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ON ISOMETRIC MINIMAL IMMERSIONS FROM WARPED PRODUCTS INTO REAL SPACE FORMS

BANG-YEN CHEN

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA (bychen@math.msu.edu)

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Abstract We establish a general sharp inequality for warped products in real space form. As applications, we show that if the warping function f of a warped product $N_1 \times_f N_2$ is a harmonic function, then

(1) every isometric minimal immersion of $N_1 \times_f N_2$ into a Euclidean space is locally a warped-product immersion, and

(2) there are no isometric minimal immersions from $N_1 \times_f N_2$ into hyperbolic spaces.

Moreover, we prove that if either N_1 is compact or the warping function f is an eigenfunction of the Laplacian with positive eigenvalue, then $N_1 \times_f N_2$ admits no isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension. We also provide examples to show that our results are sharp.

Keywords: warped products; warped-product immersion; inequality; minimal immersions; real space forms; eigenfunction of the Laplacian

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1. Introduction

Let B and F be two Riemannian manifolds of positive dimensions equipped with Riemannian metrics g_B and g_F, respectively, and let f be a positive function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The *warped product* $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_*(X)\|^2 + f^2(\pi(x))\|\eta_*(X)\|^2 \tag{1.1}$$

for any tangent vector $X \in T_xM$. Thus, we have $g = g_B + f^2 g_F$. The function f is called the *warping function* of the warped product. It is well known that the notion of warped products plays some important roles in differential geometry as well as in physics. For instance, the best relativistic model of the Schwarzschild space-time that describes the outer space around a massive star or a black hole is given as a warped product (cf. [**9**, pp. 364–367]). (For a recent survey on warped products as Riemannian submanifolds, see [**4**].)

For a warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibres, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ is obtained by tangent vectors of N_2 via the vertical lift. Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold with constant sectional curvature c. Denote by h the second fundamental form of ϕ. The immersion ϕ is called *mixed totally geodesic* if $h(X, Z) = 0$ for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$.

One of the most fundamental problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean m-space E^m (or, more generally, in a space form $R^m(c)$ of constant curvature c). According to a well-known theorem of Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension. The Nash Theorem was aimed for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity of using extrinsic help.

Based on Nash's Theorem, one of my research programs is 'to search for control of extrinsic quantities in relation to intrinsic quantities of Riemannian manifolds via Nash's Theorem and to search for their applications'.

Since Nash's Theorem implies that every warped product $N_1 \times_f N_2$ can always be regarded as a Riemannian submanifold in some Euclidean space, a special case of the research program is thus to study the two following fundamental problems.

Problem 1.1.

$$\forall N_1 \times_f N_2 \xrightarrow[\text{immersion}]{\text{isometric}} E^m \quad \text{or} \quad R^m(c) \Longrightarrow \mathbf{???}$$

Problem 1.2. Let $N_1 \times_f N_2$ be an arbitrary warped product isometrically immersed in E^m (or in $R^m(c)$) as a Riemannian submanifold. What are the relationships between the warping function f and the extrinsic structures of $N_1 \times_f N_2$?

In view of Nash's Theorem, it is natural to impose a natural condition on the immersibility problem. For example, if one imposes the minimality condition on the immersions, it leads to the following problem.

Problem 1.3. Given a warped product $N_1 \times_f N_2$, what are the necessary conditions for the warped product to admit a minimal isometric immersion in a Euclidean m-space E^m (or in $R^m(c)$)?

In this paper we prove the following results, which provide some solutions to these fundamental problems.

Theorem 1.4. *Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \leqslant \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c, \tag{1.2}$$

where $n_i = \dim N_i$, $i = 1, 2$, H^2 is the squared mean curvature of ϕ, and Δ is the Laplacian operator of N_1.

The equality sign of (1.2) holds identically if and only if $\phi : N_1 \times_f N_2 \to R^m(c)$ is a mixed totally geodesic immersion with $\operatorname{tr} h_1 = \operatorname{tr} h_2$, where $\operatorname{tr} h_1$ and $\operatorname{tr} h_2$ denote the trace of h restricted to N_1 and N_2, respectively.

As applications of Theorem 1.4 we have the following theorems.

Theorem 1.5. *Let $N_1 \times_f N_2$ be a warped product whose warping function f is a harmonic function. Then*

(1) *$N_1 \times_f N_2$ admits no isometric minimal immersion into a hyperbolic space for any codimension; and*

(2) *every isometric minimal immersion from $N_1 \times_f N_2$ into a Euclidean space is a warped-product immersion.*

Theorem 1.6. *If f is an eigenfunction of the Laplacian on N_1 with eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension.*

Theorem 1.7. *If N_1 is a compact Riemannian manifold, then every warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension.*

In the last section, we provide some examples to show that these results are best possible.

2. Preliminaries

Let N be an n-dimensional submanifold of a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c. We choose a local field of orthonormal frame

$$e_1, \dots, e_n, e_{n+1}, \dots, e_m \quad \text{in } R^m(c)$$

such that, restricted to N, the vectors $e_1, \dots, e_n$ are tangent to N and $e_{n+1}, \dots, e_m$ are normal to N.

Let $K(e_i \wedge e_j)$, $1 \leqslant i < j \leqslant n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of N is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j). \tag{2.1}$$

For a submanifold N in $R^m(c)$ we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of N and $R^m(c)$, respectively. The Gauss and Weingarten formulae are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.3}$$

for vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold. Let $\{h^r_{ij}\}$, $i,j=1,\dots,n$, $r=n+1,\dots,m$, denote the coefficients of the second fundamental form h with respect to $e_1,\dots,e_n,e_{n+1},\dots,e_m$.

The mean curvature vector $\boldsymbol{H}$ is defined by

$$\boldsymbol{H}=\frac{1}{n}\operatorname{tr}h=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i), \tag{2.4}$$

where $\{e_1,\dots,e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. The squared mean curvature is given by $H^2=\langle\boldsymbol{H},\boldsymbol{H}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner product. A submanifold N is called minimal in $R^m(c)$ if the mean curvature vector of N in $R^m(c)$ vanishes identically.

Denote by R the Riemann curvature tensor of N. Then the *equation of Gauss* is given by (see, for example, [**1**])

$$\begin{aligned}R(X,Y;Z,W)=c\{\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle\}\\+\langle h(X,W),h(Y,Z)\rangle-\langle h(X,Z),h(Y,W)\rangle,\end{aligned} \tag{2.5}$$

for vectors X, Y, Z, W tangent to N.

Let M be a Riemannian p-manifold and $\{e_1,\dots,e_p\}$ be an orthonormal frame field on M. For a differentiable function φ on M, the Laplacian of φ is defined by

$$\Delta\varphi=\sum_{j=1}^{p}\{(\nabla_{e_j}e_j)\varphi-e_je_j\varphi\}. \tag{2.6}$$

Recall that if M is compact, every eigenvalue of Δ is non-negative.

Let $\phi: N_1\times_f N_2\to R^m(c)$ denote an isometric immersion of a warped product $N_1\times_f N_2$ into a Riemannian manifold with constant sectional curvature c. Denote by $\operatorname{tr}h_1$ and $\operatorname{tr}h_2$ the trace of h restricted to N_1 and N_2, respectively, that is

$$\operatorname{tr}h_1=\sum_{\alpha=1}^{n_1}h(e_\alpha,e_\alpha),\qquad \operatorname{tr}h_2=\sum_{t=n_1+1}^{n_1+n_2}h(e_t,e_t) \tag{2.7}$$

for some orthonormal frame fields $e_1,\dots,e_{n_1}$ and $e_{n_1+1},\dots,e_{n_1+n_2}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

If $M_1\times_\rho M_2$ is a warped product of two Riemannian manifolds and $\phi_i: N_i\to M_i$, $i=1,2$, are isometric immersions from Riemannian manifolds N_1, N_2 into Riemannian manifolds M_1, M_2, respectively. Define a positive function σ on N_1 by $\sigma=\rho\circ\phi_1$. Then the map

$$\phi: N_1\times_\sigma N_2\to M_1\times_\rho M_2 \tag{2.8}$$

given by $\phi(x_1,x_2)=(\phi_1(x_1),\phi_2(x_2))$ is an isometric immersion, which is called a *warped-product immersion* [**8**] (see also [**5**]).

3. Proof of Theorem 1.4

Let $\phi : N = N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold of constant sectional curvature c. Denote by n_1, n_2, n the dimensions of N_1, N_2, $N_1 \times N_2$, respectively.

Since $N_1 \times_f N_2$ is a warped product, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z \tag{3.1}$$

for unit vector fields X, Z tangent to N_1, N_2, respectively. Hence, we find

$$\begin{aligned} K(X \wedge Z) &= \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z \rangle \\ &= (1/f)\{(\nabla_X X)f - X^2 f\}. \end{aligned} \tag{3.2}$$

If we chose a local orthonormal frame $e_1, \dots, e_n$ such that $e_1, \dots, e_{n_1}$ are tangent to N_1 and $e_{n_1+1}, \dots, e_n$ are tangent to N_2, then we have

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s) \tag{3.3}$$

for each $s = n_1 + 1, \dots, n$.

From the equation of Gauss, it follows that the scalar curvature τ and the squared mean curvature H^2 of N satisfy

$$2\tau = n^2 H^2 - \|h\|^2 + n(n-1)c, \tag{3.4}$$

where $\|h\|^2$ is the squared norm of the second fundamental form h of N in $R^m(c)$.

Let us put

$$\delta = 2\tau - n(n-1)c - \tfrac{1}{2} n^2 H^2. \tag{3.5}$$

Then (3.4) becomes

$$n^2 H^2 = 2\delta + 2\|h\|^2. \tag{3.6}$$

If we choose an orthonormal frame $e_{n+1}, \dots, e_m$ of the normal bundle so that e_{n+1} is in the direction of the mean curvature vector, then (3.6) becomes

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = 2\left[\delta + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \right]. \tag{3.7}$$

Equation (3.7) is equivalent to

$$\begin{aligned} (\bar{a}_1 + \bar{a}_2 + \bar{a}_3)^2 = 2\Bigg[\delta + \bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2 + 2 \sum_{1 \leqslant i < j \leqslant n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \\ - 2 \sum_{2 \leqslant j < k \leqslant n_1} h_{jj}^{n+1} h_{kk}^{n+1} - 2 \sum_{n_1+1 \leqslant s < t \leqslant n} h_{ss}^{n+1} h_{tt}^{n+1} \Bigg], \end{aligned} \tag{3.8}$$

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where

$$\bar{a}_1 = h_{11}^{n+1}, \qquad \bar{a}_2 = h_{22}^{n+1} + \cdots + h_{n_1 n_1}^{n+1}, \qquad \bar{a}_3 = h_{n_1+1 n_1+1}^{n+1} + \cdots + h_{nn}^{n+1}. \tag{3.9}$$

Applying Lemma 3.1 of [**2**] or of [**3**] to (3.8) yields

$$\sum_{1\leqslant j<k\leqslant n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1\leqslant s<t\leqslant n} h_{ss}^{n+1} h_{tt}^{n+1} \geqslant \tfrac{1}{2}\delta + \sum_{1\leqslant \alpha<\beta\leqslant n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^{r})^2, \tag{3.10}$$

with equality holding if and only if we have

$$h_{11}^{n+1} + \cdots + h_{n_1 n_1}^{n+1} = h_{n_1+1 n_1+1}^{n+1} + \cdots + h_{nn}^{n+1}. \tag{3.11}$$

From the equation of Gauss and (3.3), we have

$$\begin{aligned} \frac{n_2 \Delta f}{f} &= \tau - \sum_{1\leqslant j<k\leqslant n_1} K(e_j \wedge e_k) - \sum_{n_1+1\leqslant s<t\leqslant n} K(e_s \wedge e_t) \\ &= \tau - \tfrac{1}{2}(n_1(n_1-1))c - \sum_{r=n+1}^{m}\sum_{1\leqslant j<k\leqslant n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) \\ &\quad - \tfrac{1}{2}(n_2(n_2-1))c - \sum_{r=n+1}^{m}\sum_{n_1+1\leqslant s<t<n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2). \end{aligned} \tag{3.12}$$

Therefore, by (3.5), (3.10) and (3.12), we obtain

$$\begin{aligned} \frac{n_2 \Delta f}{f} &\leqslant \tau - \tfrac{1}{2}(n(n-1))c + n_1 n_2 c - \tfrac{1}{2}\delta - \sum_{\substack{1\leqslant j\leqslant n_1;\\ n_1+1\leqslant t\leqslant n}} (h_{jt}^{n+1})^2 \\ &\qquad - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^r)^2 + \sum_{r=n+2}^{m}\sum_{1\leqslant j<k\leqslant n_1} ((h_{jk}^r)^2 - h_{jj}^r h_{kk}^r) \\ &\qquad + \sum_{r=n+2}^{m}\sum_{n_1+1\leqslant s<t<n} ((h_{st}^r)^2 - h_{ss}^r h_{tt}^r) \\ &= \tau - \tfrac{1}{2}(n(n-1))c + n_1 n_2 c - \tfrac{1}{2}\delta - \sum_{r=n+1}^{m}\sum_{1\leqslant j\leqslant n_1}\sum_{n_1+1\leqslant t\leqslant n} (h_{jt}^r)^2 \\ &\qquad - \frac{1}{2}\sum_{r=n+2}^{m}\biggl(\sum_{1\leqslant j\leqslant n_1} h_{jj}^r\biggr)^2 - \frac{1}{2}\sum_{r=n+2}^{m}\biggl(\sum_{n_1+1\leqslant t\leqslant n} h_{tt}^r\biggr)^2 \\ &\leqslant \tau - \tfrac{1}{2}(n(n-1))c + n_1 n_2 c - \tfrac{1}{2}\delta \\ &= \tfrac{1}{4}n^2 H^2 + n_1 n_2 c, \end{aligned} \tag{3.13}$$

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which proves inequality (1.2). From (3.11) and (3.13) we know that the equality sign of (1.2) holds if and only if

$$h^r_{jt} = 0 \quad \text{for } n+1 \leqslant r \leqslant m, \tag{3.14}$$

and

$$h^r_{11} + \cdots + h^r_{n_1 n_1} = h^r_{n_1+1 n_1+1} + \cdots + h^r_{nn} = 0 \tag{3.15}$$

for $1 \leqslant j \leqslant n_1$, $n_1 + 1 \leqslant t \leqslant n$, $n + 2 \leqslant r \leqslant m$.

Condition (3.14) implies that the second fundamental form of $N_1 \times_f N_2$ in $R^m(c)$ satisfies $h(\mathcal{D}_1, \mathcal{D}_2) = \{0\}$. Thus, the immersion ϕ is mixed totally geodesic. Hence, by applying a result of Nölker [**8**], we know that, locally, there exists a warped-product representation $M_1 \times_\rho M_2$ of $R^m(c)$ such that $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2 = R^m(c)$ is a warped-product immersion of $\phi_1 : N_1 \to M_1$ and $\phi_2 : N_2 \to M_2$; so that we have $\phi(x_1, x_2) = (\phi_1(x_1), \phi(x_2))$, for $x_1 \in N_1$, $x_2 \in N_2$. Moreover, from (3.11) and (3.15), we obtain

$$\sum_{j=1}^{n_1} h(e_j, e_j) = \sum_{s=n_1+1}^{n_1+n_2} h(e_s, e_s). \tag{3.16}$$

Hence, we have $\operatorname{tr} h_1 = \operatorname{tr} h_2$.

Conversely, suppose that $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2 = R^m(c)$ is a mixed totally geodesic immersion with $\operatorname{tr} h_1 = \operatorname{tr} h_2$. Then all the inequalities in (3.10), (3.13) become equalities. Hence, by (3.13) we obtain the equality sign of (1.2). □

4. Proofs of Theorems 1.5–1.7

Assume that $\phi : N_1 \times_f N_2 \to R^m(c)$ is an isometric minimal immersion of a warped product $N_1 \times_f N_2$ into a complete simply connected Riemannian manifold $R^m(c)$ of constant sectional curvature c.

If f is a harmonic function on N_1, then inequality (1.2) of Theorem 1.4 implies $c \geqslant 0$. In particular, this shows that the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into a hyperbolic space.

When $c = 0$. The minimality of $N_1 \times_f N_2$ and the harmonicity of f imply that the equality sign of (1.2) holds identically. Thus, the immersion is mixed totally geodesic according to Theorem 1.4. Hence, by applying a result of [**8**], we know that ϕ is locally a warped-product immersion. This proves Theorem 1.5.

When f is an eigenfunction of the Laplacian on N_1 with eigenvalue $\lambda > 0$. Then inequality (1.2) implies that $n_1 c \geqslant \lambda > 0$. Hence, the ambient space $R^m(c)$ cannot be a Euclidean space or a hyperbolic space. Therefore, we have Theorem 1.6.

Now, we assume that N_1 is compact and $\phi : N_1 \times_f N_2 \to R^m(c)$ is an isometric minimal immersion with $c \leqslant 0$. Then, inequality (1.2) implies that $(\Delta f)/f \leqslant 0$. Since the warping function is a positive function, we have $\Delta f \leqslant 0$. Hence, by applying Hopf's Lemma and the compactness of N_1, we know that f is a positive constant. Therefore, the warped product $N_1 \times_f N_2$ is a Riemannian product of the Riemannian manifold (N_1, g_1) and the

Riemannian manifold $(N_2, f^2 g_2)$, equipped with the metric $f^2 g_2$ which is homothetic to the original metric g_2 on N_2. Because f is constant and $\phi : N_1 \times_f N_2 \to R^m(c)$ is an isometric minimal immersion, inequality (1.2) implies $c = 0$.

By using the constancy of f, minimality of the warped product, and $c = 0$, we obtain the equality sign of (1.2). Hence, the immersion is mixed totally geodesic. Therefore, by applying a result of Moore [**7**], we know that ϕ is a product immersion, say

$$\phi = (\phi_1, \phi_2) : (N_1, g_1) \times (N_2, f^2 g_2) \to E^{m_1} \times E^{m_2} = E^m.$$

Since ϕ is minimal in E^m, the first factor $\phi_1 : N_1 \to E^{m_1}$ is also an isometric minimal immersion, which is impossible due to compactness of N_1. Hence, when N_1 is compact, $N_1 \times_f N_2$ does not admit any isometric minimal immersions into Euclidean space and hyperbolic space regardless of codimension. □

5. Some remarks

In view of Theorems 1.4–1.7, we provide the following remarks.

Remark 5.1. There exist many minimal submanifolds in Euclidean space which are warped products with harmonic warping function. For example, if N_2 is a minimal submanifold of the unit $(m-1)$-sphere $S^{m-1} \subset E^m$, the minimal cone $C(N_2)$ over N_2 with vertex at the origin of E^m is the warped product $R_+ \times_s N_2$ whose warping function $f = s$ is a harmonic function. Here s is the coordinate function of the positive real line R_+.

Remark 5.2. In view of Theorem 1.6, it is interesting to point out that there exist isometric minimal immersions from warped products $N_1 \times_f N_2$ into a hyperbolic space such that the warping function f is an eigenfunction with negative eigenvalue. For example, $R \times_{e^x} E^{n-1}$ admits an isometric minimal immersion into the hyperbolic space $H^{n+1}(-1)$ of constant sectional curvature -1.

Remark 5.3. In view of Theorem 1.7, it is interesting to point out that there do exist many isometric minimal immersions from $N_1 \times_f N_2$ into Euclidean space with compact N_2. For example, a hypercaternoid in E^{n+1} is a minimal hypersurface which is isometric to a warped product $R \times_f S^{n-1}$. Also, for any compact minimal submanifold N_2 of $S^{m-1} \subset E^m$, the minimal cone $C(N_2)$ is a warped product $R_+ \times_s N_2$, which is also such an example.

Remark 5.4. In contrast to Euclidean and hyperbolic spaces, the standard m-sphere S^m admits warped-product minimal submanifolds $N_1 \times_f N_2$ such that N_1, N_2 are both compact. The simplest of such examples are minimal Clifford tori $M_{k,n-k}$, $k = 2, \dots, n-1$, in S^{n+1} defined by

$$M_{k,n-k} = S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{(n-k)}{n}}\right).$$

Remark 5.5. Ejiri constructed in [**6**] many examples of warped-product minimal immersions into complete simply connected Riemannian manifolds of constant sectional curvature.

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The above remarks show that Theorems 1.4–1.7 are best possible.

Remark 5.6. Problem 1.2 has also been studied in [**10**] from a different aspect for the class of warped products.

Remark 5.7. Inequality (1.2) also holds for warped products isometrically immersed in complex hyperbolic m-space $\mathrm{CH}^m(4c)$ of constant holomorphic sectional curvature $4c < 0$ as well as for warped products isometrically immersed in complex projective m-space $\mathrm{CP}^m(4c)$ as totally real submanifolds.

Remark 5.8. The same proof as for Theorem 1.6 shows that for any positive function f on N_1 with $(\Delta f)/f$ being positive at some points, the warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into Euclidean space or hyperbolic space for any codimension.

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