x, \partial O)^\gamma$ and $Q(f(x)) = \varepsilon^{2/(\alpha-\delta)} d(f(x), \partial O)^\gamma$. Equivalently, $d(x, \partial O) \leq r_0 \leq r_0^u$ and $d(f(x), \partial O) \leq r_0 \leq r_0^s$. Taking Assumption US(ii) and

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(iii) into consideration, we find in this case that

$$\left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma^u} \leq \frac{1}{C^u}(m(Df|_{E^u(x)})^{\kappa^u}-1)+1$$

and

$$\begin{aligned}\left(\frac{d(x,\partial O)}{d(f(x),\partial O)}\right)^{\gamma^s} &\leq \frac{1}{C^s}(m(Df^{-1}|_{E^s(f(x))})^{\kappa^s}-1)+1\\ &= \frac{1}{C^s}(\|Df_{E^s(x)}\|^{-\kappa^s}-1)+1,\end{aligned}$$

where in the last equality, we used that $m(Df^{-1}|_{E^s(f(x))})^{\kappa^s} = \|Df|_{E^s(x)}\|^{-\kappa^s}$.

If $(d(f(x),\partial O)/d(x,\partial O)) \geq 1$, on the one hand,

$$\frac{Q(f(x))}{Q(x)} = \left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma} > 1 \geq \frac{C}{\|Df|_{E^s(x)}\|^{-\kappa}+(C-1)}.$$

On the other hand, since $\gamma \leq \gamma^u$, $C \leq C^u$, $\kappa^u \leq \kappa$, we have

$$\begin{aligned}\frac{Q(f(x))}{Q(x)} &= \left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma}\\ &\leq \left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma^u}\\ &\leq \frac{1}{C^u}(m(Df|_{E^u(x)})^{\kappa^u}-1)+1\\ &\leq \frac{1}{C}(m(Df|_{E^u(x)})^{\kappa}-1)+1.\end{aligned}$$

Therefore, the companion inequality in equation (3.5) holds when $(d(f(x),\partial O)/d(x,\partial O)) \geq 1$ in Case 2.

Now if $0 < (d(f(x),\partial O)/d(x,\partial O)) \leq 1$, then

$$\frac{Q(f(x))}{Q(x)} \leq 1 \leq \frac{1}{C}(m(Df|_{E^u(x)})^{\kappa}-1)+1.$$

Using $\gamma \leq \gamma^s$, $C \leq C^s$, $\kappa^s \leq \kappa$, we obtain

$$\begin{aligned}\frac{Q(f(x))}{Q(x)} &= \left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma}\\ &\geq \left(\frac{d(f(x),\partial O)}{d(x,\partial O)}\right)^{\gamma^s}\\ &\geq \left[\frac{1}{C^s}(\|Df_{E^s(x)}\|^{-\kappa^s}-1)+1\right]^{-1}\\ &\geq \left[\frac{1}{C}(\|Df_{E^s(x)}\|^{-\kappa}-1)+1\right]^{-1}\\ &= \frac{C}{\|Df|_{E^s(x)}\|^{-\kappa}+(C-1)}.\end{aligned}$$

This completes the proof of equation (3.5) in Case 2.

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Case 3. $Q(x) = \varepsilon^{2/(\alpha-\delta)} d(x, \partial O)^\gamma$ and $Q(f(x)) = \varepsilon^{2/(\alpha-\delta)} r_0^\gamma$. Now $d(x, \partial O)^\gamma \leq r_0^\gamma \leq d(f(x), \partial O)^\gamma$, so it follows that

$$\frac{Q(f(x))}{Q(x)} = \frac{r_0^\gamma}{d(x, \partial O)^\gamma} \geq 1 \geq \frac{C}{\|Df|_{E^s(x)}\|^{-\kappa} + (C-1)}.$$

In this case, since $d(x, \partial O) \leq r_0^u$ and $(d(f(x), \partial O)/d(x, \partial O)) \geq 1$, we have

$$\begin{aligned}\frac{Q(f(x))}{Q(x)} &\leq \left(\frac{d(f(x), \partial O)}{d(x, \partial O)}\right)^\gamma \\ &\leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1\end{aligned}$$

as in Case 2 when $(d(f(x), \partial O)d(x, \partial O)) \geq 1$.

Case 4. $Q(x) = \varepsilon^{2/(\alpha-\delta)} r_0^\gamma$ and $Q(f(x)) = \varepsilon^{2/(\alpha-\delta)} d(f(x), \partial O)^\gamma$. In this case, $d(f(x), \partial O)^\gamma \leq r_0^\gamma \leq d(x, \partial O)^\gamma$, then

$$\frac{Q(f(x))}{Q(x)} = \frac{d(f(x), \partial O)^\gamma}{r_0^\gamma} \leq 1 \leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1.$$

Notice that $d(f(x), \partial O) \leq r_0^s$ and $0 < (d(f(x), \partial O)/d(x, \partial O)) \leq 1$, so we obtain

$$\begin{aligned}\frac{Q(f(x))}{Q(x)} &\geq \left(\frac{d(f(x), \partial O)}{d(x, \partial O)}\right)^\gamma \\ &\geq \frac{C}{\|Df|_{E^s(x)}\|^{-\kappa} + (C-1)}\end{aligned}$$

as in Case 2 when $0 < (d(f(x), \partial O)/d(x, \partial O)) \leq 1$.

Hence, the companion inequality in equation (3.5) is proved. From Lemma 3.1, it remains to show that $C\varepsilon(x) \leq 1 - \|Df|_{E^s(x)}\|^\kappa$ for small $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \min\left\{1, \frac{C_0^\kappa - 1}{Cr_0^\beta}, \frac{1 - C_0^{-\kappa}}{Cr_0^\beta}, \frac{1}{A_s^\beta (C^f)^{\kappa^s}}\right\},$$

where $A_s := [(1/C^s)((C^f)^{\kappa^s} - 1) + 1]^{1/\gamma^s} > 1$ and $C^f := \max_{x \in M}\{\max\{\|D_x f\|, \|D_x f^{-1}\|\}\}$.

For any $x \in O$, $f(x) \in O$, there are the following two cases.

Case 1. $\varepsilon(f(x)) = \varepsilon r_0^\beta$. In this case, observe that $f(x) \in O(r_0)$, so using equation (3.1), we get

$$C_0 \leq m(Df^{-1}|_{E^s(f(x))}) = \|Df|_{E^s(x)}\|^{-1},$$

and hence

$$C\varepsilon(x) \leq C\varepsilon r_0^\beta \leq 1 - C_0^{-\kappa} \leq 1 - \|Df|_{E^s(x)}\|^\kappa.$$

Case 2. $\varepsilon(f(x)) = \varepsilon d(f(x), \partial O)^\beta$. Now $d(f(x), \partial O) \leq r_0^s$, and from Assumption US(iii), we find

$$\left(\frac{d(x, \partial O)}{d(f(x), \partial O)}\right)^\beta \leq A_s^\beta$$

and

$$\begin{aligned} C^s d(f(x), \partial O)^{\beta^s} &\le m(Df^{-1}|_{E^s(f(x))})^{\kappa^s}(1 - m(Df^{-1}|_{E^s(f(x))})^{-\kappa^s}) \\ &= m(Df^{-1}|_{E^s(f(x))})^{\kappa^s}(1 - \|Df|_{E^s(x)}\|^{\kappa^s}) \\ &\le (C^f)^{\kappa^s}(1 - \|Df|_{E^s(x)}\|^{\kappa^s}). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} C\varepsilon(x) &\le \varepsilon C d(x, \partial O)^{\beta} \\ &\le \varepsilon A_s^{\beta} C^s d(f(x), \partial O)^{\beta} \\ &\le \varepsilon A_s^{\beta} (C^f)^{\kappa^s}(1 - \|Df|_{E^s(x)}\|^{\kappa^s}) \\ &\le 1 - \|Df|_{E^s(x)}\|^{\kappa}. \end{aligned}$$

□

Before we begin, recall the following basic fact from Riemannian geometry: since M is compact, there exist $r(M), \rho(M) > 0$ such that for every $z \in M$, the exponential map $\exp_z$ maps $T_zM(r(M)) := \{v \in T_zM : \|v\| < r(M)\}$ diffeomorphically (that is, $\exp_z$ is a bijective C^∞ map whose $\exp_z^{-1}$ is also C^∞) onto a neighborhood of $B(z, \rho(M)) := \{y \in M : d(y, z) < \rho(M)\}$ in a 2-bi-Lipschitz way.

For any $x \in O$, let $B_x^u(Q(x))$ and $B_x^s(Q(x))$ denote the closed balls of radius $Q(x)$ about the origin in $E^u(x)$ and $E^s(x)$, respectively. Denote the $Q(x)$-square centered at x by

$$S_x = B_x^u(Q(x)) \oplus B_x^s(Q(x)). \tag{3.7}$$

Fix $\varepsilon > 0$ small enough such that $Q(x) < \min\{r(M)/\sqrt{2}, \rho(M)/2\sqrt{2}\}$, and $\exp_x$ maps $S_x \subset T_xM(r(M))$ diffeomorphically into a subset of $B(x, 2\sqrt{2}Q(x)) \subset B(x, \rho(M))$. It follows that $f \circ \exp_x$ maps S_x $C^{1+\alpha}$ diffeomorphically into a subset of $B(f(x), 2\sqrt{2}C^f Q(x))$. Moreover, if necessary, choosing a smaller ε such that $Q(x) < (\rho(M)/2\sqrt{2}C^f)$, then for $y \in O$ satisfying $d(f(x), y) < \rho(M) - 2\sqrt{2}C^f Q(x)$, $f \circ \exp_x$ maps S_x $C^{1+\alpha}$ diffeomorphically into $B(f(x), 2\sqrt{2}C^f Q(x)) \subset B(y, d(y, f(x)) + 2\sqrt{2}C^f Q(x))$, which is a subset of $B(y, \rho(M))$. Since $\exp_y^{-1}$ is a diffeomorphism on $B(y, \rho(M))$, the map $F_{x,y} : S_x \to T_yM$ defined by

$$F_{x,y} := \exp_y^{-1} \circ f \circ \exp_x \tag{3.8}$$

is well defined and a $C^{1+\alpha}$ diffeomorphism onto its image.

For $x \in O$, denote by P_x^u and P_x^s the projections from T_xM to $E^u(x)$ and $E^s(x)$, respectively. Whenever $F_{x,y}$ is defined, with respect to $E^u \oplus E^s$, we have

$$D_0F_{x,y} = \begin{pmatrix} D_{x,y}^{uu} & D_{x,y}^{us} \\ D_{x,y}^{su} & D_{x,y}^{ss} \end{pmatrix}, \tag{3.9}$$

where $D_{x,y}^{\tau t} = P_y^{\tau} \circ D_{f(x)} \exp_y^{-1} \circ D_x f|_{E^t(x)} : B_x^t(Q(x)) \to E^{\tau}(y)$, $\tau, t = u, s$.

We write $F_{x,y}|_{S_x} = D_0F_{x,y} + \phi_{x,y}$ in the form

$$\begin{aligned} &F_{x,y}(v_x^u, v_x^s) \\ &\quad = (D_{x,y}^{uu}(v_x^u) + D_{x,y}^{us}(v_x^s) + \phi_{x,y}^u(v_x^u, v_x^s), D_{x,y}^{ss}(v_x^s) + D_{x,y}^{su}(v_x^u) + \phi_{x,y}^s(v_x^u, v_x^s)) \end{aligned} \tag{3.10}$$

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under the basis (E^u, E^s) for all $v_x = (v_x^u, v_x^s) \in S_x$. Note that $\|\phi_{x,y}(0)\| = d(f(x), y)$ and $D_0\phi_{x,y} = 0$.

Since the distribution E^u is continuous under Assumption US(i), for $\varepsilon > 0$ small enough and any $x \in O$, there exists a *local monotone function* $\delta^u(\cdot) : (0, \varepsilon(x)] \to (0, \rho(M) - 2\sqrt{2}C^f Q(x))$ in the sense that $\delta^u(a) \le \delta^u(b)$ if $0 < a \le b \le \varepsilon(x)$ such that for all $y \in B(f(x), \delta^u(\varepsilon(x)))$, we have

$$\|D^{us}_{x,y}\| \le \varepsilon\varepsilon(x) \quad \text{and} \quad |m(D^{uu}_{x,y}) - m(Df|_{E^u(x)})| \le \varepsilon\varepsilon(x). \tag{3.11}$$

Similarly, we have

$$\|D^{su}_{x,y}\| \le \varepsilon\varepsilon(x), \quad \text{and} \quad |\|Df|_{E^s(x)}\| - \|D^{ss}_{x,y}\|| \le \varepsilon\varepsilon(x). \tag{3.12}$$

By the same reason, for any $y \in O$ and any $x \in B(f^{-1}(y), \delta^s(\varepsilon(y))$, where $\delta^s(\varepsilon(y))$ is defined analogous to $\delta^u(\varepsilon(x))$, by considering $F^{-1}_{x,y} = \exp_x^{-1} \circ f^{-1} \circ \exp_y$ instead of $F_{x,y}$, we can get that $F^{-1}_{x,y}$ is well defined and a $C^{1+\alpha}$ diffeomorphism from S_y onto its image, also to get four similar inequalities like equations (3.11) and (3.12).

PROPOSITION 3.1. *The following holds for all $\varepsilon > 0$ small enough. For all $x \in O$, there is a local monotone function $0 < \delta^u(\varepsilon(x)) < \varepsilon\varepsilon(x)$ such that for $y \in O$ satisfying $d(f(x), y) < \delta^u(\varepsilon(x))$, $F_{x,y}|_{S_x} = D_0F_{x,y} + \phi_{x,y}$ can be put in the form*

$$\begin{aligned} &F_{x,y}(v_x^u, v_x^s) \\ &\quad = (D^{uu}_{x,y}(v_x^u) + D^{us}_{x,y}(v_x^s) + \phi^u_{x,y}(v_x^u, v_x^s), D^{ss}_{x,y}(v_x^s) + D^{su}_{x,y}(v_x^u) + \phi^s_{x,y}(v_x^u, v_x^s)) \end{aligned}$$

under the basis (E^u, E^s) for all $v_x = (v_x^u, v_x^s) \in S_x$, where $\phi_{x,y} := (\phi^u_{x,y}, \phi^s_{x,y})$ satisfies:

(1) $\|\phi_{x,y}(0)\| = d(f(x), y)$;

(2) $\|D\phi_{x,y}\|_{C^0} \le \varepsilon\varepsilon(x)$;

(3) $\|D_v\phi_{x,y} - D_w\phi_{x,y}\| \le \varepsilon\varepsilon(x)\|v - w\|^{\delta/2}$ *for all* $v, w \in S_x$.

A similar statement holds for $F^{-1}_{x,y} = \exp_x^{-1} \circ f^{-1} \circ \exp_y$ defined on S_y for all $x \in B(f^{-1}(y), \delta^s(\varepsilon(y)))$.

Proof. Since $F_{x,y}$ is $C^{1+\alpha}$, there is $\hat{C} > 0$ such that for any $v, w \in S_x$,

$$\|D_vF_{x,y} - D_wF_{x,y}\| \le \hat{C}\|v - w\|^\alpha.$$

Fix $\varepsilon > 0$ small enough so that

$$\varepsilon^{\delta/(\alpha-\delta)} \le \min\left\{1, \frac{1}{(2\sqrt{2})^{\alpha-\delta/2}\hat{C}}\right\}.$$

Part (1) is free and part (2) is a special case of part (3) when $w = 0$ since $D_0\phi_{x,y} = 0$. Then what we need to check is property (3).

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In fact, we have

$$\begin{aligned}\|D_v\phi_{x,y} - D_w\phi_{x,y}\| &= \|D_v F_{x,y} - D_w F_{x,y}\| \\ &\le \hat{C}\|v-w\|^{\alpha-\delta/2}\|v-w\|^{\delta/2} \\ &\le (2\sqrt{2})^{\alpha-\delta/2}\hat{C}Q(x)^{\delta/2}Q(x)^{\alpha-\delta}\|v-w\|^{\delta/2} \\ &\le (2\sqrt{2})^{\alpha-\delta/2}\hat{C}\varepsilon^{\delta/(\alpha-\delta)}\cdot\varepsilon\varepsilon(x)\|v-w\|^{\delta/2} \\ &\le \varepsilon\varepsilon(x)\|v-w\|^{\delta/2}.\end{aligned}$$

By considering f^{-1} in the charts $\exp_y$ of point y and $\exp_x^{-1}$ of $f^{-1}(y)$ when x is near $f^{-1}(y)$, we can apply a similar treatment to $F_{x,y}^{-1}$. The details are left to the reader. □

4. *Admissible manifolds and graph transform along the edge*

In this section, denote $t := u, s$. As usual, we give the canonical definition of *t-manifolds* first.

Definition 4.1. Suppose $x \in O$.

(1) A u-manifold at x is a manifold $V_x^u = \exp_x\{(v^u, \sigma_x^u(v^u)) \in T_xM : \|v^u\| \le Q(x)\} \subset M$, where $\sigma^u : B_x^u(Q(x)) \to E^s(x)$ is a $C^{1+\delta/2}$ function such that $\|\sigma_x^u\|_{C^0} \le Q(x)$.

(2) Similarly, an s-manifold at x is a manifold $V_x^s = \exp_x\{(\sigma_x^s(v^s), v^s) : \|v^s\| \le Q(x)\}$ where $\sigma_x^s : B_x^s(Q(x)) \to E^u(x)$ is a $C^{1+\delta/2}$ function such that $\|\sigma_x^s\|_{C^0} \le Q(x)$.

Thus, every t-manifold at x is contained in $\exp_x(S_x)$. For any $x \in O$, we call σ_x^t the *representing functions* of V_x^t.

Among the t-manifolds, we are interested in those that can eventually represent unstable and stable manifolds, and for that, we require some extra conditions on the representing functions. We call them *admissible manifolds*.

Definition 4.2. For any $x \in O$, a *t-admissible manifold* at x is a t-manifold V_x^t where the representing function σ_x^t satisfies the following three conditions:

(AM1) $\|\sigma_x^t(0)\| \le 10^{-3}Q(x)$;

(AM2) $\|D\sigma_x^t\|_{C^0} \le \sqrt{\varepsilon}$;

(AM3) $\|D\sigma_x^t\|_{C^{\delta/2}} := \|D\sigma_x^t\|_{C^0} + \mathrm{Hol}_{\delta/2}(D\sigma_x^t) \le \frac{1}{2}$, with norms taken in $B_x^t(Q(x))$.

Observe that the upper bound of $\|\sigma_x^t(0)\|$ is linear in $Q(x)$, while the estimate on the derivative of a representing function is constant in our setting rather than of the order of $Q(x)^{\delta/2}$ as in [22, Definition 4.8]. This distinction will be clear when we start to establish an estimate in the *graph transform along the edge* (Definition 4.4).

Let $\mathscr{V}_x^t$ be the space of all t-admissible manifolds at $x \in O$. We introduce two metrics on $\mathscr{V}_x^t$ as follows: for $V_1^t, V_2^t \in \mathscr{V}_x^t$ with representing functions σ_1^t, σ_2^t, respectively, put

$$d_{C^i}(V_1^t, V_2^t) := \|\sigma_1^t - \sigma_2^t\|_{C^i},\ i = 0, 1,$$

where

$$\begin{aligned}\|\sigma_1^t - \sigma_2^t\|_{C^0} &= \max_{v_x \in B_x^t(Q(x))}\|\sigma_1^t(v_x) - \sigma_2^t(v_x)\|, \\ \|\sigma_1^t - \sigma_2^t\|_{C^1} &= \|\sigma_1^t - \sigma_2^t\|_{C^0} + \|D\sigma_1^t - D\sigma_2^t\|_{C^0},\end{aligned}$$

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and

$$\|D\sigma_1^t - D\sigma_2^t\|_{C^0} = \max_{v_x \in B_x^t(Q(x))} \|D_{v_x}\sigma_1^t - D_{v_x}\sigma_2^t\|.$$

If $\varepsilon > 0$ is small enough, from conditions (AM1) and (AM2), we have $\|\sigma_x^t\|_{C^0} \le Q(x)$, because for all $v_x \in B_x^t(Q(x))$,

$$\begin{aligned}\|\sigma_x^t(v_x)\| \le \|\sigma_x^t(0)\| + \mathrm{Lip}(\sigma_x^t)\|v_x\| &\le 10^{-3}Q(x) \\ &+ \sqrt{\varepsilon}\|v_x\| \le (10^{-3} + \sqrt{\varepsilon})Q(x) < 10^{-2}Q(x),\end{aligned} \tag{4.1}$$

where in the last inequality, we require $\varepsilon > 0$ to be small. In particular, σ_x^u maps the ball $B_x^u(10^{-2}Q(x))$ into $B_x^s(10^{-2}Q(x))$, and maps $B_x^u(Q(x))$ into $B_x^s(Q(x))$.

PROPOSITION 4.1. *For every $x \in O$, the following holds for $\varepsilon > 0$ small enough. For every $V_x^u \in \mathscr{V}_x^u$ and $V_x^s \in \mathscr{V}_x^s$, the following hold:*

(1) *V_x^u and V_x^s intersect at a single point $P = P(V_x^u, V_x^s) := \exp_x(w) \in O$, and $\|w\| \le 50^{-1}Q(x)$;*

(2) *P is a Lipschitz function of (V_x^u, V_x^s), with Lipschitz constant less than $2/(1-\sqrt{\varepsilon})$.*

Proof. We follow the same strategy of [22, Proposition 4.11].

(1) To prove the non-emptiness of $V_x^u \cap V_x^s$, we fix V_x^t with representing function σ_x^t. From a pedagogical perspective, the ideas of the proof are very simple. If we suppose the intersections of V_x^u and V_x^s are non-empty, it corresponds to points (v^u, v^s) that satisfy $\exp_x(v^u, \sigma_x^u(v^u)) = \exp_x(\sigma_x^s(v^s), v^s)$, so it is the solution of the equation:

$$\begin{cases} v^u = \sigma_x^s(v^s), \\ v^s = \sigma_x^u(v^u). \end{cases}$$

Hence, the trick of the proof is to find the fixed points of $\sigma_x^s \circ \sigma_x^u$ on $B_x^u(Q(x))$.

We first note that $\sigma_x^s \circ \sigma_x^u$ maps the ball $B_x^u(10^{-2}Q(x))$ into itself from equation (4.1) if $\varepsilon > 0$ is small enough. It follows that $\sigma_x^s \circ \sigma_x^u$ is an ε-contraction of $B_x^u(10^{-2}Q(x))$ into itself. Indeed, from condition (AM2), we have $\|D(\sigma_x^s \circ \sigma_x^u)\|_{C^0} \le \|D\sigma_x^s\|_{C^0}\|D\sigma_x^u\|_{C^0} \le \varepsilon$. By the Banach fixed point theorem, $\sigma_x^s \circ \sigma_x^u$ has a unique fixed point: $(\sigma_x^s \circ \sigma_x^u)(v_0^u) = v_0^u$, where $v_0^u \in B_x^u(10^{-2}Q(x))$.

Let $v_0^s := \sigma_x^u(v_0^u)$, $w := (v_0^u, v_0^s)$, then V_x^u intersects V_x^s at $P := \exp_x(w)$. Indeed, $P \in V_x^u$ is evident because $v_0^s = \sigma_x^u(v_0^u)$ and $\|v_0^u\| \le 10^{-2}Q(x) < Q(x)$. Since $v_0^u = (\sigma_x^s \circ \sigma_x^u)(v_0^u) = \sigma_x^s(v_0^s)$ and

$$\|v_0^s\| \le \|\sigma_x^u(0)\| + \mathrm{Lip}(\sigma_x^u)\|v_0^u\| \le 10^{-3}Q(x) + \sqrt{\varepsilon}\cdot 10^{-2}Q(x) \le 10^{-2}Q(x) < Q(x),$$

we have $P \in V_x^s$. Moreover, we also see that $\|w\| \le \|v_0^u\| + \|v_0^s\| \le 50^{-1}Q(x) < d(x, \partial O)$, and thus $P \in O$.

To prove the uniqueness in the bigger ball $B_x^u(Q(x))$, we note that $\sigma_x^s \circ \sigma_x^u$ maps $B_x^u(Q(x))$ into itself if $\varepsilon > 0$ is small enough, and the same calculation as above shows that such a map has a unique fixed point even in $B_x^u(Q(x))$.

(2) Now we verify that P is a Lipschitz function of V_x^u, V_x^s. For every $x \in O$, suppose $V_i^t \in \mathscr{V}_x^t$ are represented by σ_i^t, $i = 1, 2$. Let $P_i = \exp_x(w_i) = \exp_x(v_i^u, v_i^s)$ denote the

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intersection points of $V_i^u \cap V_i^s$. We saw above that $v_i^s = \sigma_i^u(v_i^u)$ and that $v_i^u = \sigma_i^s(v_i^s)$ because v_i^u is a fixed point of $\sigma_i^s \circ \sigma_i^u$.

Since $\exp_x$ is 2-Lipschitz, this gives that $d(P_1, P_2) \le 2\|w_1 - w_2\|$. We have

$$\begin{aligned}\|w_1 - w_2\| &\le \|v_1^u - v_2^u\| + \|v_1^s - v_2^s\| \\ &= \|\sigma_1^s(v_1^s) - \sigma_2^s(v_2^s)\| + \|\sigma_1^u(v_1^u) - \sigma_2^u(v_2^u)\| \\ &\le \|\sigma_1^s(v_1^s) - \sigma_1^s(v_2^s)\| + \|\sigma_1^s(v_2^s) - \sigma_2^s(v_2^s)\| \\ &\quad + \|\sigma_1^u(v_1^u) - \sigma_1^u(v_2^u)\| + \|\sigma_1^u(v_2^u) - \sigma_2^u(v_2^u)\| \\ &\le \sqrt{\varepsilon}(\|v_1^s - v_2^s\| + \|v_1^u - v_2^u\|) + \|\sigma_1^s - \sigma_2^s\|_{C^0} + \|\sigma_1^u - \sigma_2^u\|_{C^0}.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}d(P_1, P_2) &\le 2(\|v_1^u - v_2^u\| + \|v_1^s - v_2^s\|) \\ &\le \frac{2}{1-\sqrt{\varepsilon}}(d_{C^0}(V_1^u, V_2^u) + d_{C^0}(V_1^s, V_2^s)).\end{aligned}$$ □

Although the spaces $\mathscr{V}^t$ and the action of f or f^{-1} are hard to work with due to the nonlinearity, from the definition of admissible manifold, through the inverse of the exponential map, the action of f or f^{-1} on an admissible manifold can be locally lifted to the tangent space. For example, for the action of f, if V_x^u is a u-admissible manifold at x, then $f(V_x^u) = \exp_y\{F_{x,y}(v_x^u, \sigma_x^u(v_x^u)) : \|v_x^u\| \le Q(x)\}$ if $d(f(x), y) < \delta^u(\varepsilon(x))$, where $\delta^u(\cdot)$ is given by Proposition 3.1. To prove that $f(V_x^u)$ contains a u-admissible manifold V_y^u at y, we write $F_{x,y}|_{S_x} = D_0F_{x,y} + \phi_{x,y}$ in the form

$$F_{x,y}(v_x^u, \sigma_x^u(v_x^u)) = (G_{x,y}^{\sigma_x^u}(v_x^u), L_{x,y}^{\sigma_x^u}(v_x^u)), \tag{4.2}$$

where

$$\begin{aligned}G_{x,y}^{\sigma_x^u}(\cdot) &:= D_{x,y}^{uu}(\cdot) + D_{x,y}^{us} \circ \sigma_x^u(\cdot) + \phi_{x,y}^u(\cdot, \sigma_x^u(\cdot)), \\ L_{x,y}^{\sigma_x^u}(\cdot) &:= D_{x,y}^{ss} \circ \sigma_x^u(\cdot) + D_{x,y}^{su}(\cdot) + \phi_{x,y}^s(\cdot, \sigma_x^u(\cdot)).\end{aligned} \tag{4.3}$$

If $B_y^u(Q(y)) \subset G_{x,y}^{\sigma_x^u}(B_x^u(Q(x)))$ and $G_{x,y}^{\sigma_x^u}$ is injective, we can define $\sigma_y^u := L_{x,y}^{\sigma_x^u} \circ (G_{x,y}^{\sigma_x^u})^{-1}$, and we have

$$\begin{aligned}f(V_x^u) &= \exp_y\{(G_{x,y}^{\sigma_x^u}(v_x^u),\ L_{x,y}^{\sigma_x^u} \circ (G_{x,y}^{\sigma_x^u})^{-1} \circ G_{x,y}^{\sigma_x^u}(v_x^u)) : \|v_x^u\| \le Q(x)\} \\ &\supset \exp_y\{(v_y^u, \sigma_y^u(v_y^u)) : \|v_y^u\| \le Q(y)\},\end{aligned}$$

that is, $f(V_x^u)$ contains a u-admissible manifold $V_y^u = \exp_y\{(v_y^u, \sigma_y^u(v_y^u)) : \|v_y^u\| \le Q(y)\}$ when σ_y^u satisfies properties (AM1)–(AM3). If V_y^u obtained in this way is unique, then the *graph transform* $\mathscr{F}_{x,y}^u$ (see Definition 4.4 below) sends a u-admissible manifold at x with representing function σ_x^u to a u-admissible manifold at y whose graph of the representing function σ_y^u is contained in the action of $F_{x,y}$ on the graph of σ_x^u.

Similarly, when $d(f^{-1}(y), x) < \delta^s(\varepsilon(y))$, the case of $f^{-1}(V_y^s)$ is symmetric and the corresponding *graph transform* $\mathscr{F}_{x,y}^s$ maps an s-admissible manifold at y to an s-admissible manifold at x.

We start with a definition and a basic lemma which will be used many times throughout the paper.

Definition 4.3. For $x, y \in O$, the *edge* $x \to y$ means

$$d(f(x), y) < \delta^u(\varepsilon(x))Q(x)^2Q(f(x)) \quad \text{and} \quad d(f^{-1}(y), x) < \delta^s(\varepsilon(y))Q(y)^2Q(f^{-1}(y)).$$

LEMMA 4.1. *Fix any positive constants C_1, C_2, C_3, C_4, and suppose $x \to y$. If $\varepsilon > 0$ is small enough, we have:*

(1) $C_1Q(y) + d(f(x), y) \leq C_1Q(x)(m(Df|_{E^u(x)})^\kappa - C_2\varepsilon\varepsilon(x))$;

(2) $C_3Q(y) - d(f(x), y) \geq C_3Q(x)(\|Df|_{E^s(x)}\|^\kappa + C_4\varepsilon\varepsilon(x))$.

A similar statement holds for f^{-1}.

Proof. We give the proof for part (1), the proof of part (2) follows in a similar manner and is left to the reader.

First, we fix

$$0 < \varepsilon \leq \min\left\{1, \frac{1}{\gamma - 1}, (2\gamma C_1)^{-(\alpha-\delta)/2}, \left(\frac{C_1}{2}\right)^{(\alpha-\delta)/4}, \frac{C-1}{1+C_2}\right\}.$$

Since $d(f(x), y) < \delta^u(\varepsilon(x))Q(x)^2Q(f(x))$, we have $(d(f(x), y)/d(f(x), \partial O)) < \delta^u(\varepsilon(x))Q(x)^2 < \varepsilon$. By the fact that $(1+a)^\gamma \leq 1 + 2\gamma a$ if $0 < a \leq (1/(\gamma - 1))$, we obtain

$$\begin{aligned}
Q(y) &\leq \varepsilon^{2/(\alpha-\delta)} \min\{r_0^\gamma, (d(f(x), \partial O) + d(y, f(x)))^\gamma\} \\
&\leq \varepsilon^{2/(\alpha-\delta)} \min\left\{r_0^\gamma, d(f(x), \partial O)^\gamma\left(1 + 2\gamma\frac{d(y, f(x))}{d(f(x), \partial O)}\right)\right\} \\
&\leq Q(f(x)) + 2\gamma\varepsilon^{2/(\alpha-\delta)}d(y, f(x)) \\
&\leq Q(f(x)) + \frac{d(y, f(x))}{C_1}.
\end{aligned}$$

Recall that $0 < \delta^u(\varepsilon(x)) < \varepsilon\varepsilon(x)$ in Proposition 3.1 and using Lemma 3.2, we have

$$\begin{aligned}
&\frac{C_1Q(y) + d(f(x), y) + C_1C_2\varepsilon\varepsilon(x)Q(x)}{C_1Q(x)} \\
&\leq \frac{Q(f(x))}{Q(x)} + \frac{2d(y, f(x))}{C_1Q(x)} + C_2\varepsilon\varepsilon(x) \\
&\leq \frac{Q(f(x))}{Q(x)} + \frac{2}{C_1}\varepsilon^{4/(\alpha-\delta)}\delta^u(\varepsilon(x)) + C_2\varepsilon\varepsilon(x) \\
&\leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1 + (1 + C_2)\varepsilon\varepsilon(x) \\
&\leq \frac{1}{C}(m(Df|_{E^u(x)})^\kappa - 1) + 1 + \frac{C-1}{C}(m(Df|_{E^u(x)})^\kappa - 1) \\
&= m(Df|_{E^u(x)})^\kappa,
\end{aligned}$$

proving part (1). □

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PROPOSITION 4.2. *The following hold for all $\varepsilon > 0$ small enough. Suppose $x \to y$, then:*

(1) *the map $G^{\sigma^u_x}_{x,y} : B^u_x(Q(x)) \to E^u(y)$ defined by*

$$G^{\sigma^u_x}_{x,y}(v^u_x) = D^{uu}_{x,y}(v^u_x) + D^{us}_{x,y} \circ \sigma^u_x(v^u_x) + \phi^u_{x,y}(v^u_x, \sigma^u_x(v^u_x)) \quad \text{for all } v^u_x \in B^u_x(Q(x)),$$

is injective, and $G^{\sigma^u_x}_{x,y}(B^u_x(Q(x))) \supset B^u_y(Q(y))$;

(2) *the map $\sigma^u_y := L^{\sigma^u_x}_{x,y} \circ (G^{\sigma^u_x}_{x,y})^{-1}$ defined on $G^{\sigma^u_x}_{x,y}(B^u_x(Q(x)))$ satisfies properties (AM1)–(AM3), where the map $L^{\sigma^u_x}_{x,y} : B^u_x(Q(x)) \to E^u(y)$ is defined by*

$$L^{\sigma^u_x}_{x,y}(v^u_x) := D^{ss}_{x,y} \circ \sigma^u_x(v^u_x) + D^{su}_{x,y}(v^u_x) + \phi^s_{x,y}(v^u_x, \sigma^u_x(v^u_x)) \quad \text{for all } v^u_x \in B^u_x(Q(x)).$$

The symmetric statement holds for $F^{-1}_{x,y}$.

Proof. The proof is a straightforward adaption of arguments in [22, Proposition 4.12]. We give the proof in the case of $F_{x,y}$. The case of $F^{-1}_{x,y}$ is symmetric.

(1) The well-defined nature of $(G^{\sigma^u_x}_{x,y})^{-1}$ is a consequence of the expansion property of $Df|_{E^u(x)}$. For every $v^u_x \in B^u_x(Q(x))$, by property (AM2), we know that $\|D_{v^u_x}\sigma^u_x\| \le 1$ if $0 < \varepsilon \le \min\{1, C/4\}$. Therefore, from Proposition 3.1(2) and Lemma 3.1, we have

$$\begin{aligned}\|D_{v^u_x} G^{\sigma^u_x}_{x,y}\| &\ge m(D^{uu}_{x,y}) - \|D^{us}_{x,y}\|\,\|D_{v^u_x}\sigma^u_x\| - \|D_{(v^u_x,\sigma^u_x(v^u_x))}\phi^u_{x,y}\|(1 + \|D_{v^u_x}\sigma^u_x\|)\\ &\ge m(Df|_{E^u(x)}) - \varepsilon\varepsilon(x) - \varepsilon\varepsilon(x) - 2\varepsilon\varepsilon(x)\\ &\ge m(Df|_{E^u(x)})^{\kappa} - 4\varepsilon\varepsilon(x) > 1.\end{aligned}$$

It follows that $G^{\sigma^u_x}_{x,y}$ is $[m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)]$-expanding and hence one-to-one, and $(G^{\sigma^u_x}_{x,y})^{-1}$ is well defined on $G^{\sigma^u_x}_{x,y}(B^u_x(Q(x)))$.

By invariance of the domain, to prove $G^{\sigma^u_x}_{x,y}(B^u_x(Q(x))) \supset B^u_y(Q(y))$, the problem reduces to show that

$$\|G^{\sigma^u_x}_{x,y}(v^u_x)\| \ge Q(y)$$

for any $v^u_x \in \partial B^u_x(Q(x))$.

According to

$$\begin{aligned}\|G^{\sigma^u_x}_{x,y}(0)\| &= \|D^{us}_{x,y} \circ \sigma^u_x(0) + \phi^u_{x,y}(0, \sigma^u_x(0))\|\\ &\le \|D^{us}_{x,y}\|\,\|\sigma^u_x(0)\| + \|\phi_{x,y}(0,0)\| + \mathrm{Lip}(\phi_{x,y})\|\sigma^u_x(0)\|,\end{aligned}$$

by property (AM1) and Proposition 3.1(2), we have

$$\|G^{\sigma^u_x}_{x,y}(0)\| \le 2\varepsilon\varepsilon(x) \times 10^{-3} Q(x) + d(f(x), y). \tag{4.4}$$

In particular, using Lemma 4.1 for $C_1 = 1, C_2 = 5$, and the fact that $G^{\sigma^u_x}_{x,y}$ is $[m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)]$-expanding, we have that for any $v^u_x \in \partial B^u_x(Q(x))$,

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$$
\begin{aligned}
\|G^{\sigma^u_x}_{x,y}(v^u_x)\| &\geq \|G^{\sigma^u_x}_{x,y}(v^u_x) - G^{\sigma^u_x}_{x,y}(0)\| - \|G^{\sigma^u_x}_{x,y}(0)\| \\
&\geq (m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x))Q(x) - 2\varepsilon\varepsilon(x) \times 10^{-3} Q(x) - d(f(x), y) \\
&\geq (m(Df|_{E^u(x)})^\kappa - 5\varepsilon\varepsilon(x))Q(x) - d(f(x), y) \\
&\geq Q(y).
\end{aligned}
$$

(2) We first prove σ^u_y satisfies property (AM1). Denote $v^u_{0x} := (G^{\sigma^u_x}_{x,y})^{-1}(0)$, since $G^{\sigma^u_x}_{x,y}$ is $[m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)]$-expanding on $B^u_x(Q(x))$, in particular,

$$
\|D_{v^u_y}(G^{\sigma^u_x}_{x,y})^{-1})\| < 1 \quad \text{for all } v^u_y \in G^{\sigma^u_x}_{x,y}(B^u_x(Q(x))).
$$

By equation (4.4), we have

$$
\begin{aligned}
\|v^u_{0x}\| &= \|(G^{\sigma^u_x}_{x,y})^{-1}(0) - (G^{\sigma^u_x}_{x,y})^{-1} \circ G^{\sigma^u_x}_{x,y}(0)\| \\
&\leq \|D(G^{\sigma^u_x}_{x,y})^{-1})\|_{C^0} \|G^{\sigma^u_x}_{x,y}(0)\| \\
&\leq 2\varepsilon\varepsilon(x) \times 10^{-3} Q(x) + d(f(x), y) \\
&\leq 3\varepsilon\varepsilon(x) \times 10^{-3} Q(x).
\end{aligned}
$$

Note that from equation (3.6) of Lemma 3.2, it follows that $\|D^{ss}_{x,y}\| \leq \|Df|_{E^s(x)}\|^\kappa + \varepsilon\varepsilon(x) \leq 1$ if $0 < \varepsilon \leq \min\{1, C\}$. According to properties (AM1), (AM2), and Lemma 4.1 for $C_3 = 10^{-3}$, $C_4 = 13$, we obtain

$$
\begin{aligned}
\|\sigma^u_y(0)\| &= \|L^{\sigma^u_x}_{x,y}(v^u_{0x})\| \\
&= \|D^{ss}_{x,y} \circ \sigma^u_x(v^u_{0x}) + D^{su}_{x,y}(v^u_{0x}) + \phi^s_{x,y}(v^u_{0x}, \sigma^u_x(v^u_{0x}))\| \\
&\leq \|D^{ss}_{x,y}\|(\|\sigma^u_x(0)\| + \mathrm{Lip}(\sigma^u_x)\|v^u_{0x}\|) + \|D^{su}_{x,y}\| \, \|v^u_{0x}\| + \|\phi^s_{x,y}(0, 0)\| \\
&\quad + \mathrm{Lip}(\phi^s_{x,y})(1 + \mathrm{Lip}(\sigma^u_x))\|v^u_{0x}\| \\
&\leq (\|Df|_{E^s(x)}\|^\kappa + \varepsilon\varepsilon(x))10^{-3} Q(x) + 12\varepsilon\varepsilon(x) \times 10^{-3} Q(x) + d(f(x), y) \\
&\leq 10^{-3} Q(y),
\end{aligned}
$$

proving property (AM1).

Next we prove $\|D\sigma^u_y\|_{C^0} \leq \sqrt{\varepsilon}$, where the norm is taken in $G^{\sigma^u_x}_{x,y}(B^u_x(Q(x)))$.

For all $v^u_y \in G^{\sigma^u_x}_{x,y}(B^u_x(Q(x)))$, $v^u_x = (G^{\sigma^u_x}_{x,y})^{-1}(v^u_y) \in B^u_x(Q(x))$. Thanks to $\|D_{v^u_y}(G^{\sigma^u_x}_{x,y})^{-1})\| < 1$, property (AM2) is easy to be verified from

$$
\begin{aligned}
\|D_{v^u_y}\sigma^u_y\| &= \|D^{ss}_{x,y} \cdot D_{v^u_x}\sigma^u_x \cdot D_{v^u_y}(G^{\sigma^u_x}_{x,y})^{-1} + D^{su}_{x,y} \cdot D_{v^u_y}(G^{\sigma^u_x}_{x,y})^{-1} \\
&\quad + D_{v^u_x}\phi^s_{x,y} \cdot D_{v^u_y}(G^{\sigma^u_x}_{x,y})^{-1}\| \\
&\leq \|D^{ss}_{x,y}\| \, \|D\sigma^u_x\|_{C^0} + 3\varepsilon\varepsilon(x) \\
&\leq \|Df|_{E^s(x)}\|\sqrt{\varepsilon} + 4\varepsilon\varepsilon(x) \\
&\leq \sqrt{\varepsilon}
\end{aligned}
$$

if $0 < \varepsilon \leq \min\{1, (C/4)^2\}$.

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Finally, we have to show that σ_y^u satisfies property (AM3) also. From

$$\sigma_y^u = L_{x,y}^{\sigma_x^u} \circ (G_{x,y}^{\sigma_x^u})^{-1} = [D_{x,y}^{ss} \circ \sigma_x^u(\cdot) + D_{x,y}^{su}(\cdot) + \phi_{x,y}^s(\cdot, \sigma_x^u(\cdot))] \circ (G_{x,y}^{\sigma_x^u})^{-1},$$

we have

$$\begin{aligned}\|D\sigma_y^u\|_{C^{\delta/2}} &= \|(D_{x,y}^{ss} \cdot D\sigma_x^u + D_{x,y}^{su} + D\phi_{x,y}^s) \cdot D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^{\delta/2}} \\ &\le (\|D_{x,y}^{ss}\| \cdot \|D\sigma_x^u\|_{C^{\delta/2}} + \|D_{x,y}^{su}\| + 2\|D\phi_{x,y}\|_{C^{\delta/2}(S_x)})\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^{\delta/2}}.\end{aligned}$$

CLAIM. *If* $\varepsilon > 0$ *is small enough,* $\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^{\delta/2}} < 1$.

Proof of Claim. For $v_y^u, w_y^u \in G_{x,y}^{\sigma_x^u}(B_x^u(Q(x)))$, notice that $D_{v_y^u}(G_{x,y}^{\sigma_x^u})^{-1} \cdot D_{v_x^u} G_{x,y}^{\sigma_x^u}$ and $D_{w_x^u} G_{x,y}^{\sigma_x^u} \cdot D_{w_y^u}(G_{x,y}^{\sigma_x^u})^{-1}$ are identity maps, where $v_y^u = G_{x,y}^{\sigma_x^u}(v_x^u)$, $w_x^u = (G_{x,y}^{\sigma_x^u})^{-1}(w_y^u)$. Thus, we have

$$\begin{aligned}&\|D_{v_y^u}(G_{x,y}^{\sigma_x^u})^{-1} - D_{w_y^u}(G_{x,y}^{\sigma_x^u})^{-1}\| \\ &\quad = \|D_{v_y^u}(G_{x,y}^{\sigma_x^u})^{-1} \cdot (D_{w_x^u} G_{x,y}^{\sigma_x^u} - D_{v_x^u} G_{x,y}^{\sigma_x^u}) \cdot D_{w_y^u}(G_{x,y}^{\sigma_x^u})^{-1}\| \\ &\quad = \|D_{v_y^u}(G_{x,y}^{\sigma_x^u})^{-1} \cdot (D_{x,y}^{us} D_{w_x^u}\sigma_x^u + D_{w_x^u}\phi_{x,y}^u \\ &\qquad - D_{x,y}^{us} D_{v_x^u}\sigma_x^u - D_{v_x^u}\phi_{x,y}^u) \cdot D_{w_y^u}(G_{x,y}^{\sigma_x^u})^{-1}\|.\end{aligned}$$

First, to estimate $\|D_{w_x^u}\phi_{x,y}^u - D_{v_x^u}\phi_{x,y}^u\|$ in parentheses, a little manipulation is needed. If $0 < \varepsilon \le 1$, by property (AM3) and Proposition 3.1(3), we can get

$$\begin{aligned}&\|D_{w_x^u}\phi_{x,y}^u(\cdot, \sigma_x^u(\cdot)) - D_{v_x^u}\phi_{x,y}^u(\cdot, \sigma_x^u(\cdot))\| \\ &\quad = \left\| D_{(w_x^u, \sigma_x^u(w_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{w_x^u}\sigma_x^u \end{bmatrix} - D_{(v_x^u, \sigma_x^u(v_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{v_x^u}\sigma_x^u \end{bmatrix} \right\| \\ &\quad \le \left\| D_{(w_x^u, \sigma_x^u(w_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{w_x^u}\sigma_x^u \end{bmatrix} - D_{(w_x^u, \sigma_x^u(w_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{v_x^u}\sigma_x^u \end{bmatrix} \right\| \\ &\qquad + \left\| D_{(w_x^u, \sigma_x^u(w_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{v_x^u}\sigma_x^u \end{bmatrix} - D_{(v_x^u, \sigma_x^u(v_x^u))}\phi_{x,y}^u \begin{bmatrix} I \\ D_{v_x^u}\sigma_x^u \end{bmatrix} \right\| \\ &\quad \le \frac{\varepsilon\varepsilon(x)}{2}\|w_x^u - v_x^u\|^{\delta/2} + 4\varepsilon\varepsilon(x)\|w_x^u - v_x^u\|^{\delta/2} \\ &\quad = \frac{9\varepsilon\varepsilon(x)}{2}\|w_x^u - v_x^u\|^{\delta/2}.\end{aligned}$$

For the rest in the parentheses, from property (AM3), we find

$$\|D_{x,y}^{us} D_{w_x^u}\sigma_x^u - D_{x,y}^{us} D_{v_x^u}\sigma_x^u\| \le \tfrac{1}{2}\varepsilon\varepsilon(x)\|w_x^u - v_x^u\|^{\delta/2};$$

therefore, we obtain that

$$\begin{aligned}&\|D_{v_y^u}(G_{x,y}^{\sigma_x^u})^{-1} - D_{w_y^u}(G_{x,y}^{\sigma_x^u})^{-1}\| \\ &\quad \le \|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^0}^2\, 5\varepsilon\varepsilon(x)\|w_x^u - v_x^u\|^{\delta/2} \\ &\quad \le 5\varepsilon\varepsilon(x)\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^0}^{2+\delta/2}\|w_y^u - v_y^u\|^{\delta/2}\end{aligned}$$

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or

$$\mathrm{Hol}_{\delta/2}(D(G_{x,y}^{\sigma_x^u})^{-1}) \le 5\varepsilon\varepsilon(x)\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^0}^{2+\delta/2} \le 5\varepsilon\varepsilon(x).$$

Using $G_{x,y}^{\sigma_x^u}$ is $[m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)]$-expanding again, that is,

$$\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^0} \le \frac{1}{m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)},$$

we get finally

$$\begin{aligned}\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^{\delta/2}} &= \|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^0} + \mathrm{Hol}_{\delta/2}(D(G_{x,y}^{\sigma_x^u})^{-1})\\ &\le \frac{1}{m(Df|_{E^u(x)}) - 4\varepsilon\varepsilon(x)} + 5\varepsilon\varepsilon(x)\\ &< 1\end{aligned}$$

if $0 < \varepsilon < \min\{1, C/5(C^f + 4)\}$. This completes the proof of the claim. □

We are now turning to the proof of $\|D\sigma_y^u\|_{C^{\delta/2}} \le \frac{1}{2}$. From Proposition 3.1(2) and (3), we have $\|D\phi_{x,y}\|_{C^{\delta/2}(S_x)} \le 2\varepsilon\varepsilon(x)$. By using equation (3.6) of Lemma 3.2 and combining with the claim above, we obtain that

$$\begin{aligned}\|D\sigma_y^u\|_{C^{\delta/2}} &\le (\|D_{x,y}^{ss}\| \cdot \|D\sigma_x^u\|_{C^{\delta/2}} + \|D_{x,y}^{su}\| + 2\|D\phi_{x,y}\|_{C^{\delta/2}(S_x)})\|D(G_{x,y}^{\sigma_x^u})^{-1}\|_{C^{\delta/2}}\\ &\le \|D_{x,y}^{ss}\| \cdot \|D\sigma_x^u\|_{C^{\delta/2}} + \|D_{x,y}^{su}\| + 2\|D\phi_{x,y}\|_{C^{\delta/2}(S_x)}\\ &\le \tfrac{1}{2}(\|Df|_{E^s(x)}\| + \varepsilon\varepsilon(x)) + 5\varepsilon\varepsilon(x)\\ &\le \tfrac{1}{2}\end{aligned}$$

when $0 < \varepsilon \le \min\{1, C/11\}$, and the proof of this proposition is completed. □

As has been said, since

$$f(V_x^u) = \exp_y\{(G_{x,y}^{\sigma_x^u}(v_x^u),\ \sigma_y^u \circ G_{x,y}^{\sigma_x^u}(v_x^u)) : \|v_x^u\| \le Q(x)\}$$

and $G_{x,y}^{\sigma_x^u}(B_x^u(Q(x))) \supset B_y^u(Q(y))$, Proposition 4.2 above shows that $f(V_x^u)$ restricts to a unique u-admissible manifold $\widehat{V}_y^u$ at y, where

$$\widehat{V}_y^u = \exp_y\{(v_y^u, \sigma_y^u(v_y^u)) : \|v_y^u\| \le Q(y)\}.$$

Thus, we can summarize what we have proved as the following definition which is well defined for $\varepsilon > 0$ small enough.

Definition 4.4. Suppose $x \to y$.

(1) The *unstable graph transform* $\mathscr{F}_{x,y}^u : \mathscr{V}_x^u \to \mathscr{V}_y^u$ is the map that sends $V_x^u \in \mathscr{V}_x^u$ to the unique $\mathscr{F}_{x,y}^u[V_x^u] \in \mathscr{V}_y^u$ with representing function σ_y^u such that $\mathscr{F}_{x,y}^u[V_x^u] = \exp_y\{(v^u, \sigma_y^u(v^u)) : \|v^u\| \le Q(y)\} \subset f(V_x^u)$.
(2) The *stable graph transform* $\mathscr{F}_{x,y}^s : \mathscr{V}_y^s \to \mathscr{V}_x^s$ is the map that sends $V_y^s \in \mathscr{V}_y^s$ to the unique $\mathscr{F}_{x,y}^s[V_y^s] \in \mathscr{V}_x^s$ with representing function σ_x^s such that $\mathscr{F}_{x,y}^s[V_y^s] = \exp_x\{(\sigma_x^s(v^s), v^s) : \|v^s\| \le Q(x)\} \subset f^{-1}(V_y^s)$.

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COROLLARY 4.1. *The following holds for $\varepsilon > 0$ small enough. Suppose $x \to y$ and $V_x^u \in \mathscr{V}_x^u$. Then:*

(1) *if V_x^u is represented by the function σ_x^u and $p := \exp_x(0, \sigma_x^u(0))$, then $f(p) \in \mathscr{F}_{x,y}^u[V_x^u]$;*

(2) *$f(V_x^u)$ intersects any $V_y^s \in \mathscr{V}_y^s$ at a unique point.*

Similar statements hold for the f^{-1}-image of an s-admissible manifold at y.

Proof.

(1) Since $\mathscr{F}_{x,y}^u[V_x^u] = \exp_y\{(v_y^u, \sigma_y^u(v_y^u)) : \|v_y^u\| \le Q(y)\}$, it suffices to check that

$$F_{x,y}(0, \sigma_x^u(0)) = (G_{x,y}^{\sigma_x^u}(0), L_{x,y}^{\sigma_x^u}(0)) \in \{(v_y^u, \sigma_y^u(v_y^u)) : \|v_y^u\| \le Q(y)\},$$

or check $\|G_{x,y}^{\sigma_x^u}(0)\| \le Q(y)$, which is straightforward.

(2) For any $V_y^s \in \mathscr{V}_y^s$ with the representing function σ_y^s, by Proposition 4.1(1), $\mathscr{F}_{x,y}^u[V_x^u]$ and V_y^s intersect at a single point. According to Definition 4.4, $f(V_x^u)$ contains a u-manifold $\mathscr{F}_{x,y}^u[V_x^u] \in \mathscr{V}_y^u$; therefore, $f(V_x^u)$ and V_y^s intersect at least at one point.

Since

$$f(V_x^u) = \exp_y\{(v_y^u, \sigma_y^u(v_y^u)) : v_y^u \in G_{x,y}^{\sigma_x^u}(B_x^u(Q(x)))\}$$

with $\mathrm{Lip}(\sigma_y^u) \le \sqrt{\varepsilon}$, by the Kirszbraun theorem, we can extend σ_y^s to a Lipschitz function $\tilde{\sigma}_y^s$ on $G_{x,y}^{\sigma_x^u}(B_x^u(Q(x)))$ with $\mathrm{Lip}(\tilde{\sigma}_y^s) = \mathrm{Lip}(\sigma_y^s) \le \sqrt{\varepsilon}$. The extension represents a Lipschitz manifold $\tilde{V}_y^s \supset V_y^s$. Hence, the proof is quite similar to that given earlier in Proposition 4.1(1) to conclude that $f(V_x^u) \cap \tilde{V}_y^s$ consists of a single point and so is omitted. □

PROPOSITION 4.3. *If $\varepsilon > 0$ is small enough, then the following hold. Suppose $x \to y$, if $V_1^u, V_2^u \in \mathscr{V}_x^u$, we have:*

(1) $d_{C^0}(\mathscr{F}_{x,y}^u(V_1^u), \mathscr{F}_{x,y}^u(V_2^u)) \le \|Df|_{E^s(x)}\|^{1/2} d_{C^0}(V_1^u, V_2^u)$;

(2) $d_{C^1}(\mathscr{F}_{x,y}^u(V_1^u), \mathscr{F}_{x,y}^u(V_2^u)) \le \|Df|_{E^s(x)}\|^{1/2}(d_{C^1}(V_1^u, V_2^u) + d_{C^0}(V_1^u, V_2^u)^{\delta/2})$.

Similar statements hold for $\mathscr{F}_{x,y}^s$.

Proof. The proof is similar to the proof of [22, Proposition 4.14]. We prove the proposition for $\mathscr{F}_{x,y}^u$ and leave the case of $\mathscr{F}_{x,y}^s$ to the reader.

Let V_1, V_2 with representing functions σ_1, σ_2, and let $\tilde{\sigma}_1, \tilde{\sigma}_2$ be the representing functions of $\mathscr{F}_{x,y}^u(V_1), \mathscr{F}_{x,y}^u(V_2)$, that is, $\tilde{\sigma}_i = L_{x,y}^{\sigma_i} \circ (G_{x,y}^{\sigma_i})^{-1}|_{B_y^u(Q(y))}$, $i = 1, 2$.

(1) For any $v_y^u \in B_y^u(Q(y))$, denote $v_{1x}^u = (G_{x,y}^{\sigma_1})^{-1}(v_y^u)$, $v_{2x}^u = (G_{x,y}^{\sigma_2})^{-1}(v_y^u)$. Some tedious manipulation yields

$$\begin{aligned}
\|\tilde{\sigma}_1(v_y^u) - \tilde{\sigma}_2(v_y^u)\| &= \|\tilde{\sigma}_1 \circ G_{x,y}^{\sigma_1}(v_{1x}^u) - \tilde{\sigma}_2 \circ G_{x,y}^{\sigma_1}(v_{1x}^u)\| \\
&\le \|\tilde{\sigma}_1 \circ G_{x,y}^{\sigma_1}(v_{1x}^u) - \tilde{\sigma}_2 \circ G_{x,y}^{\sigma_2}(v_{1x}^u)\| + \|\tilde{\sigma}_2 \circ G_{x,y}^{\sigma_2}(v_{1x}^u) - \tilde{\sigma}_2 \circ G_{x,y}^{\sigma_1}(v_{1x}^u)\| \\
&\le \|D_{x,y}^{ss}(\sigma_1(v_{1x}^u) - \sigma_2(v_{1x}^u))\| + \|\phi_{x,y}^s(v_{1x}^u, \sigma_1(v_{1x}^u)) - \phi_{x,y}^s(v_{1x}^u, \sigma_2(v_{1x}^u))\| \\
&\quad + \mathrm{Lip}(\tilde{\sigma}_2)\|G_{x,y}^{\sigma_2}(v_{1x}^u) - G_{x,y}^{\sigma_1}(v_{1x}^u)\|
\end{aligned}$$

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$$
\begin{aligned}
&+ \operatorname{Lip}(\phi_{x,y}))\|\sigma_2(v^u_{1x}) - \sigma_1(v^u_{1x})\| \\
&\le (\|D^{ss}_{x,y}\| + \|D^{us}_{x,y}\| + 2\operatorname{Lip}(\phi_{x,y}))\|\sigma_2(v^u_{1x}) - \sigma_1(v^u_{1x})\| \\
&\le (\|Df|_{E^s(x)}\| + 6\varepsilon\varepsilon(x))\|\sigma_2(v^u_{1x}) - \sigma_1(v^u_{1x})\|.
\end{aligned}
$$

We remind the reader that $C\varepsilon(x) \le 1 - \|Df|_{E^s(x)}\|^{\kappa} \le 1 - \|Df|_{E^s(x)}\|$ and introduce a new constant

$$
a := \inf_{x\in O} \frac{\|Df|_{E^s(x)}\|^{1/2} - \|Df|_{E^s(x)}\|}{1 - \|Df|_{E^s(x)}\|} > 0, \tag{4.5}
$$

then if $0 < \varepsilon \le (aC/6)$, we get finally

$$
\begin{aligned}
\|\tilde\sigma_1 - \tilde\sigma_2\|_{C^0} &\le (\|Df|_{E^s(x)}\| + aC\varepsilon(x))\|\sigma_1 - \sigma_2\|_{C^0} \\
&\le \|Df|_{E^s(x)}\|^{1/2}\|\sigma_1 - \sigma_2\|_{C^0}.
\end{aligned}
$$

(2) In view of part (1), our task now is to estimate $\|D\tilde\sigma_1 - D\tilde\sigma_2\|_{C^0}$ in terms of $\|D\sigma_1 - D\sigma_2\|_{C^0}$ and $\|\sigma_1 - \sigma_2\|^{\delta/2}_{C^0}$. We will show that

$$
\begin{aligned}
&\|D\tilde\sigma_1 - D\tilde\sigma_2\|_{C^0} \\
&\quad\le (\|Df|_{E^s(x)}\| + 3\varepsilon\varepsilon(x))\|D\sigma_1 - D\sigma_2\|_{C^0} + \tfrac{1}{2}(\|Df|_{E^s(x)}\| + 25\varepsilon\varepsilon(x))\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0}.
\end{aligned}
$$

For any $v^u_y \in B^u_y(Q(y))$, let $v^u_{ix} = (G^{\sigma_i}_{x,y})^{-1}(v^u_y)$ as part (1), where $i = 1, 2$. We have

$$
\begin{aligned}
&\|D_{v^u_y}\tilde\sigma_1 - D_{v^u_y}\tilde\sigma_2\| \\
&\quad= \|D_{v^u_{1x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_1}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_2}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad\le \|D_{v^u_{1x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_1}_{x,y})^{-1} - D_{v^u_{1x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\qquad+ \|D_{v^u_{1x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\qquad+ \|D_{v^u_{2x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_2}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\|,
\end{aligned}
$$

so we need to estimate each term above. We divide our proof of part (2) into four steps.

First, note that the last term is easy to estimate because $\|D(G^{\sigma_2}_{x,y})^{-1}\| < 1$ and $L^{\sigma_1}_{x,y}(\cdot) = D^{ss}_{x,y}\circ\sigma_1(\cdot) + D^{su}_{x,y}(\cdot) + \phi^s_{x,y}(\cdot, \sigma_1(\cdot))$,

$$
\begin{aligned}
&\|D_{v^u_{2x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_2}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad\le \|D_{v^u_{2x}}L^{\sigma_1}_{x,y} - D_{v^u_{2x}}L^{\sigma_2}_{x,y}\| \\
&\quad\le \|D^{ss}_{x,y}\|\cdot\|D_{v^u_{2x}}\sigma_1 - D_{v^u_{2x}}\sigma_2\| + \|D_{v^u_{2x}}\phi^s_{x,y}(\cdot, \sigma_1(\cdot)) - D_{v^u_{2x}}\phi^s_{x,y}(\cdot, \sigma_2(\cdot))\|.
\end{aligned}
$$

However, if $0 < \varepsilon \le 1$, using Proposition 3.1(2), (3), and property (AM3), we get

$$
\begin{aligned}
&\|D_{v^u_{2x}}\phi^s_{x,y}(\cdot, \sigma_1(\cdot)) - D_{v^u_{2x}}\phi^s_{x,y}(\cdot, \sigma_2(\cdot))\| \\
&\quad= \left\| D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1 \end{bmatrix} - D_{(v^u_{2x},\sigma_2(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_2 \end{bmatrix}\right\|
\end{aligned}
$$

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$$
\begin{aligned}
&\leq \left\| D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1\end{bmatrix} - D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_2\end{bmatrix}\right\| \\
&\quad + \left\| D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_2\end{bmatrix} - D_{(v^u_{2x},\sigma_2(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_2\end{bmatrix}\right\| \\
&\leq \varepsilon\varepsilon(x)\|D\sigma_1 - D\sigma_2\|_{C^0} + 4\varepsilon\varepsilon(x)\|\sigma_1-\sigma_2\|^{\delta/2}_{C^0}.
\end{aligned}
$$

Consequently, we infer that the last term satisfies

$$
\begin{aligned}
&\|D_{v^u_{2x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_2}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \leq (\|Df|_{E^s(x)}\| + 2\varepsilon\varepsilon(x))\|D\sigma_1 - D\sigma_2\|_{C^0} + 4\varepsilon\varepsilon(x)\|\sigma_1-\sigma_2\|^{\delta/2}_{C^0}. \qquad (4.6)
\end{aligned}
$$

The next thing to do is to estimate the second term:

$$
\begin{aligned}
&\|D_{v^u_{1x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}}L^{\sigma_1}_{x,y}\cdot D_{v^u_y}(G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \leq \|D_{v^u_{1x}}L^{\sigma_1}_{x,y} - D_{v^u_{2x}}L^{\sigma_1}_{x,y}\| \\
&\quad \leq \|D^{ss}_{x,y}\|\cdot\|D_{v^u_{1x}}\sigma_1 - D_{v^u_{2x}}\sigma_1\| + \|D_{v^u_{1x}}\phi^s_{x,y}(\cdot,\sigma_1(\cdot)) - D_{v^u_{2x}}\phi^s_{x,y}(\cdot,\sigma_1(\cdot))\|.
\end{aligned}
$$

Similarly, if $0 < \varepsilon \leq 1$, using Proposition 3.1(2), (3), and property (AM3), we have

$$
\begin{aligned}
&\|D_{v^u_{1x}}\phi^s_{x,y}(\cdot,\sigma_1(\cdot)) - D_{v^u_{2x}}\phi^s_{x,y}(\cdot,\sigma_1(\cdot))\| \\
&\quad = \left\| D_{(v^u_{1x},\sigma_1(v^u_{1x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{1x}}\sigma_1\end{bmatrix} - D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1\end{bmatrix}\right\| \\
&\quad \leq \left\| D_{(v^u_{1x},\sigma_1(v^u_{1x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{1x}}\sigma_1\end{bmatrix} - D_{(v^u_{1x},\sigma_1(v^u_{1x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1\end{bmatrix}\right\| \\
&\qquad + \left\| D_{(v^u_{1x},\sigma_1(v^u_{1x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1\end{bmatrix} - D_{(v^u_{2x},\sigma_1(v^u_{2x}))}\phi^s_{x,y}\begin{bmatrix} I \\ D_{v^u_{2x}}\sigma_1\end{bmatrix}\right\| \\
&\quad \leq \frac{\varepsilon\varepsilon(x)}{2}\|v^u_{1x} - v^u_{2x}\|^{\delta/2} + 2\varepsilon\varepsilon(x)\|v^u_{1x} - v^u_{2x}\|^{\delta/2} \\
&\quad = \frac{5\varepsilon\varepsilon(x)}{2}\|v^u_{1x} - v^u_{2x}\|^{\delta/2}.
\end{aligned}
$$

Due to $G^{\sigma_i}_{x,y}(\cdot) = D^{uu}_{x,y}(\cdot) + D^{us}_{x,y}\sigma_i(\cdot) + \phi^u_{x,y}(\cdot,\sigma_i(\cdot))$, we obtain

$$
\begin{aligned}
\|v^u_{1x} - v^u_{2x}\| &= \|(G^{\sigma_1}_{x,y})^{-1}(v^u_y) - (G^{\sigma_2}_{x,y})^{-1}(v^u_y)\| \\
&= \|(G^{\sigma_1}_{x,y})^{-1}\circ(G^{\sigma_2}_{x,y}\circ(G^{\sigma_2}_{x,y})^{-1}(v^u_y)) - (G^{\sigma_1}_{x,y})^{-1}\circ(G^{\sigma_1}_{x,y}\circ(G^{\sigma_2}_{x,y})^{-1}(v^u_y))\| \\
&\leq \|D(G^{\sigma_1}_{x,y})^{-1}\|_{C^0}\|G^{\sigma_2}_{x,y}(v^u_{2x}) - G^{\sigma_1}_{x,y}(v^u_{2x})\| \\
&\leq \|D^{us}_{x,y}\|\ \|\sigma_2(v^u_{2x}) - \sigma_1(v^u_{2x})\| + \mathrm{Lip}(\phi_{x,y})\|\sigma_2(v^u_{2x}) - \sigma_1(v^u_{2x})\| \\
&\leq 2\varepsilon\varepsilon(x)\|\sigma_2-\sigma_1\|_{C^0} \\
&\leq \|\sigma_2-\sigma_1\|_{C^0}. \qquad (4.7)
\end{aligned}
$$

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Hence, the second term is estimated as

$$\begin{aligned}
&\|D_{v^u_{1x}} L^{\sigma_1}_{x,y} \circ D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1} - D_{v^u_{2x}} L^{\sigma_1}_{x,y} \circ D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \le \|D^{ss}_{x,y}\| \cdot \frac{1}{2}\|v^u_{1x} - v^u_{2x}\|^{\delta/2} + \frac{5\varepsilon\varepsilon(x)}{2}\|v^u_{1x} - v^u_{2x}\|^{\delta/2} \\
&\quad \le (\tfrac{1}{2}\|Df|_{E^s(x)}\| + 3\varepsilon\varepsilon(x))\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0}.
\end{aligned} \tag{4.8}$$

Another step in the proof of part (2) is to establish an estimate of the first term. Since $\|D_{v^u_{1x}} L^{\sigma_1}_{x,y}\| < 1$ and $\|D_{v^u_y}(G^{\sigma_i}_{x,y})^{-1}\| < 1$, $i = 1, 2$, we have

$$\begin{aligned}
&\|D_{v^u_{1x}} L^{\sigma_1}_{x,y} \cdot D_{v^u_y} (G^{\sigma_1}_{x,y})^{-1} - D_{v^u_{1x}} L^{\sigma_1}_{x,y} \cdot D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \le \|D_{v^u_y} (G^{\sigma_1}_{x,y})^{-1} - D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad = \|D_{v^u_y} (G^{\sigma_1}_{x,y})^{-1}(D_{v^u_{2x}} G^{\sigma_2}_{x,y} - D_{v^u_{1x}} G^{\sigma_1}_{x,y}) D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \le \|D_{v^u_{2x}} G^{\sigma_2}_{x,y} - D_{v^u_{1x}} G^{\sigma_2}_{x,y}\| + \|D_{v^u_{1x}} G^{\sigma_2}_{x,y} - D_{v^u_{1x}} G^{\sigma_1}_{x,y}\|.
\end{aligned}$$

From the proof in property (AM3) of Proposition 4.2 and equation (4.7) above, we obtain that

$$\|D_{v^u_{2x}} G^{\sigma_2}_{x,y} - D_{v^u_{1x}} G^{\sigma_2}_{x,y}\| \le 5\varepsilon\varepsilon(x)\|v^u_{2x} - v^u_{1x}\|^{\delta/2} \le 5\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0}.$$

At the same time, if $0 < \varepsilon \le 1$, using Proposition 3.1(2), (3), and property (AM3), we get

$$\begin{aligned}
&\|D_{v^u_{1x}} \phi^u_{x,y}(\cdot, \sigma_2(\cdot)) - D_{v^u_{1x}} \phi^u_{x,y}(\cdot, \sigma_1(\cdot))\| \\
&\quad = \left\| D_{(v^u_{1x}, \sigma_2(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_2 \end{bmatrix} - D_{(v^u_{1x}, \sigma_1(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_1 \end{bmatrix} \right\| \\
&\quad \le \left\| D_{(v^u_{1x}, \sigma_2(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_2 \end{bmatrix} - D_{(v^u_{1x}, \sigma_2(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_1 \end{bmatrix} \right\| \\
&\qquad + \left\| D_{(v^u_{1x}, \sigma_2(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_1 \end{bmatrix} - D_{(v^u_{1x}, \sigma_1(v^u_{1x}))} \phi^u_{x,y} \begin{bmatrix} I \\ D_{v^u_{1x}} \sigma_1 \end{bmatrix} \right\| \\
&\quad \le \varepsilon\varepsilon(x)\|D\sigma_2 - D\sigma_1\|_{C^0} + 2\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0}.
\end{aligned}$$

Hence, by using property (AM3) and equation (4.7) again, we get for this term that

$$\begin{aligned}
&\|D_{v^u_{1x}} L^{\sigma_1}_{x,y} \cdot D_{v^u_y} (G^{\sigma_1}_{x,y})^{-1} - D_{v^u_{1x}} L^{\sigma_1}_{x,y} \cdot D_{v^u_y} (G^{\sigma_2}_{x,y})^{-1}\| \\
&\quad \le \|D_{v^u_{2x}} G^{\sigma_2}_{x,y} - D_{v^u_{1x}} G^{\sigma_2}_{x,y}\| + \|D^{us}_{x,y} D_{v^u_{1x}} \sigma_2 - D^{us}_{x,y} D_{v^u_{1x}} \sigma_2\| \\
&\qquad + \|D_{v^u_{1x}} \phi^u_{x,y}(\cdot, \sigma_2(\cdot)) - D_{v^u_{1x}} \phi^u_{x,y}(\cdot, \sigma_1(\cdot))\| \\
&\quad \le 5\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2} + \tfrac{1}{2}\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0} \\
&\qquad + \varepsilon\varepsilon(x)\|D\sigma_2 - D\sigma_1\|_{C^0} + 2\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0} \\
&\quad = \varepsilon\varepsilon(x)\|D\sigma_2 - D\sigma_1\|_{C^0} + \frac{15}{2}\varepsilon\varepsilon(x)\|\sigma_2 - \sigma_1\|^{\delta/2}_{C^0}.
\end{aligned} \tag{4.9}$$

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Finally, plugging all estimates in equations (4.6), (4.8), and (4.9) together, it holds that

$$\begin{aligned}
&\|D\tilde{\sigma}_1 - D\tilde{\sigma}_2\|_{C^0} \\
&\quad\le \varepsilon\varepsilon(x)\|D\sigma_2 - D\sigma_1\|_{C^0} + \frac{15}{2}\varepsilon\varepsilon(x)\|\sigma_2-\sigma_1\|_{C^0}^{\delta/2} \\
&\qquad + \big(\tfrac{1}{2}\|Df|_{E^s(x)}\| + 3\varepsilon\varepsilon(x)\big)\|\sigma_2-\sigma_1\|_{C^0}^{\delta/2} \\
&\qquad + (\|Df|_{E^s(x)}\| + 2\varepsilon\varepsilon(x))\|D\sigma_1 - D\sigma_2\|_{C^0} + 4\varepsilon\varepsilon(x)\|\sigma_1-\sigma_2\|_{C^0}^{\delta/2} \\
&\quad\le (\|Df|_{E^s(x)}\| + 3\varepsilon\varepsilon(x))\|D\sigma_1 - D\sigma_2\|_{C^0} + \frac{1}{2}(\|Df|_{E^s(x)}\| + 29\varepsilon\varepsilon(x))\|\sigma_2-\sigma_1\|_{C^0}^{\delta/2} \\
&\quad\le \|Df|_{E^s(x)}\|^{1/2}(\|D\sigma_1 - D\sigma_2\|_{C^0} + \|\sigma_2-\sigma_1\|_{C^0}^{\delta/2})
\end{aligned}$$

if $0 < \varepsilon \le \min\{1, aC/29\}$, where a is defined in equation (4.5). With the estimates of part (1), we conclude finally that

$$\|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{C^1} \le \|Df|_{E^s(x)}\|^{1/2}(\|\sigma_1-\sigma_2\|_{C^1} + \|\sigma_1-\sigma_2\|_{C^0}^{\delta/2}),$$

proving part (2). □

5. *Shadowing lemma and an infinite-to-one coding for f*

Since O is second countable, we would like to replace O by a countable subset $\mathscr{A}$ which will be used to construct the set of vertices of a directed graph related to the dynamics of f.

PROPOSITION 5.1. *For all $\varepsilon > 0$ sufficiently small, there exists a countable subset $\mathscr{A} \subset O$ with the following properties.*

(1) *Discreteness: for all $r > 0$, the set $\{x \in \mathscr{A} : d(x, \partial O) \ge r\}$ is finite.*

(2) *Sufficiency: for every $x \in O$, there is a sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathscr{A}$ such that for every $n \in \mathbb{Z}$:*
   (a) $e^{-\beta\varepsilon} \le (\varepsilon(f^n(x))/\varepsilon(x_n)) \le e^{\beta\varepsilon}$ *and* $e^{-\gamma\varepsilon} \le (Q(f^n(x))/Q(x_n)) \le e^{\gamma\varepsilon}$;
   (b) $x_n \to x_{n+1}$.

Proof. Fix a countable subset $\mathscr{A}$ of O such that

$$O = \bigcup_{z\in\mathscr{A}} B(z, r(z)),$$

where $r(z) = \varepsilon \min\{\delta^u(\varepsilon\cdot\varepsilon(z)), \delta^s(\varepsilon\cdot\varepsilon(z))\}\cdot Q(z)^2 \cdot \min\{Q(f(z)), Q(f^{-1}(z))\}$.

(1) For all $r > 0$, since $O(r) = \{x \in O : d(x, \partial O) \ge r\}$ is compact, if necessary, we can remove some elements of $\mathscr{A}$ in $O(r)$ to make the set $\{x \in \mathscr{A} : d(x, \partial O) \ge r\}$ finite and maintain the above denseness assumption.

(2) As mentioned above, we can fix a countable subset $\mathscr{A}$ of O such that $O = \bigcup_{z\in\mathscr{A}} B(z, r(z))$ and satisfying discreteness. For every $x \in O$ and any $n \in \mathbb{Z}$, since $f^n(x) \in O$, there exist $\{x_n\}_{n\in\mathbb{Z}} \in \mathscr{A}^{\mathbb{Z}}$ such that $d(f^n(x), x_n) \le r(x_n)$ for all $n \in \mathbb{Z}$. For $\varepsilon > 0$ sufficiently small, since $r(x_n) \le (\varepsilon/2)Q(x_n) \le (\varepsilon/2)d(x_n, \partial O)$, it follows that

$$\begin{aligned}
d(f^n(x), \partial O) &\le d(f^n(x), x_n) + d(x_n, \partial O) \\
&\le e^{\varepsilon} d(x_n, \partial O),
\end{aligned}$$

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and hence

$$\varepsilon(f^n(x)) \le e^{\beta\varepsilon}\varepsilon(x_n) \quad \text{and} \quad Q(f^n(x)) \le e^{\gamma\varepsilon}Q(x_n).$$

Analogously, from $d(f^n(x), \partial O) \ge d(x_n, \partial O) - d(f^n(x), x_n) \ge e^{-\varepsilon}d(x_n, \partial O)$, we have

$$\varepsilon(f^n(x)) \ge e^{-\beta\varepsilon}\varepsilon(x_n), \;\; Q(f^n(x)) \ge e^{-\gamma\varepsilon}Q(x_n).$$

Therefore, we obtain $e^{-\beta\varepsilon} \le \varepsilon(f^n(x))/\varepsilon(x_n) \le e^{\beta\varepsilon}$ and $e^{-\gamma\varepsilon} \le Q(f^n(x))/Q(x_n) \le e^{\gamma\varepsilon}$ if $\varepsilon > 0$ is small enough. It remains to prove part (b).

First we claim that

$$\varepsilon(x_{n+1}) \le A_u^\beta e^{2\beta\varepsilon}\varepsilon(x_n), \tag{5.1}$$

where $A_u := [(1/C^u)((C^f)^{\kappa^u} - 1) + 1]^{1/\gamma^u} > 1$.

In practice, by using $e^{-\varepsilon} \le (d(f^n(x), \partial O)/d(x_n, \partial O)) \le e^{\varepsilon}$, equation (5.1) holds in the case $\varepsilon(f^n(x)) = \varepsilon r_0^\beta$ since

$$\begin{aligned}\varepsilon(x_{n+1}) &\le \varepsilon r_0^\beta \\ &\le \varepsilon d(f^n(x), \partial O)^\beta \\ &\le \varepsilon e^{\beta\varepsilon} d(x_n, \partial O)^\beta,\end{aligned}$$

which implies that $\varepsilon(x_{n+1}) \le A_u^\beta e^{2\beta\varepsilon}\varepsilon(x_n)$ when $0 < \varepsilon \le A_u^\beta$.

However, if $\varepsilon(f^n(x)) = \varepsilon d(f^n(x), \partial O)^\beta$, from Assumption US(ii) and $e^{-\varepsilon} \le (d(f^n(x), \partial O)/d(x_n, \partial O)) \le e^{\varepsilon}$ again, we obtain for all $n \in \mathbb{Z}$,

$$\begin{aligned}d(x_{n+1}, \partial O) &\le e^{\varepsilon} d(f^{n+1}(x), \partial O) \\ &\le A_u e^{\varepsilon} d(f^n(x), \partial O) \\ &\le A_u e^{2\varepsilon} d(x_n, \partial O).\end{aligned}$$

Then $\varepsilon(x_{n+1}) \le A_u^\beta e^{2\beta\varepsilon}\varepsilon(x_n)$, which completes the proof of the claim.

Denote $A := (1/C)((C^f)^\kappa - 1) + 1 > 1$, and according to property (a) and equation (3.5) of Lemma 3.2, we get for all $n \in \mathbb{Z}$,

$$\begin{aligned}Q(x_{n+1}) &\le e^{\gamma\varepsilon} Q(f^{n+1}(x)) \\ &\le A e^{\gamma\varepsilon} Q(f^n(x)) \\ &\le A e^{2\gamma\varepsilon} Q(x_n),\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}Q(f(x_{n+1})) &\le A Q(x_{n+1}) \\ &\le A^2 e^{2\gamma\varepsilon} Q(x_n) \\ &\le A^3 e^{2\gamma\varepsilon} Q(f(x_n)).\end{aligned} \tag{5.3}$$

Combining equations (5.1), (5.2), and (5.3), we have

$$\begin{aligned}d(f^{n+1}(x), x_{n+1}) &\le r(x_{n+1}) \\ &\le \varepsilon\delta^u(\varepsilon\varepsilon(x_{n+1}))Q(x_{n+1})^2 Q(f(x_{n+1}))\end{aligned}$$

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$$\leq \varepsilon\delta^u(\varepsilon A_u^\beta e^{2\beta\varepsilon}\varepsilon(x_n)) \cdot A^2 e^{4\gamma\varepsilon} Q(x_n)^2 \cdot A^3 e^{2\gamma\varepsilon} Q(f(x_n))$$
$$= A^5 \cdot \varepsilon e^{6\gamma\varepsilon}\delta^u(A_u^\beta \cdot \varepsilon e^{2\beta\varepsilon}\varepsilon(x_n)) Q(x_n)^2 Q(f(x_n)).$$

Now we can verify the half of $x_n \to x_{n+1}$ for all $n \in \mathbb{Z}$ from the local monotonicity of $\delta^u(\cdot)$ if $\varepsilon > 0$ is small enough. In fact,

$$\begin{aligned} d(f(x_n), x_{n+1}) &\leq d(f(x_n), f^{n+1}(x)) + d(f^{n+1}(x), x_{n+1}) \\ &\leq C^f d(x_n, f^n(x)) + d(f^{n+1}(x), x_{n+1}) \\ &\leq \varepsilon[C^f \delta^u(\varepsilon\varepsilon(x_n)) + A^5 \cdot e^{6\gamma\varepsilon}\delta^u(A_u^\beta \cdot \varepsilon e^{2\beta\varepsilon}\varepsilon(x_n))] Q(x_n)^2 Q(f(x_n)) \\ &\leq \delta^u(\varepsilon(x_n)) Q(x_n)^2 Q(f(x_n)), \end{aligned}$$

where the last inequality holds if $0 < \varepsilon \leq \min\{1, 1/A_u^\beta e^{2\beta}, 1/(C^f + A^5 e^{6\gamma})\}$.

Similarly, we can perform as above to conclude that

$$d(f^{-1}(x_{n+1}), x_n) \leq \delta^s(\varepsilon(x_{n+1})) Q(x_{n+1})^2 Q(f^{-1}(x_{n+1}))$$

if $\varepsilon > 0$ is small enough.

Therefore, $x_n \to x_{n+1}$ for all $n \in \mathbb{Z}$ when $\varepsilon > 0$ is small enough. The proof is completed. □

Let $\mathscr{G} = (V, E)$ be the *directed graph* with *vertex set* $V = \mathscr{A}$ and *edge set* $E = \{x \to y : x, y \in V\}$, with $x \to y$ defined in Definition 4.3 for $x, y \in V$. This is a countable directed graph. Every vertex of $\mathscr{G}$ has finite degree if $\varepsilon > 0$ is small enough. In fact, for every $x \in V$, if $y \in V$ satisfies $x \to y$, we have

$$\begin{aligned} d(y, \partial O) &\geq d(f(x), \partial O) - d(y, f(x)) \\ &\geq d(f(x), \partial O) - \frac{\varepsilon}{2} d(f(x), \partial O) \\ &\geq e^{-\varepsilon} d(f(x), \partial O) \\ &\geq A_u e^{-\varepsilon} d(x, \partial O), \end{aligned}$$

and from the discreteness of V (Proposition 5.1(1)), there are only finitely many $y \in V$ such that $x \to y$.

We define $\Sigma := \Sigma(\mathscr{G})$ to be the set of $\mathbb{Z}$-*indexed paths* on $\mathscr{G}$:

$$\Sigma := \{\{x_n\}_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : x_n \to x_{n+1} \text{ for all } n \in \mathbb{Z}\}.$$

An element of Σ is denoted by $\underline{x} := \{x_n\}_{n\in\mathbb{Z}}$. We equip Σ with the metric $d(\underline{x}, \underline{y}) = \exp[-\min\{|n| : n \in \mathbb{Z} \text{ such that } x_n \neq y_n\}]$ and the action of the left shift map $\sigma : \Sigma \to \Sigma$, $\sigma : \{x_n\}_{n\in\mathbb{Z}} \mapsto \{x_{n+1}\}_{n\in\mathbb{Z}}$ as before. Therefore, we get a TMS (Σ, σ), it is locally compact since every vertex of $\mathscr{G}$ has finite degree.

Let $\Sigma^\#$ denote the *recurrent set* of Σ, defined by

$$\begin{aligned} \Sigma^\# := \{\underline{x} \in \Sigma : &\text{ there exists } x, y \in V, \text{there exists } n_k, m_k \uparrow \infty \text{ such that } x_{n_k} = x \\ &\text{and } x_{-m_k} = y \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

By the Poincaré recurrence theorem, every σ-invariant probability measure gives $\Sigma^\#$ full measure.

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Definition 5.1. An *ε-recurrent-pointwise-pseudo-orbit* (or *ε-rppo* for short) in O is a sequence $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$.

Along an ε-rppo $\{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$, we can construct the *local unstable* and *stable manifolds* at x_0 since the pointwise dominated splitting holds for the property of recurrence. The ideas are simple. Let $\{V^{(n)}\}_{n=0}^{\infty}$ denote a sequence of t-manifolds at x_0. We say that $V^{(n)}$ converges to a t-manifold V at x_0 if

$$d_{C^0}(V^{(n)}, V) = \|\sigma^{(n)} - \sigma\|_{C^0} \underset{n\to\infty}{\longrightarrow} 0,$$

where $\sigma^{(n)}$ and σ are the representing functions of $V^{(n)}$ and V, respectively.

Call a sequence $\underline{x}^+ = \{x_n\}_{n\geq 0}$ a *positive ε-rppo* if $x_n \to x_{n+1}$ for all $n \geq 0$ and $\{x_n\}_{n\geq 0}$ has constant subsequence. Similarly, a *negative ε-rppo* is a sequence $\underline{x}^- = \{x_{-n}\}_{n\geq 0}$ such that $x_{-n} \to x_{-n+1}$ for all $n \geq 1$ and $\{x_{-n}\}_{n\geq 0}$ has constant subsequence.

For any $n \geq 1$, let V^u_{-n} be a u-admissible manifold at x_{-n}. The unstable graph transform along $x_{-n} \to x_{-n+1}$ maps V^u_{-n} to a u-admissible manifold at x_{-n+1}, which we denote by $\mathscr{F}^u_{-n,-n+1}[V^u_{-n}]$. Another application of the unstable graph transform, this time relative to $x_{-n+1} \to x_{-n+2}$, maps $\mathscr{F}^u_{-n,-n+1}[V_{-n}] \in \mathscr{V}^u_{-n+1}$ to a u-admissible manifold at x_{-n+2}, which we denote by $\mathscr{F}^u_{-n+1,-n+2} \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}]$. Continuing this way, we eventually reach a u-admissible manifold at x_0 which we denote by $\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n+1,-n+2} \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}]$. If this limit converges to a u-admissible manifold as $n \to \infty$, we call it a *local unstable manifold* of a negative ε-rppo $\underline{x}^-$. Similarly, any s-admissible manifold V^s_n at x_n is mapped by n applications of a stable graph transform $\mathscr{F}^s$ to an s-admissible manifold $\mathscr{F}^s_{0,1} \circ \cdots \circ \mathscr{F}^s_{n-2,n-1} \circ \mathscr{F}^s_{n-1,n}[V^s_n]$ at x_0. We call it a *local stable manifold* of a positive ε-rppo $\underline{x}^+$ if this limit converges to an s-admissible manifold as $n \to \infty$. Set

$$V^u(\underline{x}^-) := \lim_{n\to\infty} (\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1})[V^u_{-n}]$$

and

$$V^s(\underline{x}^+) := \lim_{n\to\infty} (\mathscr{F}^s_{0,1} \circ \cdots \circ \mathscr{F}^s_{n-1,n})[V_n].$$

PROPOSITION 5.2. *The following holds for all $\varepsilon > 0$ small enough. Suppose $\{x_n\}_{n\in\mathbb{Z}}$ is an ε-rppo, and choose arbitrary u-admissible manifolds V^u_{-n} at x_{-n} and arbitrary s-admissible manifolds V^s_n at x_n for $n \geq 0$. Then:*

(1) *$V^u(\underline{x}^-)$ is a u-admissible manifold at x_0, and $V^s(\underline{x}^+)$ is an s-admissible manifold at x_0;*

(2) *$f^{-1}[V^u(\{x_{-n}\}_{n\geq 0})] \subset V^u(\{x_{-n-1}\}_{n\geq 0})$ and $f[V^s(\{x_n\}_{n\geq 0})] \subset V^s(\{x_{n+1}\}_{n\geq 0})$;*

(3) *if $y, z \in V^u(\underline{x}^-)$, then for all $k \geq 0$,*

$$d(f^{-k}(y), f^{-k}(z)) \leq 8Q(x_0) \prod_{i=0}^{k-1} \|Df^{-1}|_{E^u(x_{-i})}\|^{1/2}.$$

Similarly, if $y, z \in V^s(\underline{x}^+)$, for all $k \geq 0$, we have

$$d(f^k(y), f^k(z)) \leq 8Q(x_0) \prod_{i=0}^{k-1} \|Df|_{E^s(x_i)}\|^{1/2};$$

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(4) *we have the following respective shadowing property along the negative and positive direction:*

$$V^u(\underline{x}^-) = \{x \in \exp_{x_0}(S_{x_0}) \text{ for all } k \geq 0,\ f^{-k}(x) \in \exp_{x_{-k}}(S_{x_{-k}})\},$$
$$V^s(\underline{x}^+) = \{x \in \exp_{x_0}(S_{x_0}) \text{ for all } k \geq 0,\ f^{k}(x) \in \exp_{x_{k}}(S_{x_{k}})\};$$

(5) *if* $\{x_n\}_{n\in\mathbb{Z}}, \{y_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ *satisfy* $x_i = y_i$ *for* $i = -N, \ldots, N$, *then*

$$d_{C^1}(V^u(\underline{x}^-), V^u(\underline{y}^-)) \leq 2(N+1) \prod_{i=1}^{N} \|Df|_{E^s(x_{-i})}\|^{\delta/4},$$

$$d_{C^1}(V^s(\underline{x}^+), V^s(\underline{y}^+)) \leq 2(N+1) \prod_{i=1}^{N} \|Df^{-1}|_{E^u(x_{i})}\|^{\delta/4}.$$

Remark 5.1. Unlike in [22, Proposition 4.15], since the recurrence might be very slow, the convergence of $\prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|$ or $\prod_{i=1}^N \|Df^{-1}|_{E^u(x_i)}\|$ to zero might be very slow and hence the continuity of maps $\{x_n\}_{n\in\mathbb{Z}} \mapsto V^u(\underline{x}^-), V^s(\underline{x}^+)$ are not guaranteed by Assumptions US and R.

Proof. The proof is quite similar to the proof of [22, Proposition 4.15]. We treat the case of u-admissible manifolds and s-admissible manifolds in the proof of items (1,2,4,5) and (3), respectively. By suitable modification to the proof, all the other cases are similar in its opposite direction.

(1) We divide our proof into three steps. First, we verify that if the limit exists, then it is independent of the choice of V^u_{-n}. Since $\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}]$ is a u-admissible manifold at x_0, by using Proposition 4.3, for any other choice of u-admissible manifolds W^u_{-n} at x_{-n}, we have

$$\begin{aligned} &d_{C^0}(\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}], \mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[W^u_{-n}]) \\ &\quad \leq \prod_{i=1}^{n} \|Df|_{E^s(x_{-i})}\|^{1/2} d_{C^0}(V^u_{-n}, W^u_{-n}) \\ &\quad \leq \prod_{i=1}^{n} \|Df|_{E^s(x_{-i})}\|^{1/2} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

since $d_{C^0}(V^u_{-n}, W^u_{-n}) \leq 2Q(x_{-n}) < 1$ if $\varepsilon > 0$ is small enough and $\{x_{-n}\}_{n\geq 0}$ have constant subsequence. Thus, if the limit exists, then it is independent of V^u_{-n}.

The next thing to do is to prove that the limit exists. For every $m > n$, we have

$$\begin{aligned} &d_{C^0}(\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}], \mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-m,-m+1}[V^u_{-m}]) \\ &\quad \leq \prod_{i=1}^{n} \|Df|_{E^s(x_{-i})}\|^{1/2} d_{C^0}(V^u_{-n}, \mathscr{F}^u_{-n-1,-n} \circ \cdots \circ \mathscr{F}^u_{-m,-m+1}[V^u_{-m}]) \\ &\quad \leq \prod_{i=1}^{n} \|Df|_{E^s(x_{-i})}\|^{1/2} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence, $\{\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}]\}_{n\geq 1}$ is a Cauchy sequence and therefore converges.

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Finally, we have to show the admissibility of the limit. Since $\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}] \underset{n\to\infty}{\longrightarrow} V^u[\{x_{-n}\}_{n\geq 0}]$, we have $\sigma^{(n)} \underset{n\to\infty}{\longrightarrow} \sigma$ uniformly on $B^u_{x_0}(Q(x_0))$, where σ and $\sigma^{(n)}$ represent $V^u[\{x_{-n}\}_{n\geq 0}]$ and $\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1}[V^u_{-n}]$, respectively.

Due to property (AM3), that is, $\mathrm{Hol}_{\delta/2}(D\sigma^{(n)}) \leq \frac{1}{2}$ and $\|D\sigma^{(n)}\|_{C^0} \leq \frac{1}{2}$, by the Arzelà–Ascoli theorem, there exists $n_k \uparrow \infty$ such that $D\sigma^{(n_k)} \underset{k\to\infty}{\longrightarrow} \omega$ uniformly where $\|\omega\|_{C^{\delta/2}} \leq \frac{1}{2}$. Thus, for any $v \in B^u_{x_0}(Q(x_0))$,

$$\sigma^{(n_k)}(v) = \sigma^{(n_k)}(0) + \int_0^1 D_{\lambda v}\sigma^{(n_k)} \cdot v d\lambda \underset{k\to\infty}{\longrightarrow} \sigma(0) + \int_0^1 \omega(\lambda v) \cdot v d\lambda$$

and hence σ is differentiable, and $D\sigma = \omega$. We also see that $\{D\sigma^{(n)}\}$ can only have one limit point. Consequently, $D\sigma^{(n)} \underset{n\to\infty}{\longrightarrow} D\sigma$ uniformly. It follows that $\|\sigma(0)\| \leq 10^{-3}Q(x_0)$, $\|D\sigma\|_{C^0} \leq \sqrt{\varepsilon}$, and $\|D\sigma\|_{C^{\delta/2}} \leq \frac{1}{2}$; hence the u-admissibility of $V^u(\underline{x}^-)$.

(2) By the continuity of $\mathscr{F}^u_{-1,0}$ and the definition of $\mathscr{F}^u_{-1,0}$, we have

$$\begin{aligned} V^u(\{x_{-n}\}_{n\geq 0}) &= \lim_{n\to\infty} (\mathscr{F}^u_{-1,0} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1})[V^u_{-n}] \\ &= \mathscr{F}^u_{-1,0}\Big[\lim_{n\to\infty} (\mathscr{F}^u_{-2,-1} \circ \cdots \circ \mathscr{F}^u_{-n,-n+1})[V^u_{-n}]\Big] \\ &= \mathscr{F}^u_{-1,0}[V^u(\{x_{-n-1}\}_{n\geq 0})] \\ &\subset f[V^u(\{x_{-n-1}\}_{n\geq 0})]. \end{aligned}$$

(3) We write

$$V^s(\{x_n\}_{n\geq 0}) = \exp_{x_0}\{(\sigma^s_0(v^s_0), v^s_0) : \|v^s_0\| \leq Q(x_0)\}$$

and

$$V^s(\{x_{n+k}\}_{n\geq 0}) = \exp_{x_k}\{(\sigma^s_k(v^s_k), v^s_k) : \|v^s_k\| \leq Q(x_k)\} \quad \text{for all } k \geq 0.$$

Admissibility implies that $\|\sigma^s_k(0)\| \leq 10^{-3}Q(x_k)$ and $Lip(\sigma^s_k) \leq \sqrt{\varepsilon}$ for all $k \geq 0$.

By part (2), $f^k[V^s(\{x_n\}_{n\geq 0})] \subset V^s(\{x_{n+k}\}_{n\geq 0}) \subset \exp_{x_k}(S_{x_k})$ for all $k \geq 0$. Therefore, for any $y, z \in V^s(\{x_n\}_{n\geq 0})$, one can write $f^k(y) = \exp_{x_k}(\sigma^s_k(v^s_k), v^s_k)$ and $f^k(z) = \exp_{x_k}(\sigma^s_k(w^s_k), w^s_k)$.

For all $k \geq 0$, since the exponential map has Lipschitz constant less than two, we get

$$\begin{aligned} d(f^k(y), f^k(z)) &\leq 2(\|v^s_k - w^s_k\| + \mathrm{Lip}(\sigma^s_k)\|v^s_k - w^s_k\|) \\ &\leq 4\|v^s_k - w^s_k\| \end{aligned}$$

if $0 < \varepsilon \leq 1$.

Notice that

$$\begin{aligned} (\sigma^s_k(v^s_k), v^s_k) &= F_{x_{k-1},x_k}(\sigma^s_{k-1}(v^s_{k-1}), v^s_{k-1}) \quad \text{and} \quad (\sigma^s_k(w^s_k), w^s_k) \\ &= F_{x_{k-1},x_k}(\sigma^s_{k-1}(w^s_{k-1}), w^s_{k-1}), \end{aligned}$$

or in the s-direction,

$$v^s_k = D^{ss}_{x_{k-1},x_k}(v^s_{k-1}) + D^{su}_{x_{k-1},x_k} \circ \sigma^s_{k-1}(v^s_{k-1}) + \phi^s_{x_{k-1},x_k}(\sigma^s_{k-1}(v^s_{k-1}), v^s_{k-1})$$

and

$$w_k^s = D^{ss}_{x_{k-1},x_k}(w^s_{k-1}) + D^{su}_{x_{k-1},x_k} \circ \sigma^s_{k-1}(w^s_{k-1}) + \phi^s_{x_{k-1},x_k}(\sigma^s_{k-1}(w^s_{k-1}), w^s_{k-1}).$$

So we have

$$\begin{aligned}\tfrac{1}{4} d(f^k(y), f^k(z)) &\leq \|v_k^s - w_k^s\| \\ &\leq (\|D^{ss}_{x_{k-1},x_k}\| + \|D^{su}_{x_{k-1},x_k}\| + 2\,\mathrm{Lip}(\phi^s_{x_{k-1},x_k}))\|v^s_{k-1} - w^s_{k-1}\| \\ &\leq \|Df|_{E^s(x_{k-1})}\|^{1/2}\|v^s_{k-1} - w^s_{k-1}\|\end{aligned}$$

as at the end of the proof of Proposition 4.3(1). Thus,

$$\begin{aligned}d(f^k(y), f^k(z)) &\leq 4\prod_{i=0}^{k-1} \|Df|_{E^s(x_i)}\|^{1/2}\|v_0^s - w_0^s\| \\ &\leq 8Q(x_0)\prod_{i=0}^{k-1} \|Df|_{E^s(x_i)}\|^{1/2}.\end{aligned}$$

(4) The inclusion $\subseteq$ is trivial. In fact, notice that $V^u(\underline{x}^-)$ is a u-admissible manifold at x_0, it is contained in $\exp_{x_0}(S_{x_0})$, that is, for every $x \in V^u(\underline{x}^-)$, $x \in \exp_{x_0}(S_{x_0})$. By part (2), for every $k \geq 0$,

$$f^{-k}(x) \in f^{-k}[V^u(\{x_{-n}\}_{n\geq 0})] \subseteq V^u(\{x_{-n-k}\}_{n\geq 0}) \subset \exp_{x_{-k}}(S_{x_{-k}})$$

because $V^u(\{x_{-n-k}\}_{n\geq 0})$ is a u-admissible manifold at x_{-k}. We have $\subseteq$.

Now we prove $\supseteq$. An outline of the proof of this inclusion is as follows. For every point x satisfying $f^{-k}(x) \in \exp_{x_{-k}}(S_{x_{-k}})$ for all $k \geq 0$, write $x = \exp_{x_0}(v_0^u, v_0^s)$. We show that $x \in V^u(\{x_{-n}\}_{n\geq 0})$ by proving that $v_0^s = \sigma^u(v_0^u)$, where σ^u is the function which represents $V^u(\{x_{-n}\}_{n\geq 0})$.

Introduce for this purpose the point $\bar{x} = \exp_{x_0}(\bar{v}_0^u, \bar{v}_0^s) \in V^u(\{x_{-n}\}_{n\geq 0})$, where $\bar{v}_0^u = v_0^u$ and $\bar{v}_0^s = \sigma^u(\bar{v}_0^u)$. Our task now is to prove $\bar{v}_0^s = v_0^s$.

For every $k \geq 0$, write

$$f^{-k}(x) = \exp_{x_{-k}}(v^u_{-k}, v^s_{-k}) \quad \text{and} \quad f^{-k}(\bar{x}) = \exp_{x_{-k}}(\bar{v}^u_{-k}, \bar{v}^s_{-k}) \in V^u(\{x_{-n-k}\}_{n\geq 0}),$$

where $\|v^u_{-k}\|, \|v^s_{-k}\|, \|\bar{v}^u_{-k}\|, \|\bar{v}^u_{-k}\| \leq Q(x_{-k})$ for all $k \geq 0$.

By Proposition 3.1 in its version for f^{-1}, for every $k \geq 0$, $F^{-1}_{x_{-k-1}x_{-k}} = \exp^{-1}_{x_{-k-1}} \circ f^{-1} \circ \exp_{x_{-k}}$ can be put in the form

$$\begin{aligned}&F^{-1}_{x_{-k-1},x_{-k}}(v^u_{-k}, v^s_{-k}) \\ &\quad = (\bar{D}^{uu}_{x_{-k},x_{-k-1}}(v^u_{-k}) + \bar{D}^{us}_{x_{-k},x_{-k-1}}(v^s_{-k}) + \bar{\phi}^u_{x_{-k},x_{-k-1}}(v^u_{-k}, v^s_{-k}), \bar{D}^{ss}_{x_{-k},x_{-k-1}}(v^s_{-k}) \\ &\qquad + \bar{D}^{su}_{x_{-k},x_{-k-1}}(v^u_{-k}) + \bar{\phi}^s_{x_{-k},x_{-k-1}}(v^u_{-k}, v^s_{-k}))\end{aligned}$$

under the basis (E^u, E^s) for all $v_{-k} = (v^u_{-k}, v^s_{-k}) \in S_{x_{-k}}$, where

$$\begin{aligned}\bar{D}^{\tau t}_{x_{-k},x_{-k-1}} = P^{\tau}_{x_{-k-1}} \circ D_{f^{-1}(x_{-k})} \exp^{-1}_{x_{-k-1}} \circ D_{x_{-k}} f^{-1}|_{E^t(x_{-k})} &: B^t_{x_{-k}}(Q(x_{-k})) \\ &\to E^{\tau}(x_{-k-1}), \tau, t = u, s.\end{aligned}$$

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Since $d(f^{-1}(x_{-k}), x_{-k-1}) < \delta^s(\varepsilon(x_{-k}))$, in the s-direction, we have

$$\|\bar{D}^{su}_{x_{-k},x_{-k-1}}\| \le \varepsilon\varepsilon(x_{-k}),$$

$$m(Df^{-1}|_{E^s(x_{-k})}) \le m(\bar{D}^{ss}_{x_{-k},x_{-k-1}}) + \varepsilon\varepsilon(x_{-k}).$$

While in the u-direction, we have

$$\|\bar{D}^{us}_{x_{-k},x_{-k-1}}\| \le \varepsilon\varepsilon(x_{-k}),$$

$$\|\bar{D}^{uu}_{x_{-k},x_{-k-1}}\| \le \|Df^{-1}|_{E^u(x_{-k})}\| + \varepsilon\varepsilon(x_{-k}),$$

and moreover,

$$\mathrm{Lip}(\bar{\phi}_{x_{-k},x_{-k-1}}) \le \varepsilon\varepsilon(x_{-k}),$$

where $\bar{\phi} = (\bar{\phi}^u, \bar{\phi}^s)$.

Let $\Delta v^u_{-k} := v^u_{-k} - \bar{v}^u_{-k}$ and $\Delta v^s_{-k} := v^s_{-k} - \bar{v}^s_{-k}$. Observe that for every $k \ge 0$, $(v^u_{-k-1}, v^s_{-k-1}) = F^{-1}_{x_{-k-1}x_{-k}}(v^u_{-k}, v^s_{-k})$, and $(\bar{v}^u_{-k-1}, \bar{v}^s_{-k-1}) = F^{-1}_{x_{-k-1}x_{-k}}(\bar{v}^u_{-k}, \bar{v}^s_{-k})$, we have

$$\begin{aligned}\|\Delta v^u_{-k-1}\| &\le \|\bar{D}^{uu}_{x_{-k},x_{-k-1}}\| \cdot \|\Delta v^u_{-k}\| + \|\bar{D}^{us}_{x_{-k},x_{-k-1}}\| \cdot \|\Delta v^s_{-k}\| \\ &\quad + \mathrm{Lip}(\bar{\phi}^u_{x_{-k},x_{-k-1}})(\|\Delta v^u_{-k}\| + \|\Delta v^s_{-k}\|) \\ &\le (\|Df^{-1}|_{E^u(x_{-k})}\| + 2\varepsilon\varepsilon(x_{-k}))\|\Delta v^u_{-k}\| + 2\varepsilon\varepsilon(x_{-k})\|\Delta v^s_{-k}\|\end{aligned}$$

and

$$\begin{aligned}\|\Delta v^s_{-k-1}\| &\ge m(\bar{D}^{ss}_{x_{-k},x_{-k-1}})\|\Delta v^s_{-k}\| - \|\bar{D}^{su}_{x_{-k},x_{-k-1}}\| \cdot \|\Delta v^u_{-k}\| \\ &\quad - \mathrm{Lip}(\bar{\phi}^s_{x_{-k},x_{-k-1}})(\|\Delta v^u_{-k}\| + \|\Delta v^s_{-k}\|) \\ &\ge (m(Df^{-1}|_{E^s(x_{-k})}) - 2\varepsilon\varepsilon(x_{-k}))\|\Delta v^s_{-k}\| - 2\varepsilon\varepsilon(x_{-k})\|\Delta v^u_{-k}\|.\end{aligned}$$

We claim that an easy induction gives $\|\Delta v^u_{-k}\| \le \|\Delta v^s_{-k}\| \le \|\Delta v^s_{-k-1}\|$ for all $k \ge 0$. In fact, for $k = 0$, this is because $\|\Delta v^u_0\| = 0$ by definition and $C\varepsilon\varepsilon(x_{-k}) \le m(Df^{-1}|_{E^s(x_{-k})}) - 1$ if $\varepsilon > 0$ is small enough from Lemma 3.1. Assume by induction that $\|\Delta v^u_{-k}\| \le \|\Delta v^s_{-k}\| \le \|\Delta v^s_{-k-1}\|$. Notice that $C\varepsilon(x_{-k}) \le 1 - \|Df^{-1}|_{E^u(x_{-k})}\|$ from the case f^{-1} of Lemma 3.2, and choosing a smaller ε if necessary such that $\varepsilon \le \min\{1, C/4\}$, then we have

$$\begin{aligned}\|\Delta v^u_{-k-1}\| &\le (\|Df^{-1}|_{E^u(x_{-k})}\| + 2\varepsilon\varepsilon(x_{-k}))\|\Delta v^u_{-k}\| + 2\varepsilon\varepsilon(x_{-k})\|\Delta v^s_{-k}\| \\ &\le (\|Df^{-1}|_{E^u(x_{-k})}\| + 4\varepsilon\varepsilon(x_{-k}))\|\Delta v^s_{-k}\| \le \|\Delta v^s_{-k}\| \le \|\Delta v^s_{-k-1}\|\end{aligned}$$

and

$$\begin{aligned}\|\Delta v^s_{-k-2}\| &\ge (m(Df^{-1}|_{E^s(x_{-k-1})}) - 2\varepsilon\varepsilon(x_{-k-1}))\|\Delta v^s_{-k-1}\| - 2\varepsilon\varepsilon(x_{-k-1})\|\Delta v^u_{-k-1}\| \\ &\ge (m(Df^{-1}|_{E^s(x_{-k-1})}) - 4\varepsilon\varepsilon(x_{-k-1}))\|\Delta v^s_{-k-1}\| \ge \|\Delta v^s_{-k-1}\|.\end{aligned}$$

Hence, the claim is verified.

We now return to the proof of this part. If we introduce a new constant

$$b := \inf_{x\in O} \frac{m(Df^{-1}|_{E^s(x)}) - m(Df^{-1}|_{E^s(x)})^{1/2}}{m(Df^{-1}|_{E^s(x)}) - 1} > 0,$$

and choosing a smaller ε if necessary such that $0 < \varepsilon \le (bC/4)$, then by Assumption US(iii), we see that

$$\begin{aligned}\|\Delta v^s_{-k-2}\| &\ge (m(Df^{-1}|_{E^s(x_{-k-1})}) - bC\varepsilon(x_{-k-1}))\|\Delta v^s_{-k-1}\| \\ &\ge m(Df^{-1}|_{E^s(x_{-k-1})})^{1/2}\|\Delta v^s_{-k-1}\|\end{aligned}$$

for all $k \ge 0$. Therefore, we get $\|\Delta v^s_{-k}\| \ge \prod_{i=0}^{k-1} m(Df^{-1}|_{E^s(x_{-i})})^{1/2}\|\Delta v^s_0\|$ and hence either $\|\Delta v^s_0\| = 0$ or $\|\Delta v^s_{-k}\| \xrightarrow[k\to\infty]{} \infty$. However, $\|\Delta v^s_{-k}\| = \|v^s_{-k} - \bar{v}^s_{-k}\| \le 2Q(x_{-k}) < 2$, so $\|\Delta v^s_0\| = 0$. Recall that $\Delta v^s_0 = v^s_0 - \bar{v}^s_0$, we have $v^s_0 = \bar{v}^s_0$ and then $v^s_0 = \bar{v}^s_0 = \sigma^u(\bar{v}^u_0) = \sigma^u(v^u_0)$ by definition. Thus, $x = \exp_{x_0}(v^u_0, \sigma^u(v^u_0)) \in V^u(\{x_{-n}\}_{n\ge 0})$.

(5) Given $n > N$, let V^u_{-n} be a u-admissible manifold at x_{-n} and let W^u_{-n} be a u-admissible manifold at y_{-n}. For the sake of simplicity in notation, let $\mathscr{F}^\ell_u(V^u_{-n})$ (respectively $\mathscr{F}^\ell_u(W^u_{-n})$) denote the result of applying $\mathscr{F}^u_{\cdot,\cdot}$ ℓ times to V^u_{-n} using the path $x_{-n} \to \cdots \to x_{-n+\ell}$ (respectively using $y_{-n} \to \cdots \to y_{-n+\ell}$).

Here, $\mathscr{F}^{n-N}_u(V^u_{-n})$ and $\mathscr{F}^{n-N}_u(W^u_{-n})$ are u-admissible manifolds at x_{-N} and y_{-N}, respectively. Let $\sigma_{-N}, \widetilde{\sigma}_{-N}$ be their representing functions. Admissibility implies that

$$\begin{aligned}&\|\sigma_{-N} - \widetilde{\sigma}_{-N}\|_{C^0} \le \|\sigma_{-N}\|_{C^0} + \|\widetilde{\sigma}_{-N}\|_{C^0} \le 2Q(x_{-N}) < 1,\\ &\|D\sigma_{-N} - D\widetilde{\sigma}_{-N}\|_{C^0} \le \|D\sigma_{-N}\|_{C^0} + \|D\widetilde{\sigma}_{-N}\|_{C^0} \le 2\sqrt{\varepsilon} < 1\end{aligned}$$

if ε is small enough.

For every $0 \le k \le N$, represent $\mathscr{F}^{n-k}_u[V^u_{-n}]$ and $\mathscr{F}^{n-k}_u[W^u_{-n}]$ by functions σ_{-k} and $\widetilde{\sigma}_{-k}$, respectively. By Proposition 4.3, we have

$$\|\sigma_{-k+1} - \widetilde{\sigma}_{-k+1}\|_{C^0} \le \|Df|_{E^s(x_{-k})}\|^{1/2}\|\sigma_{-k} - \widetilde{\sigma}_{-k}\|_{C^0} \tag{5.4}$$

and

$$\|D\sigma_{-k+1} - D\widetilde{\sigma}_{-k+1}\|_{C^0} \le \|Df|_{E^s(x_{-k})}\|^{1/2}(\|D\sigma_{-k} - D\widetilde{\sigma}_{-k}\|_{C^0} + 2\|\sigma_{-k} - \widetilde{\sigma}_{-k}\|^{\delta/2}_{C^0}). \tag{5.5}$$

Iterating equation (5.4), from $k = N$ and going down, we get

$$\begin{aligned}\|\sigma_{-k} - \widetilde{\sigma}_{-k}\|_{C^0} &\le \|Df|_{E^s(x_{-k-1})}\|^{1/2}\|\sigma_{-k-1} - \widetilde{\sigma}_{-k-1}\|_{C^0}\\ &\le \prod_{i=k+1}^{N} \|Df|_{E^s(x_{-i})}\|^{1/2}\|\sigma_{-N} - \widetilde{\sigma}_{-N}\|_{C^0}\\ &\le \prod_{i=k+1}^{N} \|Df|_{E^s(x_{-i})}\|^{1/2},\end{aligned}$$

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and hence $d_{C^0}(\mathscr{F}_u^n[V_{-n}^u], \mathscr{F}_u^n[W_{-n}^u]) \le \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{1/2}$. Passing to the limit $n \to \infty$, we obtain

$$d_{C^0}(V^u(\{x_{-n}\}_{n\ge 0}), V^u(\{y_{-n}\}_{n\ge 0})) \le \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{1/2}.$$

Apply $\|\sigma_{-k} - \widetilde{\sigma}_{-k}\|_{C^0} \le \prod_{i=k+1}^N \|Df|_{E^s(x_{-i})}\|^{1/2}$ to equation (5.5), and set $c_{-k} := \|D\sigma_{-k} - D\widetilde{\sigma}_{-k}\|_{C^0}$ for every $0 \le k \le N$. Then $c_{-k+1} \le \|Df|_{E^s(x_{-k})}\|^{1/2}(c_{-k} + 2\prod_{i=k+1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4})$ for every $0 \le k \le N$. Now an easy induction gives that for every $0 \le k \le N$,

$$\begin{aligned} c_0 \le \prod_{i=1}^k \|Df|_{E^s(x_{-i})}\|^{1/2} c_{-k} + 2 \times \bigg[&\prod_{i=1}^k \|Df|_{E^s(x_{-i})}\|^{1/2} \cdot \prod_{i=k+1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4} \\ &+ \prod_{i=1}^{k-1} \|Df|_{E^s(x_{-i})}\|^{1/2} \cdot \prod_{i=k}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4} + \cdots + \|Df|_{E^s(x_{-1})}\|^{1/2} \\ &\times \prod_{i=2}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4} \bigg]. \end{aligned}$$

We now take $k = N$, paying attention to the inequalities $\|Df|_{E^s(x_{-i})}\|^{1/2} \le \|Df|_{E^s(x_{-i})}\|^{\delta/4}$ for all $i = 1, \ldots, N$ since $0 < \delta < \min\{1, \alpha - \beta/\gamma\}$ and noting that $c_{-N} \le 1$, we have

$$c_0 \le \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{1/2} + 2N \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4} < (2N+1) \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4}.$$

Immediately, it follows that $d_{C^1}(\mathscr{F}_u^n[V_{-n}^u], \mathscr{F}_u^n[W_{-n}^u]) \le 2(N+1) \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4}$. In part (1), we saw that $\mathscr{F}_u^n[V_{-n}^u]$ and $\mathscr{F}_u^n[W_{-n}^u]$ converge to $V^u(\{x_{-n}\}_{n\ge 0})$ and $V^u(\{y_{-n}\}_{n\ge 0})$ in C^1, respectively. If we pass to the limit as $n \to \infty$, finally we get

$$d_{C^1}(V^u(\{x_{-n}\}_{n\ge 0}), V^u(\{y_{-n}\}_{n\ge 0})) \le 2(N+1) \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{\delta/4}. \qquad \square$$

We will see that Proposition 5.2 ensures that every ε-rppo $\underline{x} \in \Sigma^\#$ is associated to a unique point.

Definition 5.2. We say that an ε-rppo $\underline{x} \in \Sigma^\#$ in O *shadows* a point $x \in O$ when $f^n(x) \in \exp_{x_n}(S_{x_n})$ for all $n \in \mathbb{Z}$.

The following lemma is the so called the *shadowing lemma*, which is at the heart in the proof of Theorem 5.1. The general idea is similar to what Sarig does in [22, §4.3].

LEMMA 5.1. *Every ε-rppo $\underline{x} \in \Sigma^\#$ in O shadows a unique point $x \in O$.*

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Proof. Let $\underline{x} \in \Sigma^\#$ be an ε-rppo. From Proposition 4.1(1), there is a singleton set $\{x\} := V^u(\underline{x}^-) \cap V^s(\underline{x}^+) \subset O$, we claim that $\underline{x}$ shadows x and $x \in O$. By Proposition 5.2(2), for all $k \geq 0$, we have

$$f^{-k}(x) \in f^{-k}[V^u(\{x_{-n}\}_{n\geq 0})] \subset V^u(\{x_{-n-k}\}_{n\geq 0}) \subset \exp_{x_{-k}}(S_{x_{-k}})$$

and

$$f^k(x) \in f^k[V^s(\{x_n\}_{n\geq 0})] \subset V^s(\{x_{n+k}\}_{n\geq 0}) \subset \exp_{x_k}(S_{x_k}),$$

that is, $f^k(x) \in \exp_{x_k}(S_{x_k})$ for all $k \in \mathbb{Z}$, and hence $\underline{x}$ shadows x.

If $y \in O$ is any other point shadowed by $\underline{x}$, according to Definition 5.2 and Proposition 5.2(4), it must lie in $V^u(\underline{x}^-) \cap V^s(\underline{x}^+) = \{x\}$. □

For every ε-rppo $\underline{x} \in \Sigma^\#$, let $\pi : \Sigma^\# \to O$ be defined by

$$\{\pi(\underline{x})\} = V^u(\underline{x}^-) \cap V^s(\underline{x}^+). \tag{5.6}$$

By the shadowing lemma in Lemma 5.1, $\pi(\underline{x})$ is the shadowed point of $\underline{x}$, and the uniqueness guarantees that π is well defined for every element of $\Sigma^\#$.

Here are the main properties of the triple $(\Sigma^\#, \sigma, \pi)$.

THEOREM 5.1. *The following hold for all $\varepsilon > 0$ small enough.*

(1) $\pi \circ \sigma = f \circ \pi$ *on* $\Sigma^\#$.

(2) $\pi(\Sigma^\#) = O^\#$.

(3) $\pi : \Sigma^\# \to O$ *is continuous. More specifically, if* $\underline{x} = \{x_n\}_{n\in\mathbb{Z}}$, $\underline{y} = \{y_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ *satisfy* $x_i = y_i$ *for* $i = -N, \ldots, N$, *then*

$$d(\pi(\underline{x}), \pi(\underline{y})) \leq 6\bigg[\prod_{i=1}^{N} \|Df|_{E^s(x_{-i})}\|^{1/2} + \prod_{i=1}^{N} \|Df^{-1}|_{E^u(x_i)}\|^{1/2}\bigg].$$

Remark 5.2. $(\Sigma^\#, \sigma, \pi)$ is not a symbolic model for *f*, since π is usually infinite-to-one.

Proof. (1) Notice that for all $\underline{x} \in \Sigma^\#$, from the shadowing lemma in Lemma 5.1 and Definition 5.2, we can denote $x = \pi(\underline{x})$ which satisfies $f^n(f \circ \pi(\underline{x})) = f^{n+1}(x) \in \exp_{x_{n+1}}(S_{x_{n+1}})$ for all $n \in \mathbb{Z}$. However, since $\sigma(\underline{x}) = \{x_{n+1}\}_{n\in\mathbb{Z}} \in \Sigma^\#$, using the shadowing lemma in Lemma 5.1 again, we can denote $y = \pi \circ \sigma(\underline{x})$ which satisfies $f^n(\pi \circ \sigma(\underline{x})) = f^n(y) \in \exp_{x_{n+1}}(S_{x_{n+1}})$ for all $n \in \mathbb{Z}$ by Definition 5.2 again. From the uniqueness of the shadowing point, $y = f(x)$, that is, $\pi \circ \sigma(\underline{x}) = f \circ \pi(\underline{x})$ for all $\underline{x} \in \Sigma^\#$.

(2) First we prove $\pi(\Sigma^\#) \supset O^\#$. For every $x \in O^\# \subset O$, looking closely into the proof of Proposition 5.1(2), we see that there exists a sequence $\{x_n\}_{n\in\mathbb{Z}} \in \Sigma$ that satisfies $d(f^n(x), x_n) < \varepsilon Q(x_n)$ and $d(x_n, \partial O) \geq e^{-\varepsilon} d(f^n(x), \partial O)$ for all $n \in \mathbb{Z}$. By the definition of $O^\#$, $d(x_n, \partial O) \geq e^{-\varepsilon} d(f^n(x), \partial O)$ means there exist sequences $n_k, m_k \uparrow \infty$ for which $d(x_{n_k}, \partial O)$ and $d(x_{-m_k}, \partial O)$ are bounded away from zero. By the discreteness of $\mathscr{A}$ (Proposition 5.1(1)), x_n must repeat some symbol infinitely often in the past and (possibly a different symbol) in the future, that is, $\{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$. Since $d(f^n(x), x_n) < \varepsilon Q(x_n)$, that is, $f^n(x) \in \exp_{x_n}(T_{x_n}M(\varepsilon Q(x_n))) \subset \exp_{x_n}(S_{x_n})$ if $\varepsilon > 0$ is small enough,

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from the uniqueness of the shadowing point, we have $x = \pi(\{x_n\}_{n\in\mathbb{Z}}) \in \pi(\Sigma^\#)$ for all $n \in \mathbb{Z}$. Thus, we get $\pi(\Sigma^\#) \supset O^\#$.

In the opposite direction, for every $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$, the shadowing lemma in Lemma 5.1 claims there is a unique $x \in O$ such that $x = \pi(\underline{x})$. Since $d(f^n(x), x_n) \leq (1/50)Q(x_n) \leq (1/50)d(x_n, \partial O)$ from Proposition 4.1(1), we have

$$d(f^n(x), \partial O) \geq d(x_n, \partial O) - d(f^n(x), x_n) \geq \frac{49}{50} d(x_n, \partial O),$$

and then $x \in O^\#$. Therefore, $\pi(\Sigma^\#) \subset O^\#$.

(3) Write $\{\pi(\underline{x})\} = V^u(\underline{x}^-) \cap V^s(\underline{x}^+)$, $\{\pi(\underline{y})\} = V^u(\underline{y}^-) \cap V^s(\underline{y}^+)$. From Propositions 4.1(2), 4.3(1), and using induction, we have

$$\begin{aligned} d(\pi(\underline{x}), \pi(\underline{y})) &\leq \frac{2}{1-\sqrt{\varepsilon}}[d_{C^0}(V^u(\underline{x}^-), V^u(\underline{y}^-)) + d_{C^0}(V^s(\underline{x}^+), V^s(\underline{y}^+))] \\ &\leq \frac{2}{1-\sqrt{\varepsilon}}\bigg[\prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{1/2} + \prod_{i=1}^N \|Df^{-1}|_{E^u(x_i)}\|^{1/2}\bigg] \\ &\leq 6\bigg[\prod_{i=1}^N \|Df|_{E^s(x_{-i})}\|^{1/2} + \prod_{i=1}^N \|Df^{-1}|_{E^u(x_i)}\|^{1/2}\bigg] \end{aligned}$$

if $0 < \varepsilon \leq \frac{1}{9}$.

Finally, the continuity is obvious since

$$\lim_{N\to\infty} \prod_{i=1}^N \|Df|_{E^s(x_{-i})}\| = \lim_{N\to\infty} \prod_{i=1}^N \|Df^{-1}|_{E^u(x_i)}\| = 0. \qquad \square$$

6. *Markov partitions and symbolic dynamics*

Let (Σ, σ) be the TMS constructed in §5, and let $\pi : \Sigma^\# \to O$ as defined in equation (5.6). In what follows, we use Theorem 5.1 to construct a cover of $O^\#$ that is locally finite and satisfies a (symbolic) Markov property.

Definition 6.1. Call $\mathscr{Z} := \{Z(z) : z \in \mathscr{A}\}$ the *Markov cover*, where

$$Z(z) := \{\pi(\underline{x}) : \underline{x} \in \Sigma^\#, x_0 = z\}.$$

In other words, $\mathscr{Z}$ is the family defined by the natural partition of $\Sigma^\#$ into a cylinder at the zeroth position. Although every element $Z(z)$ of $\mathscr{Z}$ is a subset of the much smaller set $\exp_z(T_zM(2/50Q(z)))$ (Proposition 4.1(1)), it is a cover of $O^\#$ since

$$\bigcup_{Z\in\mathscr{Z}} Z = \bigcup_{z\in\mathscr{A}} Z(z) = \pi(\Sigma^\#) = O^\#.$$

Thus, $O = \bigcup_{Z\in\mathscr{Z}} Z \bmod \mu$ for every f-invariant probability measure μ supported on O.

Suppose $x \in Z(z) \in \mathscr{Z}$, then there exists $\underline{x} \in \Sigma^\#$ such that $x_0 = z$ and $\pi(\underline{x}) = x$. Associated to $\underline{x}$ are two admissible manifolds at z : $V^u(\underline{x}^-)$ and $V^s(\underline{x}^+)$. The following

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proposition says that these manifolds do not depend on the choice of $\underline{x}$: if $\underline{y} \in \Sigma^\#$ is another ε-rppo such that $y_0 = z$ and $\pi(\underline{y}) = x$, then

$$V^u(\underline{x}^-) = V^u(\underline{y}^-) \quad \text{and} \quad V^s(\underline{x}^+) = V^s(\underline{y}^+),$$

which allows us to define *invariant fibers* inside each $Z \in \mathscr{Z}$ (see Definition 6.4).

PROPOSITION 6.1. *For any $\underline{x}, \underline{y} \in \Sigma^\#$ such that $x_0 = y_0$ and $\pi(\underline{x}) = \pi(\underline{y}) = x$, we have*

$$V^u(\underline{x}^-) = V^u(\underline{y}^-) \quad \textit{and} \quad V^s(\underline{x}^+) = V^s(\underline{y}^+).$$

Proof. As proven in [22, Proposition 6.4], we show the proposition for V^s by dividing our proof into three claims, and leave the case of V^u to the reader. Suppose $\varepsilon > 0$ is small enough.

CLAIM 1. $f^k[V^s(\underline{y}^+)] \subseteq \exp_{y_k}[B^u_{y_k}(\frac{1}{8}Q(y_k)) \oplus B^s_{y_k}(\frac{1}{8}Q(y_k))]$ *for all k large enough.*

Proof of Claim 1. Recall that by Proposition 5.2(2), for all $k \geq 0$, we have

$$f^k(V^s(\underline{y}^+)) \subset V^s(\{y_{n+k}\}_{n\geq 0}) = \exp_{y_k}\{(\sigma^s_k(v^s_k), v^s_k) : \|v^s_k\| \leq Q(y_k)\} \subset \exp_{y_k}(S_{y_k}),$$

where σ^s_k is the function which represents s-admissible manifold $V^s(\{y_{n+k}\}_{n\geq 0})$ at y_k for all $k \geq 0$. Hence, for any $y \in V^s(\underline{y}^+)$, one can write $f^k(y) = \exp_{y_k}(\sigma^s_k(v^s_k), v^s_k)$, $\|v^s_k\| \leq Q(y_k)$.

Notice that $(\sigma^s_k(v^s_k), v^s_k) = F_{y_{k-1},y_k}(\sigma^s_{k-1}(v^s_{k-1}), v^s_{k-1})$, or in the s-direction,

$$v^s_k = D^{ss}_{y_{k-1},y_k}(v^s_{k-1}) + D^{su}_{y_{k-1},y_k} \circ \sigma^s_{k-1}(v^s_{k-1}) + \phi^s_{y_{k-1},y_k}(\sigma^s_{k-1}(v^s_{k-1}), v^s_{k-1}).$$

Using equations (4.1), (3.12), and $\|\phi^s_{y_{k-1},y_k}(0,0)\| \leq \varepsilon\varepsilon(y_{k-1})Q(y_{k-1})$, it follows that

$$\begin{aligned}\|v^s_k\| &\leq \|D^{ss}_{y_{k-1},y_k}\|\,\|v^s_{k-1}\| + \|D^{su}_{y_{k-1},y_k}\|\,\|\sigma^s_{k-1}(v^s_{k-1})\| + \|\phi^s_{y_{k-1},y_k}(0,0)\| \\ &\quad + \mathrm{Lip}(\phi^s_{y_{k-1},y_k})(\|\sigma^s_{k-1}(v^s_{k-1})\| + \|v^s_{k-1}\|)) \\ &\leq (\|Df|_{E^s(y_{k-1})}\| + 4\varepsilon\varepsilon(y_{k-1}))\|v^s_{k-1}\| + 2\varepsilon\varepsilon(y_{k-1})Q(y_{k-1}).\end{aligned}$$

We see that $\|v^s_k\| \leq a_k$ where a_k is defined inductively by

$$a_0 := Q(y_0) \quad \text{and} \quad a_k = (\|Df|_{E^s(y_{k-1})}\| + 4\varepsilon\varepsilon(y_{k-1}))a_{k-1} + 2\varepsilon\varepsilon(y_{k-1})Q(y_{k-1}).$$

We claim that $a_k \leq \frac{1}{8}Q(y_k)$ for some $k \geq 0$. If the assertion would not hold, then $Q(y_k) < 8a_k$ for all $k \geq 0$, and hence $a_k < (\|Df|_{E^s(y_{k-1})}\| + 20\varepsilon\varepsilon(y_{k-1}))a_{k-1}$ for all $k \geq 0$, which implies that

$$a_k < \prod_{i=0}^{k-1}(\|Df|_{E^s(y_i)}\| + 20\varepsilon\varepsilon(y_i))a_0.$$

Fix $\kappa < \kappa_1 < 1$, and introduce a new constant c_1 associated to κ_1 given by

$$c_1 := \inf_{x\in O} \frac{\|Df|_{E^s(x)}\|^{\kappa_1} - \|Df|_{E^s(x)}\|}{1 - \|Df|_{E^s(x)}\|} > 0,$$

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then if $0 < \varepsilon \leq (c_1 C/20)$, we get

$$a_k < \prod_{i=0}^{k-1} \|Df|_{E^s(y_i)}\|^{\kappa_1} a_0.$$

Whereas by assumption, $a_k > \frac{1}{8}Q(y_k)$ for all $k \geq 0$. Since $y_{k-1} \to y_k$, from Lemma 4.1 in the case $C_3 = C_4 = 1$, we obtain

$$\begin{aligned} Q(y_k) &> Q(y_k) - d(f(y_{k-1}), y_k) \\ &\geq (\|Df|_{E^s(y_{k-1})}\|^{\kappa} + \varepsilon\varepsilon(y_{k-1}))Q(y_{k-1}) \\ &> \|Df|_{E^s(y_{k-1})}\|^{\kappa} Q(y_{k-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} a_k > \tfrac{1}{8}Q(y_k) &> \tfrac{1}{8}\|Df|_{E^s(y_{k-1})}\|^{\kappa} Q(y_{k-1}) \\ &> \frac{1}{8}\prod_{i=0}^{k-1} \|Df|_{E^s(y_i)}\|^{\kappa} a_0. \end{aligned}$$

Combining with the above, we have

$$\frac{1}{8} \leq \prod_{i=0}^{k-1} \|Df|_{E^s(y_i)}\|^{\kappa_1 - \kappa} \underset{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction. Consequently, we infer that there exists $k_0 \geq 0$ such that $a_{k_0} \leq \frac{1}{8}Q(y_{k_0})$.

Moreover, we have $a_k \leq \frac{1}{8}Q(y_k)$ for all k large enough by using induction. In fact, suppose that the inequality is true for $k-1$, we have by the definition of a_k that

$$\begin{aligned} a_k &\leq (\|Df|_{E^s(y_{k-1})}\| + 4\varepsilon\varepsilon(y_{k-1}))\tfrac{1}{8}Q(y_{k-1}) + 2\varepsilon\varepsilon(y_{k-1})Q(y_{k-1}) \\ &\leq (\|Df|_{E^s(y_{k-1})}\| + 20\varepsilon\varepsilon(y_{k-1}))\tfrac{1}{8}Q(y_{k-1}) \\ &\leq \|Df|_{E^s(y_{k-1})}\|^{\kappa_1}\tfrac{1}{8}Q(y_{k-1}) \\ &\leq \|Df|_{E^s(y_{k-1})}\|^{\kappa}\tfrac{1}{8}Q(y_{k-1}) \\ &\leq \tfrac{1}{8}Q(y_k). \end{aligned}$$

Thus, $a_k \leq \frac{1}{8}Q(y_k)$ for all k large enough.

In particular, $\|v_k^s\| \leq \frac{1}{8}Q(y_k)$ for all k large enough. Since $\|\sigma_k^s(v_k^s)\| \leq \|\sigma_k^s(0)\| + \mathrm{Lip}(\sigma_k^s)\|v_k^s\| < (10^{-3} + \sqrt{\varepsilon})Q(y_k) \leq \frac{1}{8}Q(y_k)$, if ε is small enough, we have

$$f^k[V^s(\underline{y}^+)] \subseteq \exp_{y_k}\left[B^u_{y_k}\left(\frac{1}{8}Q(y_k)\right) \oplus B^s_{y_k}\left(\frac{1}{8}Q(y_k)\right)\right]$$

for all k large enough. □

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CLAIM 2. *$f^k[V^s(\underline{y}^+)] \subseteq \exp_{x_k}(S_{x_k})$ for all k large enough.*

Proof of Claim 2. For any $y \in V^s(\underline{y}^+)$, since $\pi(\underline{x}) = \pi(\underline{y}) = x$, from Proposition 4.1(1), we get

$$d(y_k, x_k) \leq d(y_k, f^k(x)) + d(f^k(x), x_k) \leq \frac{1}{50}Q(y_k) + \frac{1}{50}Q(x_k),$$

and hence $d(f^k(y), x_k) \leq d(f^k(y), y_k) + d(y_k, x_k) < \rho(M)$ and we have $f^k(y) \in B(x_k, \rho(M))$. Now for $k > 0$ large enough, by Claim 1, we have

$$\begin{aligned}\|\exp_{x_k}^{-1} f^k(y)\| &\leq \|\exp_{y_k}^{-1} f^k(y)\| + d(y_k, x_k)\\ &\leq \frac{1}{4}Q(y_k) + \frac{1}{50}Q(x_k) + \frac{1}{50}Q(y_k)\\ &\leq \tfrac{1}{4}(2Q(y_k) + Q(x_k)).\end{aligned}$$

Note that $Q(x_k) \leq \varepsilon^{2/(\alpha-\delta)} d(x_k, \partial O)$, then

$$\begin{aligned}d(f^k(x), \partial O) &\leq d(x_k, \partial O) + d(x_k, f^k(x))\\ &\leq d(x_k, \partial O) + \frac{1}{50}Q(x_k)\\ &\leq e^{\varepsilon^{2/(\alpha-\delta)}} d(x_k, \partial O)\end{aligned}$$

and we have $Q(f^k(x)) \leq e^{\gamma\varepsilon^{2/(\alpha-\delta)}} Q(x_k)$.

Similarly, from

$$\begin{aligned}d(f^k(x), \partial O) &\geq d(y_k, \partial O) - d(y_k, f^k(x))\\ &\geq d(y_k, \partial O) - \frac{1}{50}Q(y_k)\\ &\geq e^{-\varepsilon^{2/(\alpha-\delta)}} d(y_k, \partial O),\end{aligned}$$

we have $Q(f^k(x)) \geq e^{-\gamma\varepsilon^{2/(\alpha-\delta)}} Q(y_k)$. Thus, we arrive at two useful inequalities:

$$d(y_k, \partial O) \leq e^{\varepsilon^{2/(\alpha-\delta)}} d(f^k(x), \partial O) \leq e^{2\varepsilon^{2/(\alpha-\delta)}} d(x_k, \partial O) \tag{6.1}$$

and

$$Q(y_k) \leq e^{\gamma\varepsilon^{2/(\alpha-\delta)}} Q(f^k(x)) \leq e^{2\gamma\varepsilon^{2/(\alpha-\delta)}} Q(x_k). \tag{6.2}$$

This leads to

$$\begin{aligned}\|\exp_{x_k}^{-1} f^k(y)\| &\leq \tfrac{1}{4}(2Q(y_k) + Q(x_k))\\ &\leq \tfrac{1}{4}(2e^{2\gamma\varepsilon^{2/(\alpha-\delta)}} + 1)Q(x_k)\\ &\leq Q(x_k)\end{aligned}$$

if k is large enough and $0 < \varepsilon \leq ((\log 3 - \log 2)/2\gamma)^{(\alpha-\delta)/2}$.

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It is now obvious that the claim holds since we have assumed as in [10] that the metric on M is taken in such a way that for any $x \in O$, $E^u(x)$ and $E^s(x)$ are pairwise orthogonal, we thus prove

$$f^k[V^s(\underline{y}^+)] \subseteq \exp_{x_k}(S_{x_k})$$

for all k large enough. □

CLAIM 3. $V^s(\underline{x}^+) = V^s(\underline{y}^+)$.

Proof of Claim 3. The proof of this claim is made exactly as in [6, Claim 3 on p. 102] by using Claims 1 and 2. □

This completes the proof of Proposition 6.1. □

Definition 6.2. Suppose $Z = Z(z) \in \mathscr{Z}$. For any $x \in Z$, the *local unstable manifold through* x is defined by $V^u(x, Z) := V^u(\underline{x}^-)$ for some (any) $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ such that $\pi(\underline{x}) = x$ and $x_0 = z$. Similarly, the *local stable manifold through* x is defined by $V^s(x, Z) := V^s(\underline{x}^+)$ for some (any) $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ such that $\pi(\underline{x}) = x$ and $x_0 = z$.

By Proposition 6.1, the definitions above do not depend on the choice of $\underline{x}$. Since $\bigcup_{Z\in\mathscr{Z}} Z = O^\#$, we only construct locally unstable/stable manifolds on $O^\#$ rather than O as done in [10].

THEOREM 6.1. *For every $Z \in \mathscr{Z}$, $|\{Z' \in \mathscr{Z} : Z' \cap Z \neq \varnothing\}| < \infty$.*

Proof. Fix some $Z = Z(z) \in \mathscr{Z}$. If $Z' = Z'(z')$ intersects Z, then there must exist two ε-rppo $\underline{x}, \underline{y} \in \Sigma^\#$ such that $x_0 = z$, $y_0 = z'$, and $\pi(\underline{x}) = \pi(\underline{y})$. From equation (6.1), we have

$$d(z, \partial O) = d(x_0, \partial O) \leq e^{2\varepsilon^{2/(\alpha-\delta)}} d(y_0, \partial O) = e^{2\varepsilon^{2/(\alpha-\delta)}} d(z', \partial O).$$

It follows that Z' belongs to $\{Z'(z') : z' \in \mathscr{A}, d(z', \partial O) \geq e^{-2\varepsilon^{2/(\alpha-\delta)}} d(z, \partial O)\}$. This set is finite because of the discreteness of $\mathscr{A}$ (Proposition 5.1(1)). □

PROPOSITION 6.2. *Suppose $Z \in \mathscr{Z}$, then for any $x, y \in Z$, $V^u(x, Z)$ and $V^s(y, Z)$ intersect at a unique point z, and $z \in Z$.*

Proof. It follows from Proposition 4.1(1), proved as in [22, Proposition 10.5]. □

Definition 6.3. The *Smale bracket* of two points $x, y \in Z \in \mathscr{Z}$ is the unique point $[x, y]_Z \in V^u(x, Z) \cap V^s(y, Z)$.

LEMMA 6.1. *If $x, y \in Z \in \mathscr{Z}$, then*

$$V^u([x, y]_Z, Z) = V^u(x, Z) \quad \textit{and} \quad V^s([x, y]_Z, Z) = V^s(y, Z).$$

Proof. Write $Z = Z(z)$ where $z \in \mathscr{A}$. There are two ε-rppo $\underline{x}, \underline{y} \in \Sigma^\#$ such that $x_0 = y_0 = z$, $\pi(\underline{x}) = x$, $\pi(\underline{y}) = y$, and so that

$$V^u(x, Z) = V^u(\underline{x}^-) \quad \text{and} \quad V^s(y, Z) = V^s(\underline{y}^+).$$

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It holds that $\{[x, y]_Z\} = V^u(\underline{x}^-) \cap V^s(\underline{y}^+) = \{\pi(\underline{z})\} \subset Z(z)$, where $z_0 = z$, and

$$z_n = \begin{cases} x_n, & n \leq 0, \\ y_n, & n \geq 0, \end{cases} \in \Sigma^\#.$$

By Definition 6.2,

$$V^u([x, y]_Z, Z) = V^u(\underline{z}^-) = V^u(\underline{x}^-) = V^u(x, Z)$$

and

$$V^s([x, y]_Z, Z) = V^s(\underline{z}^+) = V^s(\underline{y}^+) = V^s(y, Z). \qquad \square$$

LEMMA 6.2. *Suppose $x, y \in Z(z_0)$, and $f(x), f(y) \in Z(z_1)$. If $z_0 \to z_1$, then $f([x, y]_{Z(z_0)}) = [f(x), f(y)]_{Z(z_1)}$.*

Despite the definition of *staying in window* [22, Definitions 6.1 and 6.2] and the conclusion that any two local stable manifolds either coincide or are disjoint [22, Proposition 6.4], which are used in the proof of Lemma 6.2 above in [22, Lemma 10.7], we can still circumvent the problem just by using the weaker Proposition 6.1 above to prove the conclusion, and we will see that this conclusion is just from the Markov structure of $\Sigma^\#$.

Proof. Compared with the proof of [22, Lemma 10.7], the difference is in the proof of $f[V^s(y, Z(z_0))] \subset V^s(f(y), Z(z_1))$ and $f[V^u(x, Z(z_0))] \supset V^u(f(x), Z(z_1))$. We give the details of the first half, the other half is proved in the same way.

Since $V^s_{z_1} := V^s(f(y), Z(z_1)) = V^s(\{x_{n+1}\}_{n\geq 0})$ for any $\{x_{n+1}\}_{n\geq 0} \in \Sigma^\#$ such that $x_0 = z_0$, $x_1 = z_1$, and $\pi(\{x_{n+1}\}_{n\in\mathbb{Z}}) = f(y)$, it is an s-admissible manifold at z_1. By Definition 4.4, $\mathscr{F}^s_{z_0,z_1}[V^s_{z_1}]$ is the unique s-admissible manifold at z_0 contained in $f^{-1}(V^s_{z_1})$, and thus our problem reduces to prove $V^s(y, Z(z_0)) = \mathscr{F}^s_{z_0,z_1}[V^s_{z_1}]$.

According to Theorem 5.1(1) and (2), $\mathscr{F}^s_{z_0,z_1}[V^s_{z_1}] = V^s(\{x_n\}_{n\geq 0})$, where $x_0 = z_0$ and $\pi(\{x_n\}_{n\in\mathbb{Z}}) = y$ since the shadowing point is unique. Notice that $V^s(y, Z(z_0)) = V^s(\{y_n\}_{n\geq 0})$ for any $\{y_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ such that $y_0 = z_0$, and $\pi(\{y_n\}_{n\in\mathbb{Z}}) = y$, from Proposition 6.1, so we have $V^s(y, Z(z_0)) = V^s(\{y_n\}_{n\geq 0}) = V^s(\{x_n\}_{n\geq 0}) = \mathscr{F}^s_{z_0,z_1}[V^s_{z_1}]$. $\square$

Definition 6.4. For any $x \in Z \in \mathscr{Z}$, the *u-fiber* of x *inside* Z is defined as $W^u(x, Z) := V^u(x, Z) \cap Z$, and the *s-fiber* of x *inside* Z is defined as $W^s(x, Z) := V^s(x, Z) \cap Z$.

Obviously, since $V^u(x, Z) \cap V^s(y, Z) \subset Z$, Proposition 6.2 and Lemma 6.1 hold for an u/s-fiber also, that is, $W^u(x, Z) \cap W^s(y, Z) = V^u(x, Z) \cap V^s(y, Z)$, and

$$W^u([x, y]_Z, Z) = W^u(x, Z) \quad \text{and} \quad W^s([x, y]_Z, Z) = W^s(y, Z)$$

if $x, y \in Z \in \mathscr{Z}$.

PROPOSITION 6.3. *Suppose $Z \in \mathscr{Z}$. For any $x, y \in Z$, $W^u(x, Z)$ and $W^u(y, Z)$ are either equal or they are disjoint. Similarly for $W^s(x, Z)$ and $W^s(y, Z)$.*

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Proof. We prove the proposition for a u-fiber and leave the case of an s-fiber to the reader.

For any $x, y \in Z$, assume that $W^u(x, Z) \cap W^u(y, Z) \neq \varnothing$. Let $z \in W^u(x, Z) \cap W^u(y, Z) \subset Z$. Since $x, z \in Z$, $\{[x, z]_Z\}$ is well defined and equal to $W^u(x, Z) \cap W^s(z, Z)$, notice that $z \in W^u(x, Z) \cap W^s(z, Z)$, and thus

$$z = [x, z]_Z.$$

By Lemma 6.1, we have $W^u(z, Z) = W^u([x, z]_Z, Z) = W^u(x, Z)$.

Similarly, $W^u(z, Z) = W^u(y, Z)$. Hence, $W^u(x, Z) = W^u(y, Z)$. □

The following proposition states the *symbolic Markov property* of a u/s-fiber, which also follows from the Markov structure of $\Sigma^\#$.

PROPOSITION 6.4. *If $x = \pi(\underline{x})$ where $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$, then*

$$f[W^s(x, Z(x_0))] \subset W^s(f(x), Z(x_1)) \quad \textit{and} \quad f^{-1}[W^u(f(x), Z(x_1))] \subset W^u(x, Z(x_0)).$$

Proof. The proof is the same as in [22, Proposition 10.9]. We prove the inclusion for the s-fiber. The case of the u-fiber follows by symmetry.

By Definitions 6.4 and 6.2, $W^s(x, Z(x_0)) \subset V^s(x, Z(x_0)) = V^s(\{x_n\}_{n\geq 0})$. Using Proposition 5.2(2), we have

$$f(V^s(\{x_n\}_{n\geq 0})) \subseteq V^s(\{x_{n+1}\}_{n\geq 0}),$$

the last manifold is equal to $V^s(f(x), Z(x_1))$ since $\{x_{n+1}\}_{n\geq 0} \in \Sigma^\#$ satisfies $x_{0+1} = x_1$ and

$$\pi(\{x_{n+1}\}_{n\geq 0}) = \pi \circ \sigma(\underline{x}) = f \circ \pi(\underline{x}) = f(x).$$

Therefore, $f[W^s(x, Z(x_0))] \subset V^s(f(x), Z(x_1))$.

Suppose $y \in W^s(x, Z(x_0))$. Since $y \in Z(x_0)$, there exists an ε-rppo $\underline{y} \in \Sigma^\#$ such that $y_0 = x_0$ and $\pi(\underline{y}) = y$. From $y, x \in Z(x_0)$, we know that $\{[y, x]_{Z(x_0)}\}$ is well defined and equal to $V^u(\underline{y}^-) \cap V^s(\underline{x}^+)$. However, $y \in V^s(x, Z(x_0)) = V^s(\underline{x}^+)$ and $y = \pi(\underline{y})$ implies that $y \in V^u(\underline{y}^-) \cap V^s(\underline{x}^+)$, and thus

$$y = [y, x]_{Z(x_0)} = \pi(\underline{z}),$$

where $z_0 = x_0$, and

$$z_n = \begin{cases} y_n, & n \leq 0, \\ x_n, & n \geq 0, \end{cases} \in \Sigma^\#.$$

Therefore, $f(y) \in Z(x_1)$ since $\{z_{n+1}\}_{n\in\mathbb{Z}} \in \Sigma^\#$ satisfies $z_{0+1} = z_1 = x_1$ and

$$\pi(\{z_{n+1}\}_{n\geq 0}) = \pi \circ \sigma(\underline{z}) = f \circ \pi(\underline{z}) = f(y). \qquad \square$$

LEMMA 6.3. *Suppose $Z, Z' \in \mathscr{Z}$ and $Z \cap Z' \neq \varnothing$.*

(1) *If $Z = Z(z)$ and $Z' = Z'(z')$, where $z, z' \in \mathscr{A}$, then $Z \subset \exp_{z'}(S_{z'})$.*

(2) *For any $x \in Z \cap Z'$, $W^u(x, Z) \subset V^u(x, Z')$ and $W^s(x, Z) \subset V^s(x, Z')$.*

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Proof. We follow a similar strategy in [22, Lemma 10.10].

(1) Note that by Proposition 4.1(1),

$$Z(z) \subset \exp_z\left(T_zM\left(\frac{2}{50}Q(z)\right)\right) \subset B\left(z, \frac{4}{50}Q(z)\right) \subset B\left(z', \frac{4}{50}Q(z) + d(z, z')\right).$$

Fix some $x \in Z \cap Z' \subset O$, so from equation (6.2), we have

$$Q(z) \leq e^{2\gamma\varepsilon^{2/(\alpha-\delta)}} Q(z')$$

and

$$\begin{aligned}\frac{4}{50}Q(z) + d(z, z') &\leq \frac{5}{50}Q(z) + \frac{1}{50}Q(z') \\ &\leq \left(\frac{5}{50}e^{2\gamma\varepsilon^{2/(\alpha-\delta)}} + \frac{1}{50}\right)Q(z') \\ &\leq \frac{16}{50}Q(z')\end{aligned}$$

if $0 < \varepsilon \leq (\log 3/2\gamma)^{(\alpha-\delta)/2}$.

Therefore,

$$Z(z) \subset B\left(z', \frac{4}{50}Q(z) + d(z, z')\right) \subset B\left(z', \frac{8}{25}Q(z')\right) \subset \exp_{z'}(S_{z'}).$$

(2) We show that $W^s(x, Z) \subset V^s(x, Z')$. The case of the u-direction follows by symmetry.

Since $x \in Z(z) \cap Z'(z')$, we can write $x = \pi(\underline{x})$ where $\underline{x} \in \Sigma^\#$, $x_0 = z$, and $x = \pi(\underline{y})$, where $\underline{y} \in \Sigma^\#$, $y_0 = z'$. Observe that for all $k \in \mathbb{Z}$, $f^k(x) = f^k(\pi(\underline{x})) = \pi(\sigma^k(\underline{x})) \in Z(x_k)$, we have $f^k(x) \in Z(x_k)$. Similarly, since $f^k(x) = f^k(\pi(\underline{y})) = \pi(\sigma^k(\underline{y})) \in Z'(y_k)$, we have $f^k(x) \in Z'(y_k)$, and hence $f^k(x) \in Z(x_k) \cap Z'(y_k)$. By the first part of the lemma, $Z(x_k) \subset \exp_{y_k}(S_{y_k})$.

Since $x = \pi(\underline{x})$ and $Z(z) = Z(x_0)$, we have by the symbolic Markov property of an s-fiber iterated forward k times

$$f^k[W^s(x, Z(z))] \subset W^s(f^k(x), Z(x_k)) \subset Z(x_k) \subset \exp_{y_k}(S_{y_k})$$

for all $k \geq 0$. By Proposition 5.2(4), $W^s(x, Z) \subset V^s(\underline{y}^+) = V^s(x, Z')$. □

The next step to do in the construction of a Markov partition is to refine $\mathscr{Z}$ to destroy its non-trivial intersections. The result will be a partition of $O^\#$ by sets with the (geometrical) Markov property. This method is the so called *Bowen–Sinai refinement procedure*, first developed by Sinai and Bowen for finite covers [8, 23, 24], and also works for countable covers satisfying the local finiteness property [22, §11].

Write $\mathscr{Z} = \{Z_1, Z_2, \ldots\}$. Following [8, 22], we define a partition of Z_i such that for every $Z_i, Z_j \in \mathscr{Z}$,

$$\begin{aligned} T_{ij}^{us} &:= \{x \in Z_i : W^u(x, Z_i) \cap Z_j \neq \varnothing, W^s(x, Z_i) \cap Z_j \neq \varnothing\}, \\ T_{ij}^{u\varnothing} &:= \{x \in Z_i : W^u(x, Z_i) \cap Z_j \neq \varnothing, W^s(x, Z_i) \cap Z_j = \varnothing\}, \\ T_{ij}^{\varnothing s} &:= \{x \in Z_i : W^u(x, Z_i) \cap Z_j = \varnothing, W^s(x, Z_i) \cap Z_j \neq \varnothing\}, \\ T_{ij}^{\varnothing\varnothing} &:= \{x \in Z_i : W^u(x, Z_i) \cap Z_j = \varnothing, W^s(x, Z_i) \cap Z_j = \varnothing\}. \end{aligned}$$

Let $\mathscr{T} := \{T_{ij}^{\alpha\beta} : i, j \in \mathbb{N}, Z_i \cap Z_j \neq \varnothing, \alpha \in \{u, \varnothing\}, \beta \in \{s, \varnothing\}\}$.

Clearly, $T_{ii}^{us} = Z_i$; therefore, $\mathscr{T}$ covers the same set as $\mathscr{Z}$, namely $\pi(\Sigma^\#) = O^\#$. Using Proposition 6.3 and Lemma 6.3(2) above, we prove that $T_{ij}^{us} = Z_i \cap Z_j$ as in [22, p. 386].

Definition 6.5. For every $x \in \pi(\Sigma^\#) = O^\#$, let $R(x) := \bigcap_{\substack{T \in \mathscr{T} \\ x \in T}} T$ and set $\mathscr{R} := \{R(x) : x \in O^\#\}$.

Proposition 6.5. *$\mathscr{R}$ is a countable pairwise disjoint cover of $O^\#$.*

An outline of the proof of this proposition is as follows, the details of which we omit. Proceeding as in [22, Proposition 11.2], first, from the locally finite property of $\mathscr{Z}$ (Theorem 6.1), $R(x)$ is a finite intersection of elements of $\mathscr{T}$, that is, for every $R \in \mathscr{R}$, $|\{Z \in \mathscr{Z} : Z \supset R\}| < \infty$. Second, since $\mathscr{T}$ is countable and the fact that $T_{ii}^{us} = Z_i$, we conclude that $\mathscr{R}$ is a countable cover of $O^\#$, that is, $\bigcup_{R \in \mathscr{R}} R = O^\#$. Moreover, it is a countable pairwise disjoint cover of $O^\#$. The key point to destroy the non-trivial intersection is to prove that $R(x)$ is the equivalence class of x for the following equivalence relation on $O^\#$: for every $x, y \in O^\#$, define

$$x \sim y \text{ if and only if } \text{ for all } Z, Z' \in \mathscr{Z}, \quad \begin{pmatrix} x \in Z & \Leftrightarrow & y \in Z, \\ W^u(x, Z) \cap Z' \neq \varnothing & \Leftrightarrow & W^u(y, Z) \cap Z' \neq \varnothing, \\ W^s(x, Z) \cap Z' \neq \varnothing & \Leftrightarrow & W^s(y, Z) \cap Z' \neq \varnothing \end{pmatrix}.$$

So for every $x, y \in \bigcup_{R \in \mathscr{R}} R = O^\#$, either $R(x) = R(y)$ or $R(x) \cap R(y) = \varnothing$. Thus, $\mathscr{R}$ is a Bowen–Sinai refinement of $\mathscr{Z}$. For more details, see [22, Proposition 11.2].

By Definition 6.5 and the local finiteness property of $\mathscr{Z}$, the following inclusion relation and local finiteness properties for $\mathscr{R}$ are proved exactly as in [22, Lemma 11.3].

Lemma 6.4. *$\mathscr{R}$ is a locally finite refinement of $\mathscr{Z}$:*

(1) *for every $R \in \mathscr{R}$ and $Z \in \mathscr{Z}$, if $R \cap Z \neq \varnothing$, then $R \subset Z$;*
(2) *for every $Z \in \mathscr{Z}$, $|\{R \in \mathscr{R} : R \subset Z\}| < \infty$;*
(3) *for every $R \in \mathscr{R}$, $|\{Z \in \mathscr{Z} : Z \supset R\}| < \infty$.*

Now we follow [22, §11] to show that $\mathscr{R}$ is a Markov partition in the sense of Sinai [24].

Definition 6.6. For any $R \in \mathscr{R}$ and $x \in R$, we define the *u-fiber* and *s-fiber* of $x$inside R respectively by

$$W^u(x, R) := \bigcap_{\substack{T_{ij}^{\alpha\beta} \in \mathscr{T} \\ T_{ij}^{\alpha\beta} \supset R}} W^u(x, Z_i) \cap T_{ij}^{\alpha\beta} \quad \text{and} \quad W^s(x, R) := \bigcap_{\substack{T_{ij}^{\alpha\beta} \in \mathscr{T} \\ T_{ij}^{\alpha\beta} \supset R}} W^s(x, Z_i) \cap T_{ij}^{\alpha\beta}.$$

Proposition 6.3 implies that any two u-fibers (s-fibers) either coincide or are disjoint: suppose $R \in \mathscr{R}$ and $x, y \in R$, either $W^u(x, R)$, $W^u(y, R)$ are equal or they are disjoint, and similarly for $W^s(x, R)$ and $W^s(y, R)$. The proof is straightforward as in [**22**, Proposition 11.5(2)].

PROPOSITION 6.6. *Suppose $R \in \mathscr{R}$ and $x, y \in R$.*

(1) $W^u(x, R), W^s(x, R) \subset R$ *and* $W^u(x, R) \cap W^s(x, R) = x$.

(2) *The intersection $W^u(x, R) \cap W^s(y, R)$ consists of a single point and this point belongs to R. Denote it by $[x, y]_R$ and call the Smale bracket of x, y in R.*

(3) *Markov property: let $R_0, R_1 \in \mathscr{R}$. If $x \in R_0$ and $f(x) \in R_1$, then*

$$f(W^s(x, R_0)) \subset W^s(f(x), R_1) \quad \textit{and} \quad f^{-1}(W^u(f(x), R_1)) \subset W^u(x, R_0).$$

Proof. Part (1) is proved exactly as in [**22**, Proposition 11.5(1)]. Using Proposition 6.3, part (2) is a standard result which can be found in [**22**, Proposition 11.5(3)]. Along the proof, we also obtain that $[x, y]_R = [x, y]_Z$ for every $x, y \in R$ and all $Z \in \mathscr{Z}$ containing R. Finally, part (3) is the heart to obtain a Markov partition which is proved as in [**22**, Proposition 11.7], using Propositions 6.3, 6.4, and Lemmas 6.2, 6.3(2), and the equality $[x, y]_R = [x, y]_Z$ holds for every $x, y \in R$ and every $Z \in \mathscr{Z}$ containing R. □

We now construct a new symbolic coding of f, which is a *symbolic model* of f generated by the Markov partition $\mathscr{R}$ that satisfies Theorem 2.1. Let $\widehat{\mathscr{G}} = (\widehat{V}, \widehat{E})$ be the directed graph with vertex set $\widehat{V} = \mathscr{R}$ and edge set $\widehat{E} = \{R_1 \to R_2 : R_1, R_2 \in \mathscr{R}$ such that $R_1 \cap f^{-1}R_2 \neq \varnothing\}$. This is also a countable directed graph, and every vertex has finite degree because for every $R_0 \in \mathscr{R}$, if $R_1 \in \mathscr{R}$ satisfies $R_0 \to R_1$, by Definition 6.5 and the local finiteness property (Lemma 6.4(3)) for $\mathscr{R}$, there are finite points of $\mathscr{A}$, say z such that $R_0 \subset Z(z)$. Since $R_0 \to R_1$, we can take $x \in R_0 \cap f^{-1}(R_1) \subset Z(z)$, there exist by Definition 6.1 an ε-rppo $\underline{x} \in \Sigma^\#$ such that $x_0 = z$ and $\pi(\underline{x}) = x$. Then $f(x) \in R_1 \cap Z(x_1)$ and from the inclusion relation (Lemma 6.4(1)) for $\mathscr{R}$, $R_1 \subset Z(x_1)$ and such R_1 is finite (Lemma 6.4(2)) since $Z(x_1)$ is finite for fixed R_0 (Lemma 6.4(3)). Therefore,

$$|\{R_1 \in \mathscr{R} : R_0 \to R_1\}| \leq \sum_{\substack{z \in \mathscr{A} \\ Z(z) \supset R_0}} \sum_{\substack{z' \in \mathscr{A} \\ z \to z'}} |\{R_1 \in \mathscr{R} : R_1 \subset Z(z')\}| < \infty.$$

Let $(\widehat{\Sigma}, \widehat{\sigma})$ be the TMS induced by $\widehat{\mathscr{G}}$:

$$\widehat{\Sigma} := \{\{R_n\}_{n \in \mathbb{Z}} \in \mathscr{R}^{\mathbb{Z}} : R_n \to R_{n+1} \text{ for all } n \in \mathbb{Z}\},$$

where the $\widehat{\sigma}$ is the left shift and the element of $\widehat{\Sigma}$ is denoted by $\underline{R} := \{R_n\}_{n\in\mathbb{Z}}$. We equip $\widehat{\Sigma}$ with the metric $d(\underline{R}, \underline{S}) = \exp[-\min\{|n| : n \in \mathbb{Z} \text{ such that } R_n \neq S_n\}]$ as before. Since every vertex of $\widehat{\mathscr{G}}$ has finite degree, $\widehat{\Sigma}$ is locally compact. Define as before

$$\widehat{\Sigma}^{\#} := \{\underline{R} \in \widehat{\Sigma} : \text{there exists } R, S \in \mathscr{R}, \text{there exists } n_k, m_k \uparrow \infty \\ \text{such that } R_{n_k} = R \text{ and } R_{-m_k} = S \text{ for all } k \in \mathbb{Z}\}.$$

Again by the Poincaré recurrence theorem, every $\widehat{\sigma}-$invariant probability measure gives $\widehat{\Sigma}^{\#}$ full measure. Furthermore, every periodic point of $\widehat{\sigma}$ is in $\widehat{\Sigma}^{\#}$.

For $n \in \mathbb{Z}$ and a path $R_{-n} \to \cdots \to R_n$ on $\widehat{\mathscr{G}}$, $\bigcap_{i=-n}^{n} f^{-i}(R_i) \neq \varnothing$. This follows by induction, using the Markov property of $\mathscr{R}$ (Proposition 6.6(3)), as done originally by Adler, Weiss, and Sinai [**1**, **3**, **23**, **24**]. For more details, see [**22**, Lemma 12.1].

LEMMA 6.5. *For every finite path $R_{-n} \to \cdots \to R_n$ on $\widehat{\mathscr{G}}$, there exists an ε-rppo $\underline{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$ such that for every $i = -n, \ldots, n$, $R_i \subset Z(x_i)$, and*

$$\operatorname{diam}\bigg(\bigcap_{i=-n}^{n} f^{-i}(R_i)\bigg) \leq 8Q(x_n)\prod_{j=1}^{n}\|Df^{-1}|_{E^u(x_j)}\|^{1/2} + 8Q(x_{-n})\prod_{j=1}^{n}\|Df|_{E^s(x_{-j})}\|^{1/2}.$$

Moreover, for every $\underline{R} = \{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}^{\#}$, we have

$$\operatorname{diam}\bigg(\bigcap_{i=-N}^{N} f^{-i}(R_i)\bigg) \underset{N\to\infty}{\longrightarrow} 0.$$

Proof. Fix $x, y \in \bigcap_{i=-n}^{n} f^{-i}(R_i)$. By Definition 6.5, we can pick some $z \in \mathscr{A}$ such that $R_0 \subset Z(z) \in \mathscr{Z}$. Since $x \in R_0 \subset Z(z)$, there exists by Definition 6.1 an ε-rppo $\underline{x} \in \Sigma^{\#}$ such that $x_0 = z$, $\pi(\underline{x}) = x$. From $f^i(x) = f^i \circ \pi(\underline{x}) = \pi \circ \sigma^i(\underline{x}) \in Z(x_i)$ and $f^i(x) \in R_i$, we have $f^i(x) \in R_i \cap Z(x_i)$ for all $i = -n, \ldots, n$. So we have $R_i \subset Z(x_i)$ by inclusion relation Lemma 6.4(1) for all $i = -n, \ldots, n$. It follows that

$$\bigcap_{i=-n}^{n} f^{-i}(R_i) \subset \bigcap_{i=-n}^{n} f^{-i}[Z(x_i)],$$

and hence $y \in \bigcap_{i=-n}^{n} f^{-i}[Z(x_i)] \subset Z(x_0)$.

Since $x, y \in Z(x_0)$, we can let $z' := [x, y]_{Z(x_0)} \in Z(x_0)$. Notice that $f^i(x)$, $f^i(y) \in Z(x_i)$ for all $i = -n, \ldots, n$ and $x_0 \to \cdots \to x_n$, we have by Lemma 6.2 iterated forward n times

$$\begin{aligned} f^n(z') &= f^n([x, y]_{Z(x_0)}) \\ &= f^{n-1}([f(x), f(y)]_{Z(x_1)}) \\ &= \cdots = [f^n(x), f^n(y)]_{Z(x_n)} \\ &= V^u(f^n(x), Z(x_n)) \cap V^s(f^n(y), Z(x_n)), \end{aligned}$$

and thus $f^n(z') \in V^u(f^n(x), Z(x_n)) = V^u(\{x_{-k+n}\}_{k\geq 0})$.

Similarly, since $x_{-n} \to \cdots \to x_0$, we have by Lemma 6.2 iterated backward n times

$$\begin{aligned} f^{-n}(z') &= f^{-n}([x, y]_{Z(x_0)}) \\ &= f^{-n+1}([f^{-1}(x), f^{-1}(y)]_{Z(x_{-1})}) \\ &= \cdots = [f^{-n}(x), f^{-n}(y)]_{Z(x_{-n})} \\ &= V^u(f^{-n}(x), Z(x_{-n})) \cap V^s(f^{-n}(y), Z(x_{-n})), \end{aligned}$$

and thus $f^{-n}(z') \in V^s(f^{-n}(y), Z(x_{-n})) = V^s(\{x_{k-n}\}_{k\geq 0})$.

Now using Proposition 5.2(3), for $x, y \in \bigcap_{i=-n}^{n} f^{-i}(R_i)$, we have

$$\begin{aligned} d(x, y) &\leq d(x, z') + d(z', y) \\ &= d(f^{-n}(f^n(x)), f^{-n}(f^n(z'))) + d(f^n(f^{-n}(z')), f^n(f^{-n}(y))) \\ &\leq 8Q(x_n) \prod_{i=0}^{n-1} \|Df^{-1}|_{E^u(x_{-i+n})}\|^{1/2} + 8Q(x_{-n}) \prod_{i=0}^{n-1} \|Df|_{E^s(x_{i-n})}\|^{1/2} \\ &= 8Q(x_n) \prod_{j=1}^{n} \|Df^{-1}|_{E^u(x_j)}\|^{1/2} + 8Q(x_{-n}) \prod_{j=1}^{n} \|Df|_{E^s(x_{-j})}\|^{1/2}. \end{aligned}$$

Therefore, for every finite path $R_{-n} \to \cdots \to R_n$ on $\widehat{\mathscr{G}}$, there exists an ε-rppo $\underline{x} = \{x_n\}_{n\in\mathbb{Z}}$ such that

$$\operatorname{diam}\bigg(\bigcap_{i=-n}^{n} f^{-i}(R_i)\bigg) \leq 8Q(x_n) \prod_{j=1}^{n} \|Df^{-1}|_{E^u(x_j)}\|^{1/2} + 8Q(x_{-n}) \prod_{j=1}^{n} \|Df|_{E^s(x_{-j})}\|^{1/2}.$$

Finally, if $\underline{R} = \{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}^\#$, then there exists $R \in \mathscr{R}$, such that $R_m = R$ for infinitely many $m > 0$. From Lemma 6.4(3), for this R, $|\{Z \in \mathscr{Z} : Z \supset R\}|$ is finite, so by the pigeonhole principle, there exists $\bar{x} \in \mathscr{A}$, such that $x_m = \bar{x}$ for infinitely many $m > 0$. Similarly, there exists $\bar{y} \in \mathscr{A}$, such that $x_m = \bar{y}$ for infinitely many $m < 0$. Thus, for every $\underline{R} \in \widehat{\Sigma}^\#$, there exists an ε-rppo $\underline{x} = \{x_n\}_{n\in\mathbb{Z}}$ such that

$$\operatorname{diam}\bigg(\bigcap_{i=-N}^{N} f^{-i}(R_i)\bigg) \leq 8Q(x_N) \prod_{j=1}^{N} \|Df^{-1}|_{E^u(x_j)}\|^{1/2} + 8Q(x_{-N}) \prod_{j=1}^{N} \|Df|_{E^s(x_{-j})}\|^{1/2}$$

holds for every $N \geq 1$. So we have

$$\operatorname{diam}\bigg(\bigcap_{i=-N}^{N} f^{-i}(R_i)\bigg) \underset{N\to\infty}{\longrightarrow} 0. \qquad \square$$

For every $\underline{R} \in \widehat{\Sigma}^\#$, let $\widehat{\pi} : \widehat{\Sigma}^\# \to O$ be defined by

$$\{\widehat{\pi}(\underline{R})\} := \bigcap_{n\geq 0} \overline{\bigcap_{i=-n}^{n} f^{-i}(R_i)}.$$

Here, $\widehat{\pi}$ is well defined because $\overline{\bigcap_{i=-n}^{n} f^{-i}(R_i)}$ is a non-empty compact subset of M whose diameters tend to zero as $n \to \infty$ and the intersection $\bigcap_{n\geq 0} \overline{\bigcap_{i=-n}^{n} f^{-i}(R_i)}$ is a singleton.

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It is worthy to point out that we take the closures of $\bigcap_{i=-n}^{n} f^{-i}(R_i)$ because the R_i terms are not necessarily closed. *A priori*, the image of $\widehat{\pi}$ on $\widehat{\Sigma}^{\#}$ could be bigger than $\pi(\Sigma^{\#}) = O^{\#}$. Fortunately, this is not the case for Theorem 2.1(3), or see the proof of the following Theorem 6.2(3).

The triple $(\widehat{\Sigma}^{\#}, \widehat{\sigma}, \widehat{\pi})$ is the one that satisfies Theorem 2.1, where the non-injectivity of $\widehat{\pi}$ is analyzed using the following notion of affiliation, which was introduced by Sarig [**22**, §12.3].

Definition 6.7. $R, R' \in \mathscr{R}$ are called *affiliated* if there exist $Z, Z' \in \mathscr{Z}$ such that $Z \supset R$, $Z' \supset R'$, and $Z \cap Z' \neq \varnothing$. If this occurs, we write $R \sim R'$.

For $R \in \mathscr{R}$, define $N(R) := \{(R', z) \in \mathscr{R} \times \mathscr{A} : R' \sim R$ and $Z(z) \supset R'\}$. By the local-finiteness of $\mathscr{Z}$, and Lemma 6.4(2) and (3), $N(R)$ is finite. See the proof of [**22**, Lemma 12.7].

Definition 6.8. We say that $\underline{R} \approx \underline{R}'$ if $R_n \sim R'_n$ for all $n \in \mathbb{Z}$.

LEMMA 6.6. *If $\underline{R}, \underline{R}' \in \widehat{\Sigma}^{\#}$ and $\widehat{\pi}(\underline{R}) = \widehat{\pi}(\underline{R}')$, then $\underline{R} \approx \underline{R}'$.*

Proof. Suppose $\underline{R} = \{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}^{\#}$. By Definition 6.5, we can pick some $x_0 \in \mathscr{A}$ such that $R_0 \subset Z(x_0) \in \mathscr{Z}$. Since $R_0 \to R_1$, there exists a point $x \in R_0 \cap f^{-1}(R_1) \subset Z(x_0)$. So there exists by Definition 6.1 an ε-rppo $\underline{x}' \in \Sigma^{\#}$ such that $x'_0 = x_0$, $\pi(\underline{x}') = x$. Then $f(x) \in R_1 \cap Z(x'_1)$. From Lemma 6.4(1), we have $R_1 \subset Z(x'_1)$. Set $x_1 = x'_1$, so we get $x_0 \to x_1$. Inductively forwards, one can proceed this construction to give us a path $x_0 \to \cdots \to x_n$ on Σ satisfies $R_n \subset Z(x_n)$ for every $n \geq 0$. Similarly, repeating the procedure inductively backwards gives us another path $x_{-n} \to \cdots \to x_0$ on Σ satisfies $R_{-n} \subset Z(x_{-n})$ for every $n \geq 0$. Therefore, this procedure gives us a sequence $\{x_n\}_{n\in\mathbb{Z}} \in \Sigma$. We claim that $\{x_n\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$ for $\underline{R} \in \widehat{\Sigma}^{\#}$.

In fact, since $\underline{R} \in \widehat{\Sigma}^{\#}$, there exists $R \in \mathscr{R}$, such that $R_m = R$ for infinitely many $m > 0$. From Lemma 6.4(3), for this R, $|\{Z \in \mathscr{Z} : Z \supset R\}|$ is finite, so by the pigeonhole principle, there exists $z \in \mathscr{A}$, such that $x_m = z$ for infinitely many $m > 0$. Similarly, there exists $z' \in \mathscr{A}$, such that $x_m = z'$ for infinitely many $m < 0$. Thus, we construct an ε-rppo $\underline{x} \in \Sigma^{\#}$ for $\underline{R} \in \widehat{\Sigma}^{\#}$ such that

$$z := \widehat{\pi}(\underline{R}) \in \bigcap_{n\geq 0} \overline{\bigcap_{i=-n}^{n} f^{-i}(Z(x_i))},$$

that is, $f^n(z) \in \overline{Z(x_n)}$ for all $n \in \mathbb{Z}$.

Recall that every element $Z(x_n)$ is a subset of the much smaller set $\exp_{x_n}(T_{x_n}M(2/50Q(x_n)))$ (Proposition 4.1(1)), we have $d(f^n(z), x_n) \leq \operatorname{diam}(\overline{Z(x_n)}) = \operatorname{diam}(Z(x_n)) \leq \frac{1}{2}Q(x_n)$ and then $f^n(z) \in \exp_{x_n}(S_{x_n})$ for all $n \in \mathbb{Z}$. By Definition 5.2 and the shadowing lemma (Lemma 5.1), $\pi(\underline{x}) = z = \widehat{\pi}(\underline{R})$.

Analogously, for $\underline{R}' \in \widehat{\Sigma}^{\#}$, there exist $\{y_n\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$ such that $\pi(\underline{y}) = \widehat{\pi}(\underline{R})$. Since $\widehat{\pi}(\underline{R}) = \widehat{\pi}(\underline{R}') = z$, we have for every $n \in \mathbb{Z}$, $f^n(z) = \pi \circ \sigma^n(\underline{x}) = \pi \circ \sigma^n(\underline{y})$ and so $f^n(z) \in Z(x_n) \cap Z(y_n)$, proving that $R_n \sim R'_n$ for all $n \in \mathbb{Z}$. □

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THEOREM 6.2. *The following holds for all $\varepsilon > 0$ small enough.*
(1) $\widehat{\pi} \circ \widehat{\sigma} = f \circ \widehat{\pi}$ *on* $\widehat{\Sigma}^{\#}$.
(2) $\widehat{\pi} : \widehat{\Sigma}^{\#} \to O$ *is continuous.*
(3) $\widehat{\pi}|_{\widehat{\Sigma}^{\#}} : \widehat{\Sigma}^{\#} \to O^{\#}$ *is a finite-to-one surjective map.*

Proof. (1) For any $\underline{R} \in \widehat{\Sigma}^{\#}$, because f is a homeomorphism, we have

$$\begin{aligned}\{\widehat{\pi} \circ \widehat{\sigma}(\underline{R})\} &= \left\{ \bigcap_{n=0}^{\infty} \overline{f^{n}(R_{-n+1}) \cap \cdots \cap f^{-n}(R_{n+1})} \right\} = \left\{ \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=-n}^{n} f^{-i}(R_{i+1})} \right\} \\ &\supset \left\{ \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=-n-2}^{n} f^{-(i+1)+1}(R_{i+1})} \right\} = \left\{ f\left(\bigcap_{n=0}^{\infty} \overline{\bigcap_{i=-n-2}^{n} f^{-(i+1)}(R_{i+1})} \right) \right\} \\ &= \left\{ f\left(\bigcap_{n=0}^{\infty} \overline{\bigcap_{j=-n-1}^{n+1} f^{-j}(R_{j})} \right) \right\} = \{f \circ \widehat{\pi}(\underline{R})\}.\end{aligned}$$

(2) If otherwise, then there is a $\xi > 0$ so that for every $N > 0$, one can find $\underline{R}^{(N)}$, $\underline{S}^{(N)} \in \widehat{\Sigma}^{\#}$ with $R_i^{(N)} = S_i^{(N)}$ for all $i = -N, \ldots, N$, but $d(\widehat{\pi}(\underline{R}^{(N)}), \widehat{\pi}(\underline{S}^{(N)})) \geq \xi$. However, notice that $\widehat{\pi}(\underline{R}^{(N)}), \widehat{\pi}(\underline{S}^{(N)}) \in \overline{\bigcap_{i=-N}^{N} f^{-i}(R_i^{(N)})}$, and hence by Lemma 6.5, there exists an ε-rppo $\underline{x} = \{x_n^{(N)}\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$ such that

$$\begin{aligned}&d(\widehat{\pi}(\underline{R}^{(N)}), \widehat{\pi}(\underline{S}^{(N)})) \\ &\quad\leq \operatorname{diam}\left(\overline{\bigcap_{i=-N}^{N} f^{-i}(R_i^{(N)})} \right) \\ &\quad= \operatorname{diam}(\bigcap_{i=-N}^{N} f^{-i}(R_i^{(N)})) \\ &\quad\leq 8Q(x_N^{(N)}) \prod_{j=1}^{N} \|Df^{-1}|_{E^u(x_j^{(N)})}\|^{1/2} + 8Q(x_{-N}^{(N)}) \prod_{j=1}^{N} \|Df|_{E^s(x_{-j}^{(N)})}\|^{1/2} \xrightarrow[N\to\infty]{} 0.\end{aligned}$$

This leads to a contradiction.

(3) We first prove that $O^{\#} = \widehat{\pi}(\widehat{\Sigma}^{\#})$. The proof of $O^{\#} \subset \widehat{\pi}(\widehat{\Sigma}^{\#})$ is the same as [**22**, Theorem 12.5(3)]. So it suffices to establish the opposite direction $O^{\#} \supset \widehat{\pi}(\widehat{\Sigma}^{\#})$.

Suppose $\underline{R} = \{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}^{\#}$, proceeding as in the proof of Lemma 6.6, we can construct an ε-rppo $\underline{x} \in \Sigma^{\#}$ for $\underline{R} \in \widehat{\Sigma}^{\#}$ such that

$$z := \widehat{\pi}(\underline{R}) \in \bigcap_{n\geq 0} \overline{\bigcap_{i=-n}^{n} f^{-i}(Z(x_i))},$$

that is, $f^n(z) \in \overline{Z(x_n)}$ for all $n \in \mathbb{Z}$. Therefore, $d(f^n(z), x_n) \leq \operatorname{diam}(\overline{Z(x_n)}) = \operatorname{diam}(Z(x_n)) \leq \frac{1}{2}Q(x_n) \leq \frac{1}{2}d(x_n, \partial O)$, and we have

$$d(f^n(z), \partial O) \geq d(x_n, \partial O) - d(f^n(z), x_n) \geq \tfrac{1}{2}d(x_n, \partial O).$$

Thus, $z \in O^{\#}$, that is, $\widehat{\pi}(\widehat{\Sigma}^{\#}) \subset O^{\#}$.

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Finally, we prove $\widehat{\pi}|_{\widehat{\Sigma}^\#}$ is finite-to-one. The proof is an adaptation of [**21**, Theorem 5.6(5)] and [**6**] to our context. The original proof [**22**, Theorem 12.8] has a small error that was corrected in [**21**].

Suppose $x \in O^\#$, since $\widehat{\pi}(\widehat{\Sigma}^\#) = O^\#$, x has a $\widehat{\pi}$-preimage $\underline{R} \in \widehat{\Sigma}^\#$ such that $R_n = R$ for infinitely many positive n and $R_n = S$ for infinitely many negative n. Let $N := N(R)N(S)$ and suppose by way of contradiction that there are $N+1$ different $\underline{R}^{(0)}, \underline{R}^{(1)}, \ldots, \underline{R}^{(N)} \in \widehat{\Sigma}^\#$ with $\underline{R}^{(0)} = \underline{R}$ and $\widehat{\pi}(\underline{R}^{(m)}) = x$ for all $m = 0, 1, \ldots, N$. As produced in the proof of Lemma 6.6, we can construct for each $\underline{R}^{(m)}$ an ε-rppo $\underline{x}^{(m)} \in \Sigma^\#$ such that $\pi(\underline{x}^{(m)}) = \widehat{\pi}(\underline{R}^{(m)}) = x$, and $R_n^{(m)} \subset Z(x_n^{(m)})$ for all $n \in \mathbb{Z}$ and all $m = 0, 1, \ldots, N$.

Since the sequences $\{\underline{R}^{(m)}\}_{m=0}^N$ are distinct, there exists some $n_0 > 0$ such that

$$(R_{-n_0}^{(i)}, \ldots, R_{n_0}^{(i)}) \neq (R_{-n_0}^{(j)}, \ldots, R_{n_0}^{(j)}) \quad \text{for all } 0 \leq i, j \leq N \text{ with } i \neq j.$$

Denote by n^+ the first $n \geq n_0$ such that $R_n = R$, and by n^-, the first $n \leq -n_0$ such that $R_n = S$. Lemma 6.6 implies that there are at most $N = N(R)N(S)$ distinct choices of quadruple $(R_{n^+}^{(m)}, x_{n^+}^{(m)}; R_{n^-}^{(m)}, x_{n^-}^{(m)})$ for m. By the pigeonhole principle, there are $0 \leq k, l \leq N$ with $k \neq l$ such that $(R, x_{n^+,R}) := (R_{n^+}^{(k)}, x_{n^+}^{(k)}) = (R_{n^+}^{(l)}, x_{n^+}^{(l)})$, $(S, x_{n^-,S}) := (R_{n^-}^{(k)}, x_{n^-}^{(k)}) = (R_{n^-}^{(l)}, x_{n^-}^{(l)})$, and

$$(R_{n^-}^{(k)}, \ldots, R_{n^+}^{(k)}) \neq (R_{n^-}^{(l)}, \ldots, R_{n^+}^{(l)}).$$

Now we apply the *diamond argument* [**2**, Lemma 6.7] to finish the proof of part (3). Fix $x \in \bigcap_{i=n^-}^{n^+} f^{-i}(R_i^{(k)})$, $y \in \bigcap_{i=n^-}^{n^+} f^{-i}(R_i^{(l)})$, and define z_1, z_2 by the equalities

$$f^{n^+}(z_1) := [f^{n^+}(x), f^{n^+}(y)]_R \quad \text{and} \quad f^{n^-}(z_2) := [f^{n^-}(x), f^{n^-}(y)]_S.$$

These points are uniquely defined by Proposition 6.6(2) since $f^{n^+}(x), f^{n^+}(y) \in R_{n^+}^{(k)} = R_{n^+}^{(l)} = R$, $f^{n^-}(x), f^{n^-}(y) \in R_{n^-}^{(k)} = R_{n^-}^{(l)} = S$. We will obtain a contradiction once we show that $z_1 \neq z_2$ and $z_1 = z_2$.

First, note that $\{f^{n^+}(z_1)\} = W^u(f^{n^+}(x), R_{n^+}^{(k)}) \cap W^s(f^{n^+}(y), R_{n^+}^{(l)}) \subset W^u(f^{n^+}(x), R_{n^+}^{(k)})$, by the Markov property (Proposition 6.6(3)), we have $z_1 \in \bigcap_{i=n^-}^{n^+} f^{-i}(R_i^{(k)})$. Similarly, since $\{f^{n^-}(z_2)\} = W^u(f^{n^-}(x), R_{n^-}^{(k)}) \cap W^s(f^{n^-}(y), R_{n^-}^{(l)}) \subset W^s(f^{n^-}(y), R_{n^-}^{(l)})$, we have $z_2 \in \bigcap_{i=n^-}^{n^+} f^{-i}(R_i^{(l)})$. Therefore, we get $z_1 \neq z_2$ for $(R_{n^-}^{(k)}, \ldots, R_{n^+}^{(k)}) \neq (R_{n^-}^{(l)}, \ldots, R_{n^+}^{(l)})$.

We now show that $z_1 = z_2$. Since $f^{n^+}(z_1) \in R \subset Z(x_{n^+,R})$ and $f^{n^-}(z_2) \in S \subset Z(x_{n^-,S})$, there are $\underline{x} = \{x_n\}_{n\in\mathbb{Z}}$, $\underline{y} = \{y_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ with $x_{n^+} = x_{n^+,R}$, $y_{n^-} = x_{n^-,S}$ such that $z_1 = \pi(\underline{x})$, $z_2 = \pi(\underline{y})$. Define $\underline{z}' = \{z_n'\}_{n\in\mathbb{Z}} \in \Sigma$ by

$$z_n' = \begin{cases} y_n, & n \leq n^-, \\ x_n^{(k)}, & n^- \leq n \leq n^+, \\ x_n, & n \geq n^+. \end{cases}$$

Since both $\underline{x}, \underline{y} \in \Sigma^\#$, also $\underline{z}' \in \Sigma^\#$. We claim that $z_1 = \pi(\underline{z}') = z_2$.

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In practice, note again that from $f^{n^+}(z_1) \in W^u(f^{n^+}(x), R^{(k)}_{n^+})$ and the Markov property (Proposition 6.6(3)), we have $f^{n^-}(z_1) \in W^u(f^{n^-}(x), R^{(k)}_{n^-})$. Naturally, $f^{n^-}(z_2) \in W^u(f^{n^-}(x), R^{(k)}_{n^-})$. Hence, we have by Definition 6.6,

$$f^{n^-}(z_1), f^{n^-}(z_2) \in W^u(f^{n^-}(x), R^{(k)}_{n^-}) \subset W^u(f^{n^-}(x), Z(x^{(k)}_{n^-})) = W^u(f^{n^-}(x), Z(z'_{n^-})).$$

By the symbolic Markov property (Proposition 6.4), for all $n \leq n^-$, Proposition 5.2(2) and (4) imply that

$$f^n(z_1), f^n(z_2) \in W^u(f^n(x), Z(z'_n)) \subset \exp_{z'_n}(S_{z'_n}) \quad \text{for all } n \leq n^-.$$

Similarly, we obtain that for all $n \geq n^+$,

$$f^n(z_1), f^n(z_2) \in W^s(f^n(x), Z(z'_n)) \subset \exp_{z'_n}(S_{z'_n}) \quad \text{for all } n \geq n^+.$$

The next thing is to fix $n^- \leq n \leq n^+$. Observe that we have $f^n(z_1), f^n(z_2) \in R^{(k)}_n \cup R^{(l)}_n \subset Z(x^{(k)}_n) \cup Z(x^{(l)}_n)$. Since $\pi(\underline{x}^{(k)}) = \pi(\underline{x}^{(l)}) = x$, we get $f^n(x) \in Z(x^{(k)}_n) \cap Z(x^{(l)}_n)$. Then Lemma 6.3(1) implies that $Z(x^{(k)}_n) \cup Z(x^{(l)}_n) \subset \exp_{x^{(k)}_n}(S_{x^{(k)}_n})$. Recalling $x^{(k)}_n = z'_n$ for all $n^- \leq n \leq n^+$, therefore,

$$f^n(z_1), f^n(z_2) \in \exp_{z'_n}(S_{z'_n}), \; n^- \leq n \leq n^+.$$

In conclusion, for this ε-rppo $\underline{z}' \in \Sigma^\#$, $f^n(z_1), f^n(z_2) \in \exp_{z'_n}(S_{z'_n})$ for all $n \in \mathbb{Z}$. By the uniqueness of shadowing point, $z_1 = z_2$. This leads to a contradiction. □

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