Canad. Math. Bull. Vol. **24** (2), 1981

ON GLEASON'S DEFINITION OF QUADRATIC FORMS

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§1. **Introduction.** Suppose R is a commutative ring with identity. Let M be an R-module, and suppose f is a function from M to R. How do we characterize the property that f be a quadratic form? One approach is in terms of bilinear forms as follows.

DEFINITION (Jacobson [2, Definition 6.1]). $f: M \to R$ is a *quadratic form* if, and only if, the following conditions are satisfied:

(i) f is *quadratically homogeneous*; that is

(1) $$\forall a \in R\ \forall x \in M \qquad f(ax) = a^2 f(x)$$

(ii) $F(x, y)$ is *bilinear* on $M \times M \to R$, where

(2) $$\forall x, y \in M \qquad F(x, y) = f(x+y) - f(x) - f(y).$$

In [1] Gleason proved: Suppose that R is a field, not of characteristic 2, and not the field of 3 elements. Let M be a vector space over R. If $f: M \to R$ satisfies $f(0) = 0$, and $\forall x, y \in M, \forall a \in R$

(3) $$f(ax+y) - f(ax-y) = a[f(x+y) - f(x-y)],$$

then f is a quadratic form.

As a routine computation with bilinear forms shows, all quadratic forms f satisfy $f(0) = 0$, and (3). So, at least for the fields delineated in the hypothesis of Gleason's theorem, equation (3) characterizes quadratic forms. In this paper Gleason's theorem is generalized as follows.

THEOREM. *Suppose R is a commutative ring with identity. In order that every function f, with domain an R-module, which satisfies $f(0) = 0$ and (3), be a quadratic form it is necessary and sufficient that 2 be a unit of R and the only element $k \in R$ having the property that $(a^4 - a^2)k = 0$ for all $a \in R$ is $k = 0$.*

REMARK. If, in addition to 2 being a unit in R, 3 is not a zero divisor in R then $(a^4 - a^2)k = 0$ implies $k = 0$. (Just take $a = 2$.)

It is easy to see that Gleason's theorem is a consequence of the above result. The necessity of the condition on R is proved in §2, and the sufficiency (which is more difficult) in §3.

Received by the editors June 29, 1979 and, in revised form, November 12, 1979.

§2. **Proof of the necessity.**

LEMMA 1. *If 2 is not a unit in R there is an R-module M and a function $f: M \to R$ which satisfies $f(0)=0$, and (3) which is not a quadratic form.*

Proof. Take $M=R^3$ and define

$$f(a, b, c)=\begin{cases}0 & \text{if } \ a \in 2R \ \text{ and } \ b \in 2R \ \text{ and } \ c \in 2R \\ 1 & \text{otherwise}\end{cases}$$

Then $f(0,0,0)=0$, and f satisfies (3) since $f(ax+y)=f(ax-y)$ for all $a \in R$, $x, y \in R^3$. However F is not bi-additive and so not bilinear. For, as is easily verified, $F((1,1,0),\ (0,0,1))-F((1,0,0),\ (0,0,1))-F((0,1,0),\ (0,0,1))=1$, not 0 as it would be were F additive in the first variable.

LEMMA 2. *Suppose 2 is a unit in R, and suppose there is a non-zero element $k \in R$ such that $(a^4-a^2)k=0$, for all $a \in R$. Then there is an R-module M, and a function $f: M \to R$ which satisfies $f(0)=0$, and (3) which is not a quadratic form.*

Proof. Let $M=R^2$, and let $p, q: M \to R$ be the canonical projections. Let k be as above. Set $f(x)=kp(x)^2q(x)^2$. Then $f(0)=0$. And

$$f(ax+y)-f(ax-y)=4akX(a^2Y+Z)$$

where $X=p(x)q(y)+p(y)q(x)$, $Y=p(x)q(x)$, $Z=p(y)q(y)$. Thus f satisfies (3) if, and only if, $4a^3kXY=4akXY$. But the identity(1) $(a-1)[(a+1)^4-(a+1)^2]-(a+3)[a^4-a^2]=2a^3-2a$ and the definition of k shows that $2a^3k=2ak$ for all $a \in R$. Thus f satisfies (3). But f is not a quadratic form: it does not satisfy the well-known parallelogram law

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{4}$$

Just take $x=(1,0)$, $y=(0,1)$, in (4). Then the left-hand side is $2k$, and $2k \neq 0$ since 2 is a unit and $k \neq 0$ whereas the right-hand side is 0.

§3. **Proof of the sufficiency.** In this section we assume 2 is a unit in R and the only element annihilated by all a^4-a^2 is 0. As usual M denotes an R-module.

LEMMA 3. *If $f: M \to R$ satisfies $f(0)=0$, and equation (3) then f is quadratically homogeneous.*

Proof. Letting $x=0$, and $a=2$ in (3) we deduce that $f(y)=f(-y)$, for all

(1) Supplied by the referee.

$y \in M$. Next let $y = ax$ in (3), to deduce that, using $f(x-ax) = f(ax-x)$,

$$\begin{aligned} f(2ax) &= a[f(x+ax)-f(x-ax)] \\ &= a[f(ax+x)-f(ax-x)] \\ &= a^2[f(2x)-f(0)]. \end{aligned}$$

Thus, for all $a \in R$, and for all $x \in M$, we have

$$f(2ax) = a^2 f(2x). \tag{5}$$

But 2 is a unit of R. So (5) is equivalent to (1).

The following auxiliary function is the key to the success of the proof. Let $f: M \to R$ be given. Define

$$P(x, y) = f(x+y) + f(x-y) - 2f(x) - 2f(y). \tag{6}$$

It is well known that when 2 is not a zero-divisor in the ring (see e.g. Gleason [1]) that F is bi-additive if, and only if, P is identically 0, which is, of course, the parallelogram law. [See e.g. (3)]

LEMMA 4. *If $f: M \to R$ is quadratically homogeneous then $P(ax, ay) = a^2 P(x, y)$, for all $a \in R$, all $x, y \in M$.*

Proof. Immediate.

LEMMA 5. *Let $y \in M$ be fixed. Define $\phi: M \to R$ by $\phi(x) = P(x, y)$, for all $x \in M$. If $f: M \to R$ satisfies $f(0) = 0$ and* (3) *then $\phi(0) = 0$ and ϕ satisfies* (3).

Proof. That $\phi(0) = 0$ is easy. Let $a \in R$, $x, z \in M$. Then

$$\begin{aligned} \phi(ax+z) - \phi(ax-z) &= f(ax+z+y) + f(ax+z-y) - f(ax-z-y) \\ &\quad - f(ax-z+y) - 2(f(ax+z) - f(ax-z)) \\ &= f(ax+z+y) - f(ax-(z+y)) \\ &\quad + f(ax+z-y) - f(ax-(z-y)) \\ &\quad - 2a(f(x+z) - f(x-z)). \end{aligned}$$

Hence, using the fact that f satisfies (3) again

$$\begin{aligned} \phi(ax+z) - \phi(ax-z) &= a(f(x+z+y) - f(x-z-y)) \\ &\quad + a(f(x+z-y) - f(x-z+y)) \\ &\quad - 2a(f(x+z) - f(x-z)) \\ &= a(\phi(x+z) - \phi(x-z)); \end{aligned}$$

which shows that ϕ satisfies (3).

LEMMA 6. *If $f(0) = 0$, and f satisfies* (3) *then $P(ax, y) = a^2 P(x, y)$.*

https://doi.org/10.4153/CMB-1981-036-4 Published online by Cambridge University Press

Proof. In the notation of Lemma 5 this says $\phi(ax)=a^2\phi(x)$; which is a consequence of Lemma 3, since ϕ satisfies (3).

LEMMA 7. *If* $f(0)=0$, *and* f *satisfies* (3) *then* $P(x, y)=0$, *for all* $x, y \in M$.

Proof. Let $x, y \in M$ be fixed. Then, for all $a \in R$, $a^2P(x, y)=P(ax, ay)$ (Lemma 4) $= a^2P(x, ay)$ (Lemma 6) $= a^2P(ay, x)$ (symmetry of P) $= a^4P(y, x) = a^4P(x, y)$. Hence, for all $a \in R$ $(a^4-a^2)P(x, y)=0$. So, by our assumption about the nature of R, *it follows that* $P(x, y)=0$; as claimed.

We now have all the pieces available to prove that if $f(0)=0$, and f satisfies (3), then f is a quadratic form. For as we have shown in Lemma 3, f is quadratically homogeneous. Moreover, F is bi-additive since f satisfies the parallelogram law $P=0$, (see [3]). We show finally that F is bi-homogeneous.

Define following Gleason, $H(x, y)=f(x+y)-f(x-y)$. Then (3) may be restated as $H(ax, y)=aH(x, y)$. Furthermore, $H(x, y)+P(x, y)=2F(x, y)$, so the fact that $P=0$ implies that $H=2F$, and so $2F(ax, y)=2aF(x, y)$. Finally, since 2 is a unit of R we deduce that F is homogeneous in the first variable, and so by the symmetry it is bi-homogeneous. This completes the proof that F is bilinear, and so f is a quadratic form.

I would like to thank the referee for his helpful comments and detailed suggestions.

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