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Nagoya Math. J.
Vol. 121 (1991), 185–194

A CHARACTERIZATION OF WHITE NOISE TEST FUNCTIONALS

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§ 1. Introduction and main result

In a recent paper [PS 89], two of the present authors have found a characterization of a certain space $(\mathcal{S})^*$ of generalized functionals of white noise, i.e. generalized functionals on $\mathcal{S}'(\mathbb{R})$ equipped with the σ-algebra $\mathcal{B}$ generated by its cyclinder sets and with the white noise measure μ given by

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle x, \xi\rangle) d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_2^2\right),$$

for $\xi \in \mathcal{S}(\mathbb{R})$. Here, $|\cdot|_2$ denotes the norm of $L^2(\mathbb{R})$, and $\langle\cdot, \cdot\rangle$ dual pairing. Below, we shall shortly recall the construction of the space $(\mathcal{S})^*$ as the dual of a space $(\mathcal{S})$ of "smooth" functionals on $\mathcal{S}'(\mathbb{R})$. The characterization mentioned above is of considerable power: it provides an extremely convenient way to decide whether a certain given functional is an element in $(\mathcal{S})^*$. This has been shown in [PS 89] for a number of examples (especially for certain measures on $\mathcal{S}'(\mathbb{R})$). The purpose of the present note is to give a similar characterization for the elements $(\mathcal{S})$ of test functionals. For notation, definitions, more background and references, we refer the reader to [PS 89].

Let $\Gamma(A)$ denote the second quantization of the self-adjoint $L^2(\mathbb{R})$-operator A which on $\mathcal{S}(\mathbb{R})$ is defined as

$$A\xi(u) = -\xi''(u) + (1 + u^2)\xi(u), \qquad \xi \in \mathcal{S}(\mathbb{R}),\ u \in \mathbb{R}.$$

Let $\mathcal{P}$ denote the algebra of smooth polynomials on $\mathcal{S}'(\mathbb{R})$, i.e. $\mathcal{P}$ is generated by the random variables $X_\xi = \langle\cdot, \xi\rangle$, $\xi \in \mathcal{S}(\mathbb{R})$. For $p \geq 0$, let $\mathcal{S}_p(\mathbb{R})$ denote the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm $|\xi|_{2,p} = |A^p\xi|_2$, and let $(\mathcal{S})_p$ denote the completion of $\mathcal{P}$ with respect to the norm

Received May 17, 1990.

$\|\varphi\|_{2,p} = \|\Gamma(A)^p\varphi\|_2$, where $\|\cdot\|_q$, $q \geq 1$, denotes the norm of $L^q(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu) \equiv (L^q)$. The corresponding dual spaces are denoted by $\mathcal{S}_{-p}(\mathbb{R})$ and $(\mathcal{S})_{-p}$, respectively. Then we define the space $(\mathcal{S})$ of test functionals of white noise as the projective limit of the family $\{(\mathcal{S})_p, p \in \mathbb{N}_0\}$, and denote by $(\mathcal{S})^*$ its dual. Note that $\mathcal{S}(\mathbb{R})$ is the projective limit of the family $\{\mathcal{S}_p(\mathbb{R}), p \in \mathbb{N}_0\}$, cf. e.g. [Si 71]. Every element $\varphi \in (L^2)$ admits a chaos decomposition

$$\varphi = \sum_{n=0}^{\infty} I_n(F^{(n)}), \tag{1.1}$$

where $F^{(n)} \in \widehat{L^2(\mathbb{R}^n)}$, $\hat{\cdot}$ denoting symmetrization, and $I_n(F^{(n)})$ is the multiple Wiener integral of $F^{(n)}$ of order n. It can be shown that $\varphi \in (\mathcal{S})$ implies that for every $n \in \mathbb{N}$, $F^{(n)}$ has a version belonging to $\widehat{\mathcal{S}(\mathbb{R}^n)}$. Moreover, $\varphi \in (L^2)$ corresponding to $(F^{(n)}; n \in \mathbb{N}_0)$ belongs to $(\mathcal{S})$ if and only if for every $p \in \mathbb{R}$,

$$\|\varphi\|_{2,p}^2 = \sum_{n=0}^{\infty} n!|(A^{\otimes n})^p F^{(n)}|_2^2 . \tag{1.2}$$

The norms under the last sum will also be denoted by $|F^{(n)}|_{2,p}$.

On (L^q), $q > 1$, and for $\xi \in \mathcal{S}(\mathbb{R})$, we define the $\mathcal{S}$-transform (cf. also [KT 80]) of an element φ by

$$\mathcal{S}\varphi(\xi) = \int \varphi(x + \xi) d\mu(x) .$$

Note that we have the formula

$$\mathcal{S}\varphi(\xi) = \int \varphi(x) {:} \exp\langle x, \xi\rangle {:} \, d\mu(x) ,$$

where we have set

$$:\exp\langle x, \xi\rangle := \exp\left(\langle x, \xi\rangle - \frac{1}{2}|\xi|_2^2\right) .$$

If $\varphi \in (L^2)$ has a chaos decomposition as in (1.1) then

$$\mathcal{S}\varphi(\xi) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}(u)\xi^{\otimes n}(u)du , \tag{1.3}$$

as an easy direct computation shows.

Since for all $\lambda \in \mathbb{C}$ and $\xi \in \mathcal{S}(\mathbb{R})$, $:\exp\lambda\langle \cdot, \xi\rangle : \in (\mathcal{S})$, we may extend the $\mathcal{S}$-transform consistently to $(\mathcal{S})^*$ by setting for $\Phi \in (\mathcal{S})^*$

$$\mathcal{S}\Phi(\xi) = \langle \Phi, :\exp\langle \cdot, \xi\rangle : \rangle . \tag{1.4}$$

It is not hard to see that for all $\Phi \in (\mathcal{S})^*$, and $\xi \in \mathcal{S}(\mathbb{R})$, the mapping $\lambda \mapsto \mathcal{S}\Phi(\lambda\xi)$ has an entire analytic extension to $\mathbb{C}$. Clearly, it is given by

$$z \longmapsto \langle \Phi, :\exp z\langle \cdot, \xi\rangle : \rangle, \qquad z \in \mathbb{C}.$$

In the following we shall study analytic properties of this function. To this end we introduce the following notions for mappings

$$\begin{aligned} f\colon \mathbb{C} \times \mathcal{S}(\mathbb{R}) &\longrightarrow \mathbb{C} \\ (z, \xi) &\longmapsto f(z, \xi)\,. \end{aligned}$$

For $R \geq 0$, and $\xi \in \mathcal{S}(\mathbb{R})$ set

$$M(R, \xi) = \sup_{z \in \mathbf{C}, |z| = R} |f(z, \xi)|\,.$$

Also, we shall use the notation

$$B_p^* = \{\xi \in \mathcal{S}(\mathbb{R});\ |\xi|_{2,-p} \leq 1\}\,,$$

for real p.

DEFINITION 1.1.

(a) f is called of *p-order* ρ, $\rho \geq 0$, if and only if

$$\limsup_{R \to \infty} \frac{\ln \ln M(R, \xi)}{\ln R} = \rho\,,$$

uniformly on B_p^*.

(b) Assume that f is of p-order ρ, $\rho \geq 0$. f is called of *p-type* τ, $\tau \in [0, +\infty]$, if and only if

$$\limsup_{R \to \infty} R^{-\rho} \ln M(R, \xi) = \tau\,,$$

uniformly on B_p^*.

(c) f is called of *p-growth* (ρ, τ), if and only if f is of p-order not exceeding ρ, and if of p-order ρ, then f is of p-type not exceeding τ.

Assume that F is a *ray entire function* on $\mathcal{S}(\mathbb{R})$, i.e. for every $\xi, \eta \in \mathcal{S}(\mathbb{R})$, the mapping $\lambda \mapsto F(\lambda\xi + \eta)$ has an entire analytic continuation. The function $z \mapsto F(z\xi)$ is denoted by $f(z, \xi)$. Then we shall use the notions *p-order*, *p-type* and *p-growth* also the function F.

In [PS 89] a ray entire F of $(-p)$-growth $(2, \tau)$, for some $p \geq 0$, and $\tau \geq 0$, was named a *U-functional*. The main result of [PS 89] was the fact that any element in $(\mathcal{S})^*$ has an $\mathcal{S}$-transform which is a U-functional, and

https://doi.org/10.1017/S0027763000003469 Published online by Cambridge University Press

conversely, for any U-functional F there is a unique element $\Phi \in (\mathcal{S})^*$ with $\mathcal{S}\Phi(\xi) = F(\xi)$.

The main result of the present paper is the following theorem.

THEOREM 1.2. *If $\varphi \in (\mathcal{S})$ then $\mathcal{S}\varphi$ is ray entire and for every $p \geq 0$ of p-growth (2, 0). Conversely, if F is ray entire on $\mathcal{S}(\mathbb{R})$, and for every $p \geq 0$ F is of p-growth (2, 0), then there is a unique element $\varphi \in (\mathcal{S})$ whose $\mathcal{S}$-transform is equal to F.*

The plan of the paper is as follows. Section 2 contains the proof of Theorem 1.2. In Section 3 we discuss some of consequences and examples.

ACKNOWLEDGEMENT. Two of us (J.P. and L.S.) acknowledge gratefully the warm hospitality of Professor T. Hida and the Department of Mathematics at Nagoya University.

§ 2. Proof of Theorem 1.2

Assume that F is ray entire and of p-order $\rho \geq 0$. If F is of p-type 0, we have by definition

$$\limsup_{R\to\infty} R^{-\rho} \ln M(R, \xi) = 0$$

uniformly on B_p^* and consequently

$$\lim_{R\to\infty} R^{-\rho} \ln M(R, \xi) = 0$$

uniformly on B_p^*. But this implies that for every $\varepsilon > 0$ we have

$$\lim_{R\to\infty} M(R, \xi) e^{-\varepsilon R^\rho} = 0\,, \tag{2.1}$$

uniformly on B_p^*. On the other hand, if (2.1) holds uniformly on B_p^*, then it entails that F is of p-type 0. Thus we have proved the following result.

LEMMA 2.1. *F is of p-growth (2, 0) if and only if for every $\varepsilon > 0$ and $\xi \in \mathcal{S}(\mathbb{R})$,*

$$\lim_{R\to\infty} M(R, \xi) e^{-\varepsilon R^2} = 0\,,$$

uniformly on B_p^*.

Now we prove the first statement of Theorem 1.2.

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LEMMA 2.2. *If $\varphi \in (\mathcal{S})$ then $\mathcal{S}\varphi$ is for every $p \in \mathbb{N}$ of p-growth (2, 0).*

Proof. Let $\varphi \in (\mathcal{S})$ and $\xi \in \mathcal{S}(\mathbb{R})$. Denote by $z \mapsto f(z, \xi)$ the entire extension of $\lambda \mapsto \mathcal{S}\varphi(\lambda\xi)$. The self-adjointness of $\Gamma(A)$ and Schwarz' inequality provide for every $q \geq 1$, and $z \in \mathbb{C}$, the following estimation.

$$\begin{aligned} |f(z, \xi)| &= |(\varphi, : e^{z\langle \cdot, \xi\rangle} :)| \\ &= |(\Gamma(A)^q \varphi, \Gamma(A)^{-q} : e^{z\langle \cdot, \xi\rangle} :)| \\ &= |(\Gamma(A)^q \varphi, : e^{z\langle \cdot, A^{-q}\xi\rangle} :)| \\ &\leq \|\varphi\|_{2,q} \| : e^{z\langle \cdot, A^{-q}\xi\rangle} : \|_2 . \end{aligned}$$

The last norm can easily be evaluated:

$$\| : e^{z\langle \cdot, A^{-q}\xi\rangle} : \|_2 = \exp\left(\frac{1}{2}|z|^2 |\xi|^2_{2,-q}\right).$$

Therefore we have the bound

$$|f(z, \xi)| \leq \|\varphi\|_{2,q} \exp\left(\frac{1}{2}|z|^2 |\xi|^2_{2,-q}\right),$$

which is valid for all $z \in \mathbb{C}$ and all $q \in \mathbb{R}$. Clearly, this gives the estimate

$$M(R, \xi) \leq \|\varphi\|_{2,q} \exp\left(\frac{1}{2}R^2 |\xi|^2_{2,-q}\right),$$

for all $R \geq 0$, and all real q.

Now choose q large enough so that for given $p \in \mathbb{N}$, and given $\varepsilon > 0$, $2^{2(p-q)} < \varepsilon$. Then we find for $\xi \in \mathcal{S}(\mathbb{R})$ with $|\xi|_{2,-p} \leq 1$

$$\begin{aligned} M(R, \xi) e^{-\varepsilon R^2} &\leq \|\varphi\|_{2,q} \exp\left[-\left(\varepsilon - \frac{1}{2}|A^{p-q}A^{-p}\xi|^2_2\right)R^2\right] \\ &\leq \|\varphi\|_{2,q} e^{-1/2\,\varepsilon R^2}, \end{aligned}$$

since for $q > p$ we have $\|A^{p-q}\| \leq 2^{p-q}$. Obviously this tends uniformly on B_p^* to zero. Thus an application of Lemma 2.1 concludes the proof. □

The proof of the second statement of Theorem 1.2 will be done in a sequence of lemmas.

From now on assume that F is a ray entire function on $\mathcal{S}(\mathbb{R})$, which is of p-growth (2, 0). Also denote

$$f(z, \xi) = F(z\xi), \qquad z \in \mathbb{C},\ \xi \in \mathcal{S}(\mathbb{R}).$$

The power series expansion of $f(\cdot, \xi)$ will be denoted as follows.

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$$f(z, \xi) = \sum_{n=0}^{\infty} z^n f_n(\xi) .$$

The next result has been shown in [PS 89].

LEMMA 2.3. *Let F be ray entire and of p-growth $(2, 0)$. Then F is infinitely often Gâteaux differentiable in every direction of $\mathcal{S}(\mathbb{R})$. Moreover, for all $n \in \mathbb{N}$, $\xi \in \mathcal{S}(\mathbb{R})$*

$$f_n(\xi) = \frac{1}{n!}(D_\xi^n F)(0) ,$$

where D_ξ denotes the Gâteaux derivative in direction ξ. In particular, $f^{(n)}(\xi)$ is homogeneous in ξ of degree n.

LEMMA 2.4. *If F is of p-growth $(2, 0)$, then for all $K > 0$ there is $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, and for all $\xi \in \mathcal{S}(\mathbb{R})$*

$$|f_n(\xi)| \leq e^{-Kn} n!^{-1/2} |\xi|_{2,-p}^n . \tag{2.2}$$

Proof. Assume for the moment that $|\xi|_{2,-p} = 1$. It follows from Lemma 2.1 that for every $\varepsilon > 0$

$$\lim_{R \to \infty} M(R, \xi) e^{-\varepsilon R^2} = 0 ,$$

uniformly on B_p^*.

Let $K_1 > 0$ and choose $\varepsilon = e^{-2K_1}$. Choose R_0 large enough so that for all $R \geq R_0$

$$M(R, \xi) e^{-\varepsilon R^2} \leq 1 .$$

Note that R_0 is independent of $\xi \in B_p^*$. Let $R = \max(R_0, n^{1/2} e^{K_1})$. Then for n_0 large enough we have $R = n^{1/2} e^{K_1}$ for all $n \geq n_0$. Cauchy's theorem gives the estimate

$$|f_n(\xi)| \leq (n^{1/2} e^{K_1})^{-n} e^n ,$$

for all $n \geq n_0$. Stirling's theorem shows that

$$|f_n(\xi)| \leq n!^{-1/2} e^{-(K_1 - K_2)n} ,$$

where K_2 is a certain constant. Choose K_1 above such that $K_1 - K_2 = K$ and inequality (2.2) is proved for ξ with $|\xi|_{2,-p} = 1$.

For general $\xi \in \mathcal{S}(\mathbb{R})$ inequality (2.2) follows now by homogeneity of f_n. □

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Next we construct from f_n by polarization an n-linear form $F^{(n)}$ on $\mathcal{S}(\mathbb{R})$. For $\xi_1, \cdots, \xi_n \in \mathcal{S}(\mathbb{R})$ we set

$$F^{(n)}(\xi_1, \cdots, \xi_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{l_1 < \cdots < l_k} f_n(\xi_{l_1} + \cdots + \xi_{l_k}) . \tag{2.3}$$

If we combine Lemma 2.4 with formula (2.3) we easily obtain the following estimate (an analogous result is proved in detail in [PS 89]).

LEMMA 2.5. *For every $K > 0$ there is $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, and all $\xi_1, \cdots, \xi_n \in \mathcal{S}(\mathbb{R})$*

$$|F^{(n)}(\xi_1, \cdots, \xi_n)| \leq e^{-Kn} n!^{-1/2} \prod_{k=1}^{n} |\xi_k|_{2,-p} . \tag{2.4}$$

Let $\mathcal{S}_{-p}(\mathbb{R}^n)$ denote the completion of $\mathcal{S}(\mathbb{R}^n)$ under the norm $|g|_{2,-p} \equiv |(A^{\otimes n})^{-p} g|_2$. Note that the set of elements $\{\otimes_{k=1}^n \xi_k ; \xi_k \in \mathcal{S}(\mathbb{R})\}$ is total in $\mathcal{S}_{-p}(\mathbb{R}^n)$. Then inequality (2.4) tells us that the n-linear form $F^{(n)}$, which we may consider as a linear form on $\mathcal{S}(\mathbb{R})^{\otimes n}$, has a continuous extension to $\mathcal{S}_{-p}(\mathbb{R}^n)$. We denote this extension by the same symbol. Therefore, we may now consider $F^{(n)}$ as an element in $\mathcal{S}_p(\mathbb{R}^n)$, the dual of $\mathcal{S}_{-p}(\mathbb{R}^n)$. The next lemma estimates the $|\cdot|_{2,p}$-norm of $F^{(n)}$ under a slightly stronger hypothesis.

LEMMA 2.6. *Assume that F is of $(p+q)$-growth $(2, 0)$ with $q > 1/2$. Then for all $K > 0$ there is $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies*

$$|F^{(n)}|_{2,p} \leq n!^{-1/2} e^{-Kn} .$$

Proof. Note that

$$|F^{(n)}|_{2,p}^2 = \sum_{k_1, \cdots, k_n = 0}^{\infty} |F^{(n)}(A^p e_{k_1}, \cdots, A^p e_{k_n})|^2 ,$$

where $\{e_k ; k \in \mathbb{N}_0\}$ is any CONS of $L^2(\mathbb{R})$ in the domain of A^p. Choose e_k to be the k-th Hermite function and note that $Ae_k = 2(k+1)e_k$, $k \in \mathbb{N}_0$. Then we know that for any $K_1 > 0$ there exists n_0 (depending on K_1, p, q) so that for all $n \geq n_0$ we have

$$\begin{aligned} |F^{(n)}(A^p e_{k_1}, \cdots, A^p e_{k_n})| &\leq n!^{-1/2} e^{-K_1 n} \prod_{l=1}^{n} |A^p e_{k_l}|_{2,-(p+q)} \\ &= n!^{-1/2} e^{-K_1 n} \prod_{l=1}^{n} (2(k_l + 1))^{-q} . \end{aligned}$$

Therefore,

$$\sum_{k_1,\cdots,k_n=0}^{\infty} |F^{(n)}(A^p e_{k_1}, \cdots, A^p e_{k_n})|^2 \leq n!^{-1} e^{-2K_1 n} 2^{-2qn} \left(\sum_{k=0}^{\infty} (k+1)^{-2q} \right)^n$$
$$\leq n!^{-1} e^{-2(K_1 - K_2)n},$$

for some constant $K_2 > 0$. Finally, choose K_1 so that $K_1 - K_2 = K$. □

LEMMA 2.7. *If F is of $(p+q)$-growth $(2, 0)$ for some $q > 1/2$, then there is a unique element φ in $(\mathcal{S})_p$ so that $\mathcal{S}\varphi = F$.*

Proof. First of all note that the hypothesis that F is of $(p+q)$-growth $(2, 0)$ implies that for every $n \in \mathbb{N}$, $F^{(n)} \in \mathcal{S}_{p+q}(\mathbb{R}^n)$, so in particular $F^{(n)} \in \mathcal{S}_p(\mathbb{R}^n)$. Also, by construction $F^{(n)}$ is symmetric. Now set

$$\varphi = \sum_{n=0}^{\infty} I_n(F^{(n)}).$$

By construction, $\varphi \in (\mathcal{S})_p$ if and only if (1.2) is finite. But the estimate of Lemma 2.6 shows that this is true. □

It is clear that the second statement of Theorem 1.2 follows immediately from Lemma 2.7.

§ 3. Consequences and examples

As a first illustration of our theorem let us investigate the question whether the composition of $f \circ X_\eta$ of a function f in the Schwartz space $\mathcal{S}(\mathbb{R})$ of test functions with the random variable $X_\eta = \langle \cdot, \eta \rangle$, $\eta \in \mathcal{S}(\mathbb{R})$, belongs to $(\mathcal{S})$. As a special case, let us choose f to be the Gauss function $f(u) = \exp(-\frac{1}{2}u^2)$. It is an easy exercise to compute

$$\mathcal{S}\varphi(\xi) = (1 + |\eta|_2^2)^{-1/2} - \exp\left(-\frac{1}{2} \frac{(\eta, \xi)^2}{1 + |\eta|_2^2} \right),$$

for

$$\varphi(x) = f \circ X_\eta = \exp\left(-\frac{1}{2} \langle x, \eta \rangle^2 \right).$$

It is obvious that $\mathcal{S}\varphi$ is not of type 0. Therefore φ does *not* belong to $(\mathcal{S})$, and consequently in general the answer to the above raised question is *negative.*

Based on Theorem 1.2 we can very easily establish a result which has already been proved in [PR 89] by other methods, and which describes a large class of examples in $(\mathcal{S})$ of the above type.

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THEOREM 3.1. *Let $\eta \in \mathcal{S}(\mathbb{R})$, and f be a function on $\mathbb{R}$ whose Fourier transform $\hat{f}$ belongs to $L^1(\mathbb{R})$ and has compact support. Then $f \circ X_\eta \in (\mathcal{S})$.*

Proof. By Fubini's theorem, we find that

$$\mathcal{S}f \circ X_\eta(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(p) \mathcal{S}e^{ip\langle \cdot, \eta \rangle}(\xi) dp \,.$$

It is plain to compute the $\mathcal{S}$-transform under the last integral:

$$\mathcal{S}e^{ip\langle \cdot, \eta \rangle}(\xi) = \exp\left(ip(\xi, \eta) - \frac{1}{2} p^2 |\eta|_2^2 \right),$$

where $(\cdot, \cdot)$ is the inner product of $L^2(\mathbb{R})$. Thus we have the formula $(z \in \mathbb{C})$

$$\mathcal{S}f \circ X_\eta(z\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(p) \exp\left(izp(\xi, \eta) - \frac{1}{2} p^2 |\eta|_2^2 \right) dp \,. \tag{3.1}$$

First of all note that the fact that $\hat{f}$ has compact support allows us to differentiate with respect to $z \in \mathbb{C}$ under the integral sign in (3.1). From this we obtain immediately that the function $z \mapsto \mathcal{S}f \circ X_\eta(z\xi)$ admits the Cauchy-Riemann equations and therefore $\mathcal{S}f \circ X_\eta$ is ray entire.

Assume that $\operatorname{supp} \hat{f} \subset [-N, N]$, $N > 0$. Then we can estimate for every $p \in \mathbb{N}$ as follows.

$$\begin{aligned} M(R, \xi) &\equiv \sup_{|z| = R} |\mathcal{S}f \circ X_\eta(z\xi)| \\ &\leq (2\pi)^{-1/2} |\hat{f}|_1 e^{RN|(\xi, \eta)|} \\ &\leq (2\pi)^{-1/2} |\hat{f}|_1 e^{RN|\xi|_2, -p|\eta|_2, p} \,, \end{aligned}$$

where $|\cdot|_1$ is the $L^1(\mathbb{R})$-norm. Clearly, this bound shows that $\mathcal{S}f \circ X_\eta$ is for every $p \in \mathbb{N}$ of p-order 1 and consequently of p-growth $(2, 0)$. □

EXAMPLE 3.2. Note that the function $f(u) = u^{-1} \sin u$, $u \in \mathbb{R}$ satisfies the hypothesis of Theorem 3.1. Therefore φ given by

$$\varphi(x) = \frac{\sin \langle x, \eta \rangle}{\langle x, \eta \rangle}, \qquad x \in \mathcal{S}'(\mathbb{R}),\ \eta \in \mathcal{S}(\mathbb{R}) \,,$$

belongs to $(\mathcal{S})$. Since $(\mathcal{S})$ is an algebra (e.g. [HPS 88]), so $\varphi^2 \in (\mathcal{S})$. φ^2 was used in [PR 89] to show that every smooth bounded cyclinder function can be approximated by sequences in $(\mathcal{S})$ in the topology defined by certain Dirichlet forms over $\mathcal{S}'(\mathbb{R})$.

EXAMPLE 3.3. If f is entire and of growth $(2, 0)$ (cf. e.g. [Bo 54]), and

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we pick $\eta \in \mathcal{S}(\mathbb{R})$, then for all $p \in \mathbb{N}$,

$$F(\xi) := f \circ (\eta, \xi)$$

is of p-growth $(2, 0)$. This follows as above from the estimate $|(\eta, \xi)| \leq |\eta|_{2,p}|\xi|_{2,-p}$, and the finiteness of $|\eta|_{2,p}$ for all p. Therefore any such pair (f, η) gives rise to an element in $(\mathcal{S})$. For example, we may choose for f the functions $z \mapsto \cos\sqrt{z}$ or $z \mapsto \Gamma(z)^{-1}$.

Note added in proof. After this manuscript was submitted, we learned that Ju. G. Kondrat'ev had obtained similar results in "Nuclear spaces of entire functions in problems of infinite-dimensional analysis"; Soviet Math. Dokl., 22 (1980), 588–592.

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