Modular Forms Associated to Theta Functions

Min Ho Lee

Abstract. We use the theory of Jacobi-like forms to construct modular forms for a congruence subgroup of $SL(2, \mathbb{R})$ which can be expressed as linear combinations of products of certain theta functions.

1 Introduction

Since they were introduced systematically by Jacobi in the nineteenth century, theta functions of various types have been studied extensively in connection with many branches of pure and applied mathematics. They have played an important role in the studies of linear partial differential equations of parabolic type, the Riemann zeta function, representations of integers as sums of squares, quadratic forms, and abelian varieties. Theta functions can also be used to express solutions of some integrable nonlinear partial differential equations, which include many well-known equations in mathematical physics such as the nonlinear Schrödinger equation, the Sine-Gordon equation, the Korteweg-de Vries (KdV) equation and the Kadomtsev-Petviashvili equation (*cf.* [1], [5], [9]). Moreover, in recent years numerous papers have been devoted to the application of theta functions in the areas of infinite dimensional Lie algebras and their applications, which include the theory of lattices, codes, and sphere packings (*cf.* [2], [4], [7]).

Modular forms of one variable are holomorphic functions on the Poincaré upper half plane which satisfy a certain transformation property under the action of a discrete subgroup of $SL(2, \mathbb{R})$. Such modular forms and their generalizations play an essential role in number theory, and they are related to various areas of mathematics. In addition to providing important examples of modular forms, theta functions are closely linked to modular forms in various ways. In many applications of theta functions described above, their connections with modular forms play a crucial role.

As was pointed out by Zagier in [11, p. 278], in many occurrences of modular forms in mathematical physics, especially those connected with infinite-dimensional Lie algebras, the functions that are actually involved are Jacobi forms. Jacobi forms generalize theta functions, and they were systematically introduced by Eichler and Zagier in [8]. Jacobi-like forms, introduced by Zagier (*cf.* [12]; see also [3]), are certain power series whose coefficients are complex valued functions and which satisfy a transformation formula under the action of a discrete subgroup of $SL(2, \mathbb{R})$. If the coefficients are holomorphic functions defined on the Poincaré upper half plane, then Jacobi-like forms may be considered as a generalization of Jacobi forms. In a

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recent paper [6], motivated by a problem in conformal field theory, Dong and Mason studied relations between some type of theta functions and Jacobi-like forms. In this paper we use the results of Dong and Mason and the theory of Jacobi-like forms to construct modular forms for a congruence subgroup of $SL(2, \mathbb{R})$ which can be expressed as linear combinations of products of certain theta functions.

2 Jacobi-Like Forms and Modular Forms

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ be the Poincaré upper half plane on which $SL(2, \mathbb{R})$ acts by linear fractional transformations. Thus we have

$$(2.1) \qquad \gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. Let R be the ring of holomorphic functions on $\mathcal{H}$, and denote by $R[[X]]$ the space of formal power series in X with coefficients in R. Let Γ be a discrete subgroup of $SL(2, \mathbb{R})$, and let k be a nonnegative integer.

Definition 2.1 An element $\phi(z, X) \in R[[X]]$ is a *Jacobi-like form of weight k for Γ* if

$$(2.2) \qquad \phi\big(\gamma z, (cz + d)^{-2} X\big) = (cz + d)^k e^{cX/(cz+d)} \phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Let $\phi(z, X) = \sum_{n=0}^{\infty} a_n(z) X^n \in R[[X]]$ be a Jacobi-like form of weight k for Γ, and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. By (2.2) we have

$$(cz+d)^{-k} \sum_{n=0}^{\infty} a_n(\gamma z)(cz+d)^{-2n} X^n = \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{c}{cz+d} \right)^\ell X^\ell \right) \left(\sum_{m=0}^{\infty} a_m(z) X^m \right)$$

$$= \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{c}{cz+d} \right)^\ell a_m(z) X^{\ell+m}$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{c}{cz+d} \right)^\ell a_{n-\ell}(z) X^n$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Comparing the coefficient of X^n for each nonnegative integer n, we obtain

$$(2.3) \qquad (cz+d)^{-k-2n} a_n(\gamma z) = \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{c}{cz+d} \right)^\ell a_{n-\ell}(z).$$

Definition 2.2   Given a nonnegative integer m, a *modular form of weight m for* Γ *is a holomorphic function* $f\colon \mathcal{H} \to \mathbb{C}$ *such that*

$$f(\gamma z) = (cz + d)^m f(z)$$

for all $z \in \mathcal{H}$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Note that the usual definition of modular forms also include the finiteness condition at infinity or the cusp condition. In this paper, however, we suppress this condition.

Proposition 2.3   *Let* $\phi(z, X) = \sum_{n=0}^{\infty} a_n(z) X^n \in R[[X]]$ *be a Jacobi-like form of weight k for* Γ. *For each nonnegative integer n the function* $h_n\colon \mathcal{H} \to \mathbb{C}$ *given by*

$$h_n(z) = \sum_{m=0}^{n} \frac{(-2\pi i)^m}{m!} (2n - m + k - 2)! \, a_{n-m}^{(m)}(z)$$

for all $z \in \mathcal{H}$ *is a modular form of weight* $2n + k$ *for* Γ, *where* $a_{n-m}^{(m)}(z)$ *denotes the m-th derivative of* $a_{n-m}(z)$ *with respect to z.*

Proof If $\sum_{n=0}^{\infty} a_n(z) X^n$ is a Jacobi-like form of weight k for Γ, then by Equation (17) in [12] we see that the function

$$\tilde{h}(z) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} (2n - m + k - 2)! \left(a_{n-m}/(2\pi i)^{n-m} \right)^{(m)}(z)$$

is a modular form of weight $2n + k$ for Γ. Hence it follows that the same is true for the function $h_n(z) = \tilde{h}_n(z)/(2\pi i)^n$. ∎

3   Theta Functions

Throughout the rest of this paper we fix a positive integer r, an element v of $\mathbb{C}^{2r}$ considered as a column vector, and a symmetric positive definite integral $2r \times 2r$ matrix A whose diagonal entries are even. For each nonnegative integer k, we define the theta function $\theta_k\colon \mathcal{H} \to \mathbb{C}$ by

$$(3.1) \qquad \theta_k(z) = \sum_{\ell \in \mathbb{Z}^{2r}} (v^t A \ell)^k e^{\pi i (\ell^t A \ell) z}$$

for all $z \in \mathcal{H}$.

Remark 3.1   Given a column vector $x \in \mathbb{C}^{2r}$, let V be a $2r \times 2r$ matrix such that $V^t A V$ is the identity matrix, and let $y = (y_1, \dots, y_{2r})^t = V^{-1} x \in \mathbb{C}^{2r}$. Then we have

$$x^t A x = y^t y = \sum_{\nu=1}^{2r} y_\nu^2.$$

A homogeneous polynomial $P_k(x)$ of degree k is said to be a *spherical function with respect to the positive definite form $x^t A x$* if it satisfies

$$\sum_{\nu=1}^{2r} \frac{\partial^2}{\partial y_\nu^2} P_k(V y) = 0.$$

It is known that such a polynomial is of the form

$$P_k(x) = \sum_u C_u (u^t A x)^k$$

with $C_u \in \mathbb{C}$, where the summation is extended over finitely many u satisfying $u^t A u = 0$ (see *e.g.* [7], [10]). Thus, if $v^t A v = 0$, the polynomial function $x \mapsto (v^t A x)^k$ is a spherical function and therefore $\theta_k(z)$ in (3.1) is an example of a theta series with spherical coefficients.

Now we define the formal power series $\Theta(z, X) \in R[[X]]$ associated to the sequence $\{\theta_n(z)\}_{n=0}^\infty$ of theta functions by

(3.2) $$\Theta(z, X) = \sum_{n=0}^\infty \frac{2^n (2\pi i)^n}{(2n)!} \theta_{2n}(z) X^n$$

for all $z \in \mathcal{H}$.

Let N be the smallest positive integer such that NA^{-1} is an integral matrix with even diagonal entries, and let $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$ be the associated congruence subgroup given by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

Let ε be the character on the set of positive integers defined by the Jacobi symbol

$$\varepsilon(n) = \left(\frac{(-1)^r \det A}{n} \right)$$

for each positive integer n.

Theorem 3.2   *The formal power series $\Theta(z, X) \in R[[X]]$ given by (3.2) satisfies the transformation property*

(3.3) $$\Theta\big(\gamma z, (cz+d)^{-2} X\big) = \varepsilon(d)(cz+d)^r \exp[c v^t A v X / (cz+d)] \cdot \Theta(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Proof See Theorem 1 in [6].   ∎

4 Theta Functions and Modular Forms

Let $\theta_k(z)$ with $k \geq 0$ be the theta function given by (3.1), which is associated to the vector $v \in \mathbb{C}^{2r}$ and the symmetric integral matrix A. We assume that $v^t A v \neq 0$ and define the formal power series $\vartheta(z, X) \in R[[X]]$ by

$$(4.1) \qquad \vartheta(z, X) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \left(\frac{2\pi i}{v^t A v} \right)^n \cdot \frac{\theta_{2\ell}(z)\theta_{2n-2\ell}(z)}{(2\ell)!\,(2n-2\ell)!} X^n$$

for all $z \in \mathcal{H}$.

Lemma 4.1 *The formal power series $\vartheta(z, X) \in R[[X]]$ given by (4.1) is a Jacobi-like form of weight $2r$ for $\Gamma_0(N)$.*

Proof For each $z \in \mathcal{H}$ we have

$$\vartheta(z, X) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \left(\frac{2\pi i}{v^t A v} \right)^{\ell} \frac{\theta_{2\ell}(z)}{(2\ell)!} \cdot \left(\frac{2\pi i}{v^t A v} \right)^{n-\ell} \frac{\theta_{2n-2\ell}(z)}{(2n-2\ell)!} X^n$$

$$= \left(\sum_{n=0}^{\infty} \frac{(2\pi i)^n}{(2n)!\,v^t A v} \theta_{2n}(z) X^n \right)^2$$

$$= \left(\sum_{n=0}^{\infty} \frac{2^n (2\pi i)^n}{(2n)!} \theta_{2n}(z) \left(\frac{X}{2v^t A v} \right)^n \right)^2$$

$$= \Theta\big(z, X/(2v^t A v)\big)^2.$$

Thus, using (3.3) and the fact that $\varepsilon(d)^2 = 0$, we see that

$$\vartheta\big(\gamma z, (cz+d)^{-2}X\big) = \left(\Theta\big(\gamma z, (cz+d)^{-2}X/(2v^t A v)\big) \right)^2$$

$$= \varepsilon(d)^2 (cz+d)^{2r} \exp\left[2cv^t A v X/\big(2v^t A v(cz+d)\big) \right]$$

$$\cdot \Theta\big(z, X/(2v^t A v)\big)^2$$

$$= (cz+d)^{2r} \exp[cX/(cz+d)] \cdot \vartheta(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$; hence it follows that $\vartheta(z, X)$ is a Jacobi-like form of weight $2r$ for $\Gamma_0(N)$. ∎

Given a nonnegative integer n, we set

$$(4.2) \qquad \xi_n(z) = \sum_{m=0}^{n} \sum_{\ell=0}^{n-m} \sum_{j=0}^{m} \frac{(-1)^m (2n-m+2r-2)!}{m!\,(2n-2m)!\,(v^t A v)^{n-m}} \binom{2n-2m}{2\ell} \binom{m}{j} \theta_{2\ell}^{(j)}(z)$$

$$\cdot \theta_{2n-2m-2\ell}^{(m-j)}(z)$$

for all $z \in \mathcal{H}$.

Theorem 4.2 *For each nonnegative integer n the function $\xi_n\colon \mathcal{H} \to \mathbb{C}$ given by (4.2) is a modular form of weight $2(n + r)$ for $\Gamma_0(N)$.*

Proof Given a nonnegative integer n, we set

$$(4.3) \qquad \eta_n(z) = \left(\frac{2\pi i}{v^t A v} \right)^n \sum_{\ell=0}^n \frac{\theta_{2\ell}(z)\theta_{2n-2\ell}(z)}{(2\ell)!\,(2n-2\ell)!}$$

for $z \in \mathcal{H}$. Then by (4.1) we have

$$\vartheta(z, X) = \sum_{n=0}^\infty \eta_n(z) X^n.$$

Since by Lemma 4.1 the formal power series $\vartheta(z, X)$ is a Jacobi-like form of weight $2r$ for $\Gamma_0(N)$, by Proposition 2.3 we see that the function $\tilde{\xi}\colon \mathcal{H} \to \mathbb{C}$ defined by

$$\tilde{\xi}(z) = \sum_{m=0}^n (-2\pi i)^m \frac{(2n - m + 2r - 2)!}{m!} \eta_{n-m}^{(m)}(z)$$

is a modular form of weight $2(n + r)$ for $\Gamma_0(N)$. However, we have

$$\eta_{n-m}^{(m)}(z) = \left(\frac{2\pi i}{v^t A v} \right)^{n-m} \sum_{\ell=0}^{n-m} \frac{1}{(2\ell)!\,(2n-2m-2\ell)!} \sum_{j=0}^m \binom{m}{j} \theta_{2\ell}^{(j)}(z) \cdot \theta_{2n-2m-2\ell}^{(m-j)}(z)$$

$$= \left(\frac{2\pi i}{v^t A v} \right)^{n-m} \sum_{\ell=0}^{n-m} \sum_{j=0}^m \frac{1}{(2\ell)!\,(2n-2m-2\ell)!} \binom{m}{j} \theta_{2\ell}^{(j)}(z) \cdot \theta_{2n-2m-2\ell}^{(m-j)}(z)$$

$$= \frac{(2\pi i)^{n-m}}{(2n-2m)!\,(v^t A v)^{n-m}} \sum_{\ell=0}^{n-m} \sum_{j=0}^m \binom{2n-2m}{2\ell} \binom{m}{j} \theta_{2\ell}^{(j)}(z) \cdot \theta_{2n-2m-2\ell}^{(m-j)}(z)$$

for $0 \le m \le n$. Hence using this and (4.2), we see that

$$\xi_n(z) = \sum_{m=0}^n \frac{(-1)^m (2n - m + 2r - 2)!}{m!\,(2\pi i)^{n-m}} \eta_{n-m}^{(m)}(z) = (2\pi i)^{-n} \tilde{\xi}(z)$$

for all $z \in \mathcal{H}$, and therefore the theorem follows. ∎

5   Theta Relations

Certain relations involving products of theta functions arise in the theory of sphere packings, lattices and codes (see *e.g.* [4, Chapter 4]). In this section we use the results from previous sections to establish a relation satisfied by some products of theta functions of the form $\theta_k(z)$ in (3.1).

Theorem 5.1 *Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have*

$$(5.1) \qquad \sum_{\ell=0}^{n} \binom{2n}{2\ell} \left[\theta_{2\ell}(\gamma z) \cdot \theta_{2n-2\ell}(\gamma z) - \frac{(2\ell)!}{\ell!} \left(\frac{cv^t Av}{2\pi i} \right)^{\ell} (cz+d)^{2n+2r-\ell} \right.$$

$$\left. \times \sum_{j=0}^{n-\ell} \binom{2n-2\ell}{2j} \theta_{2j}(z) \cdot \theta_{2n-2\ell-2j}(z) \right] = 0$$

for all $z \in \mathcal{H}$, where $\gamma z = (az+b)(cz+d)^{-1}$ as in (2.1).

Proof Since $\vartheta(z,X)$ is a Jacobi-like form for $\Gamma_0(N)$ of weight $2r$, if $\eta_n(z)$ is as in (4.3), by (2.3) we see that

$$(cz+d)^{-2r-2n} \frac{\eta_n(\gamma z)}{(2\pi i)^n} = \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{c}{2\pi i(cz+d)} \right)^{\ell} \frac{\eta_{n-\ell}(z)}{(2\pi i)^{n-\ell}}$$

for $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Thus, using (4.3), we obtain

$$(cz+d)^{-2r-2n} \frac{1}{(v^t Av)^n} \sum_{\ell=0}^{n} \frac{\theta_{2\ell}(\gamma z)\theta_{2n-2\ell}(\gamma z)}{(2\ell)! \, (2n-2\ell)!}$$

$$= \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(\frac{c}{2\pi i(cz+d)} \right)^{\ell} \frac{1}{(v^t Av)^{n-\ell}} \sum_{j=0}^{n-\ell} \frac{\theta_{2j}(z)\theta_{2n-2j}(z)}{(2j)! \, (2n-2\ell-2j)!}.$$

Hence we have

$$\sum_{\ell=0}^{n} \binom{2n}{2\ell} \theta_{2\ell}(\gamma z) \cdot \theta_{2n-2\ell}(\gamma z) - \left(\frac{cv^t Av}{2\pi i} \right)^{\ell} (cz+d)^{2n+2r-\ell} \frac{(2n)!}{\ell! \, (2n-2\ell)!}$$

$$\times \sum_{j=0}^{n-\ell} \binom{2n-2\ell}{2j} \theta_{2j}(z) \cdot \theta_{2n-2\ell-2j}(z) = 0,$$

and therefore the theorem follows. ∎

Example 5.2 Using (5.1) for $n=0$, $n=1$ and $n=2$, we obtain

$$\theta_0^2(\gamma z) = (cz+d)^{2r} \theta_0^2(z),$$

$$(\theta_0\theta_2)(\gamma z) = (cz+d)^{2r+2}(\theta_0\theta_2)(z) + \left(\frac{cv^t Av}{2\pi i} \right)(cz+d)^{2r+1}\theta_0^2(z),$$

$$(\theta_0\theta_4 + 3\theta_2^2)(\gamma z) = (cz+d)^{2r+4}(\theta_0\theta_4 + 3\theta_2^2)(z) + 12\left(\frac{cv^t Av}{2\pi i} \right)(cz+d)^{2r+3}(\theta_0\theta_2)(z)$$

$$+ 6\left(\frac{cv^t Av}{2\pi i} \right)^2 (cz+d)^{2r+2}\theta_0^2(z)$$

for $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

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Department of Mathematics
University of Northern Iowa
Cedar Falls, Iowa 50614
U.S.A.
e-mail: lee@math.uni.edu