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RESIDUE FIELDS OF VALUED FUNCTION FIELDS OF CONICS

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Suppose that K is a function field of a conic over a subfield K_0. Let v_0 be a valuation of K_0 with residue field k_0 of characteristic $\neq 2$. Let v be an extension of v_0 to K having residue field k. It has been proved that either k is an algebraic extension of k_0 or k is a regular function field of a conic over a finite extension of k_0. This result can also be deduced from the genus inequality of Matignon (cf. [On valued function fields I, Manuscripta Math. **65** (1989), 357–376]) which has been proved using results about vector space defect and methods of rigid analytic geometry. The proof given here is more or less self-contained requiring only elementary valuation theory.

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0. Introduction

Let v_0 be a non-trivial valuation of a field K_0 with residue field k_0 and value group G_0. Let w be an extension of v_0 to a simple transcendental extension $K_0(x)$. In 1983, Ohm [**8**] proved a conjecture made by Nagata which asserts that the residue field k of w is either an algebraic extension of k_0 or k is a simple transcendental extension of a finite extension of k_0. His method of proof leads to an explicit determination of k in the case that k/k_0 is non-algebraic. Matignon and Ohm have also solved the converse problem stated below.

If G is a totally ordered abelian group containing G_0 as an ordered subgroup with $[G:G_0]<\infty$ and if Δ is a finite extension of k_0, then there exists a valuation v of $K_0(x)$ extending v_0 such that the residue field of v is a simple transcendental extension of Δ and the value group of v is G (cf. [7, Cor. 3.2]). In this paper, we consider analogous problems for an extension $(K,v)/(K_0,v_0)$ of valued fields where K is a function field of a conic over K_0. Our method of determining the residue field of v incidentally yields that the analogous converse does not hold for function fields of conics.

1. Statements of results

Recall that for a finitely generated field extension K/K_0, K is said to be a function field of a conic over K_0 if the transcendence degree (henceforth abbreviated as tr. deg.)

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of K/K_0 is 1 and if $K=K_0(x,y)$ where x and y satisfy an irreducible polynomial relation of total degree 2 over K_0. Further, it is said to be a regular function field of a conic over K_0 if (i) K/K_0 is a separable extension, i.e., either x is separably algebraic over $K_0(y)$ or y is separably algebraic over $K_0(x)$ and (ii) K_0 is algebraically closed in K.

We shall prove:

Theorem 1.1. *Let K be a function field of a conic over a field K_0. Let v_0 be a valuation of K_0 and v be an extension of v_0 to K. Assume that the characteristic of the residue field k_0 of v_0 is $\neq 2$. Then the residue field k of v is either an algebraic extension of k_0 or k is a regular function field of a conic over a finite extension of k_0.*

At the end of the third section we give an example to show that the above result does not always hold in case $\operatorname{char} k_0=2$ even if K/K_0 is assumed to be regular.

It is well known that, for a finitely generated extension K/K_0 of tr.deg 1 with K_0 algebraically closed in K, the genus of K/K_0 is 0 if and only if K is a function field of a conic over K_0 (see [**1**, p. 302, Thm. 6]). Keeping this in view, we see that Theorem 1.1. can also be easily deduced from the genus inequality of Matignon (cf. [**6**, Thm. 4], [**4**]); the latter has been proved using methods of rigid analytic geometry and some deep results of valuation theory. The proof given here is based on elementary valuation theory and happens also quickly to yield the following theorems.

Theorem 1.2. *Let the hypothesis be as in Theorem* 1.1. *Assuming that the extension k/k_0 is not algebraic. Let Δ be the algebraic closure of k_0 in k and $G_0\subseteq G$ be the value groups of v_0 and v respectively. If k is not a purely transcendental extension of Δ, then $G=G_0$ and $\Delta=k_0$.*

Theorem 1.1 leads to the following problem:

Let v_0 be a non-trivial valuation of a field K_0 with value group G_0 and residue field k_0 of char. $\neq 2$. Given a totally ordered abelian group G containing G_0 as an ordered subgroup with $[G:G_0]<\infty$ and an extension k of k_0 which is a regular function field of a conic over a finite extension of k_0, does there exist an extension v of v_0 to an over field K which is a function field of a conic over K_0 such that the value group of v is G and its residue field is k?

It is immediate from Theorem 1.2 that the answer to the above question is "no" in general.

2. Some preliminary results

We first introduce some notation and a few definitions. Let $P(X)=P(X_1,\dots,X_n)$ be an irreducible polynomial of $\deg\geqq 1$ over a field L_0 in $n\geqq 2$ variables $X_1,\dots,X_n$. The ideal (P) generated by $P(X)$ in $L_0[X]$ is a prime ideal and the quotient field L (say) of the integral domain $L_0[X]/(P)$ may be regarded as an extension of L_0 by identifying L_0 with its canonical image in L. If x_i is the image of X_i in L, then $P(x_1,\dots,x_n)=0$ and

$L=L_0(x_1,\ldots,x_n)$. Moreover the degree of transcendence of L/L_0 is $n-1$ and $x_1,\ldots,x_n$ satisfy no non- trivial L_0-polynomial relation of degree $<\deg P$; in particular if $P(X)$ is of total degree $\geqq 2$, then $x_1,\ldots,x_n$ are non-zero.

An extension L_1 of L_0 will be said to be the *function field of* $P(X)$ *over* L_0 if L_1 is L_0-isomorphic to the quotient field of $L_0[X]/(P)$; the irreducible polynomial $P(X)$ will then be called a *defining polynomial for* L_1/L_0.

The following proposition is well known (cf. [**11**, Proposition 1.1]) and can be easily proved using ([**13**, p. 101, Theorem 29)]. We omit its proof.

Proposition A. *Let* $P(X)=P(X_1,\ldots,X_n)$ *be an irreducible polynomial over a field* L_0 *in* $n\geqq 2$ *variables. An extension* L *of* L_0 *is the function field of* $P(X)$ *over* L_0 *if tr. deg* $(L/L_0)=n-1$ *and if there exist* $x_1,\ldots,x_n$ *in* L *satisfying* $P(x_1,\ldots,x_n)=0$ *such that* $L=L_0(x_1,\ldots,x_n)$.

To any polynomial $P(X)$ of $\deg\geqq 1$ over a field L_0, one can associate a homogeneous polynomial $P^h(X_0,X)$ in $L_0[X_0,X]$ (where $(X_0,X)=(X_0,X_1,\ldots,X_n)$) having the same degree as $P(X)$ which is uniquely determined by the additional property that $P(X)=P^h(1,X)$. Since the factors of a homogeneous polynomial are again homogeneous, $P(X)$ is irreducible over L_0 if and only if $P^h(X_0,X)$ is so. Moreover for $P^h(X_0,X)$ L_0-irreducible of degree $\geqq 2$, the polynomials $P^h(1,X)$ and $P^h(X_0,\ldots,1_i,\ldots,X_n)$ all define the same (to be precise L_0-isomorphic) function field over L_0; this is a consequence of Proposition A and the observation that we can write the function field L of $P^h(1,X)$ as $L=L_0(x_1,\ldots,x_n)$, $x_i\neq 0$, where

$$0=P^h(1,x_1,\ldots,x_n)=P^h(1/x_i,\ldots,1_i,\ldots,x_n/x_i).$$

So L/L_0 is independent of the variable used to dehomogenize $P^h(X_0,X)$.

It may be remarked that the function field of an L_0-irreducible polynomial $P(X)$ of $\deg\geqq 2$ is invariant under a homogeneous change of variables, i.e., if

$$X'_i=a_{i0}X_0+\cdots+a_{in}X_n,\quad 0\leqq i\leqq n \tag{1}$$

where (a_{ij}) is an invertible matrix with entries in L_0 and if the forms P^h and Q^h are related by $Q^h(X'_0,X')=P^h(X_0,X)$, then the function field of $Q(X')=Q^h(1,X')$ is the same as the function field of $P(X)$ over L_0. For if $L_0(x_0,x)=L_0(x'_0,x')$ (here x abbreviates $(x_1,\ldots,x_n)$) is the function field of $P^h(X_0,X)$, it follows from Proposition A that $L_0(x_1/x_0,\ldots,x_n/x_0)$ is the function field of $P(X)$ and $L_0(x'_1/x'_0,\ldots,x'_n/x'_0)$ is the function field of $Q(X')$, where the x'_i are defined in terms of x_i by means of (1), and then the invertible relations between the x'_i and the x_i given by (1) show that these two fields are equal.

It is clear from the above discussion that if $P(X_1,\ldots,X_n)$ and $Q(X'_1,\ldots,X'_n)$ are two L_0-irreducible polynomials of degree 2 such that their associated quadratic forms are related by

$$P^h(X_0, X) = qQ^h(X'_0, X')$$

where q is a non-zero element of L_0 and where X_i and X'_i are related by (1), then the function fields of $P(X)$ and $Q(X')$ over L_0 are the same.

The following lemma (which is already known [**11**, Proposition 2.2., Theorem 2.3]) is an immediate consequence of what we have said above and of the fact that every quadratic form over a field of char $\neq 2$ can be diagonalised by a linear change of variables.

Lemma 2.1. *Let K_0 be a field of char $\neq 2$ and let K be a function field of a conic over K_0. Then there exist explicitly constructible elements c, $d \in K_0$ such that the K_0-irreducible polynomial $X^2 - cy^2 - d$ is a defining polynominal for K/K_0.*

Notation. Let v' be a valuation of a simple transcendental extension $K_0(y)$ of a field K_0 which extends a valuation v_0 of K_0. Let $k_0 \subseteq k'$ be the residue fields of v_0 and v' respectively. For any ξ in the valuation ring of v', we denote by ξ^* its v'-residue, i.e., the image of ξ in the residue field of v'. Suppose that k' is not an algebraic extension of k_0. For such an extension v'/v_0, we define a number E (more precisely written as $E(v'/v_0)$) by

$$E = \min \{[K_0(y): K_0(\xi)] \mid \xi \in K_0(y), v'(\xi) \geqq 0, \xi^* \text{ is tr.over } k_0\}.$$

Lemma 2.2. *Let v' be an extension of a valuation v_0 of K_0 to a simple transcendental extension $K_0(y)$. Suppose that the residue field k' of v' is not an algebraic extension of the residue field k_0 of v_0. Then to any λ in the value group of v', there corresponds a polynomial $R(y) \in K_0[y]$ of degree less than $E = E(v'/v_0)$ such that $\lambda = v'(R(y))$.*

Proof. Fix an algebraic closure $\bar{K}_0$ of K_0 and an extension v'' of v' to $\bar{K}_0(y)$. We denote by $\bar{v}_0$ the restriction of v'' to $\bar{K}_0$ and by $\bar{k}_0 \subseteq k''$ the residue fields of $\bar{v}_0$, v'' respectively. Let $G_0 \subseteq G'$ be the value groups of v_0 and v'. The extension k'/k_0 is given to be non-algebraic, therefore so is k''/k_0, and since $\bar{k}_0/k_0$ is algebraic, it follows that $k''/\bar{k}_0$ is a transcendental extension. Arguing exactly as in [**9**, p. 205, 2.5], we can easily prove that there exist α, $a \in \bar{K}_0$ such that the v''-residue $((y-\alpha)/a)^*$ of $((y-\alpha)/a)$ is transcendental over $\bar{k}_0$. We shall denote $v''(y-\alpha) = \bar{v}_0(a)$ by μ. Clearly μ is torsion mod G_0, i.e., there exists a positive integer m such that $m\mu \in G_0$. As in [**2**, Chapter 6, §10.1, Proposition 2], it can be easily shown that for any polynomial $f(y) = \sum_i c_i(y-\alpha)^i$ over $\bar{K}_0$,

$$v''(f(y)) = \min (\bar{v}_0(c_i) + i\mu),$$

since the assumption $v''(f(y)) > \min_i (\bar{v}_0(c_i) + i\mu)$ would lead to $((y-\alpha)/a)^*$ being algebraic over $\bar{k}_0$. This also shows that $v''(f(y))$ is torsion mod G_0 for any $f(y)$ in $\bar{K}_0(y)$. Define a subset D of $\bar{K}_0$ by

$$D=\{\gamma\in\bar{K}_0:\bar{v}_0(\gamma-\alpha)\geqq\mu\}.$$

Choose an element β of D such that $[K_0(\beta):K_0]\leqq[K_0(\gamma):K_0]$ for all γ in D. We shall denote by $P(y)$ the minimal polynomial of β over K_0 of degree n(say), by θ the element $v'(P(y))$ of G' and by G_1 the value group of the valuation $\bar{v}_0$ restricted to $K_0(\beta)$. As shown above θ is torsion mod G_0; let s be the smallest positive integer such that $s\theta\in G_1$. By [**5**, Theorem 1.4, Corollary 1.2] the value group G' of v' can be expressed as

$$G'=G_1+\mathbb{Z}\theta=\bigcup_{r=0}^{s-1}(G_1+r\theta). \tag{2}$$

It is clear from the proof of Theorem 1.3 of [**5**] that

$$E(v'/v_0)=sn=s\deg P(y). \tag{3}$$

Observe that α,β lie in D; also in view of the choice of β any polynomial over K_0 having degree less than n has no root in D. So by assertion (ii) of Lemma 2.1 of [**5**] for such a polynomial $g(y)$, one has

$$v'(g(y))=\bar{v}_0(g(\alpha))=\bar{v}_0(g(\beta)). \tag{4}$$

Let λ be any element of G'. In view of (2), there exists a polynomial $g(y)\in K_0[y]$ of degree less than n and an integer r, $0\leqq r\leqq s-1$, such that

$$\lambda=\bar{v}_0(g(\beta))+r\theta.$$

Keeping (4), in view, we can re-write the above equation as

$$\lambda=v'(g(y)P(y)^r).$$

We set $R(y)=g(y)P(y)^r$. Then by (3) and the fact that $\deg g(y)<n$, we see that

$$\deg R(y)<(r+1)n\leqq sn=E(v'/v_0).$$

The lemma is now proved.

Lemma 2.3. *Let the hypothesis be as in the above lemma. Let $\eta=f(y)/g(y)$ with $f(y)$ and $g(y)$ in $K_0[y]$, be an element of the valuation ring of v' having its v'-residue η^* transcendental over k_0. If $\deg f(y)\leqq E$ and $\deg g(y)\leqq 2E-1$, then η^* is a generator of the simple transcendental extension k'/Δ', where Δ' is the algebraic closure of k_o in k'.*

Proof. Let v'', $\bar{v}_0$, α, β, $P(y)$, θ and s be as in the proof of Lemma 2.2. Let $q(y)\in K_0[y]$ be a polynomial of degree less than n such that $\bar{v}_0(q(\beta))=s\theta$. By [**5**, Theorem 1.3(i)] the v'-residue of $P(y)^s/q(y)$ is a generator of the simple transcendental extension k'/Δ' and Δ' equals the residue field of the valuation $\bar{v}_0$ restricted to $K_0(\beta)$; we shall denote this generator of k'/Δ' by t.

Observe that any polynomial $h(y)\in K_0[y]$ can be uniquely written as a finite sum

$$h(y)=\sum_{i=0}^{m} h_i(y)P(y)^i$$

where, for $0\leqq i\leqq m$, the polynomial $h_i(y)\in K_0[y]$ is either zero or is of degree less than that of $P(y)$. This will be referred to as the canonical representation of $h(y)$ with respect to $P(y)$.

By hypothesis $\deg f(y)\leqq E=sn$, so the index i in the canonical representation of $f(y)$ with respect to $P(y)$ cannot vary beyond s.

Arguing similarly for $g(y)$, we can write the canonical representations of $f(y)$ and $g(y)$ with respect to $P(y)$ as

$$f(y)=\sum_{i=0}^{s} f_i(y)P(y)^i,\ g(y)=\sum_{i=0}^{2s-1} g_i(y)P(y)^i.$$

By [5, Lemma 2.1(ii), (iii)], we have

$$v'(f(y))=\min_i(\bar{v}_0(f_i(\beta))+i\theta),\ v'(g(y))=\min_i(\bar{v}_0(g_i(\beta))+i\theta).$$

Let j be the smallest index, $0\leqq j\leqq s$, such that $v'(f(y))=\bar{v}_0(f_j(\beta))+j\theta$. Since s is the smallest positive integer for which $s\theta\in\bar{v}_0(K_0(\beta))$, it follows that

$$\bar{v}_0(f_j(\beta))+j\theta<\bar{v}_0(f_i(\beta))+i\theta,\quad 0\leqq i\leqq s,\quad i\not\equiv j \bmod s \tag{5}$$

Also, we have

$$v'(g(y))=v'(f(y))=\bar{v}_0(f_j(\beta))+j\theta;$$

the same property of s shows that

$$\bar{v}_0(f_j(\beta))+j\theta\leqq\bar{v}_0(g_i(\beta))+i\theta,\quad 0\leqq i\leqq 2s-1 \tag{6}$$

and

$$\bar{v}_0(f_j(\beta))+j\theta<\bar{v}_0(g_i(\beta))+i\theta,\quad \text{if}\quad i\not\equiv j \bmod s \tag{7}$$

Write $\eta=\eta_1/\eta_2$, where

$$\eta_1=f(y)/f_j(y)P(y)^j=\sum_i f_i(y)P(y)^i/f_j(y)P(y)^j,\quad \eta_2=g(y)/f_j(y)P(y)^j.$$

Keeping (5) in view and the fact that for any non-zero polynomial $h(y) \in K_0[y]$ of degree less than n, the v''-residue of $(h(y)/h(\beta))$ is 1(cf. [5, Lemma 2.1(ii)]), we deduce immediately that

$$\eta_1^* = \begin{cases} 1 & \text{if} \quad j > 0 \\ 1 + t(f_s(\beta)q(\beta)/f_0(\beta))^*, & \text{if} \quad j = 0. \end{cases}$$

Similarly using (6) and (7), we derive that

$$\eta_2^* = \begin{cases} (g_j(\beta)/f_j(\beta))^* + t(q(\beta)g_{j+s}(\beta)/f_j(\beta))^*, & \text{if} \quad j < s \\ (g_s(\beta)/f_s(\beta))^* \left[1 + \frac{1}{t}(g_0(\beta)/g_s(\beta)q(\beta))^* \right], & \text{if} \quad j = s. \end{cases}$$

Thus it has been shown that $\eta^* = (A' + tB')/(C' + tD')$ for some A', B', C', D' in Δ'. By hypothesis $\eta^* \notin \Delta'$, so $A'D' - B'C' \neq 0$. The element t being a generator of the simple transcendental extension k'/Δ', it now follows that so is η^*.

The following lemma can be easily deduced from Theorem 17.17 and Corollary 16.6 of [3]. For the sake of completeness, we give a simple proof here.

Lemma 2.4. *Let $F = F'(\sqrt{\eta})$ be a quadratic extension of a field F' of char $\neq 2$, $\eta \in F'$. Let w' be a valuation of F' having $w'(\eta) = 0$ such that the residue field k' of w' has char $\neq 2$. Supppose that w' can be uniquely extended to a valuation w of F, then the w-residue of $\sqrt{\eta}$ is not in k'.*

Proof. Let W' be the valuation ring of w', and let M' be the maximal ideal of W'. If the w'-residue η^* of η lies in k'^2, then in the ring $W'[\eta]$ there are two maximal ideals contracting to M' (as $W'[\eta]/M'W'[\eta] \cong k'[X]/(X^2 - \eta^*) \cong k' \oplus k'$). In the integral closure T of W' in F there are maximal ideals lying over each of these maximal ideals in $W'[\eta]$, as T is integral over $W'[\eta]$. Each maximal ideal of T determines a valuation of F extending w'.

3. Proof of Theorems 1.1, 1.2

To prove the first theorem, we may assume that k/k_0 is not an algebraic extension. In view of Lemma 2.1, we may write $K = K_0(x, y)$ where (x, y) satisfies an irreducible polynomial $X^2 - cY^2 - d$ over K_0. Observe that y is transcendental over K_0 and that $[K: K_0(y)] \leqq 2$. We denote by v', the valuation v restricted to $K_0(y)$ and by k', G' the residue field and the value group of v'. Then $[k: k'] \leqq 2$ and k'/k_0 is not an algebraic extension.

When $k = k'$, the desired result follows from the Ruled Residue Theorem [8] applied

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to the simple transcendental extension $K_0(y)/K_0$ and the observation that a simple transcendental extension $L_0(t)$ of a field L_0 is the regular function field of a conic over L_0, which can be visualized by writing $L_0(t)$ as $L_0(t,1/t)$ where $(t,1/t)$ satisfies $XY-1=\theta$.

Assume now that $[k:k']=2$. Let Δ', Δ denote the algebraic closures of k_0 in k' and k respectively. By the Ruled Residue Theorem k' is a simple transcendental extension of Δ' and Δ' is a finite extension of k_0. If $\Delta'\subsetneqq\Delta$, then

$$k'=\Delta'(t)\subsetneqq\Delta(t)\subseteq k.$$

In view of the assumption that $[k:k']=2$, it is now clear that in the present case

$$[\Delta:\Delta']=2 \quad \text{and} \quad k=\Delta(t).$$

The theorem remains to be proved when $\Delta'=\Delta$ and $[k:k']=2$. Since

$$[K:K_0(y)]=[k:k']=2, \tag{8}$$

it follows from the fundamental inequality (cf. [**2**, Chapter 6, §8.3, Theorem 1(b)]) relating the degree of extension with the ramification indices and residual degrees that the value group of v is G'; in particular $v(x)\in G'$. By Lemma 2.2, there exists a non-zero polynomial $R(y)\in K_0[y]$ of degree less than $E=E(v'/v_0)$ such that $v(x)=v'(R(y))$. Set

$$Z=x/R(y) \quad \text{and} \quad \eta=(cy^2+d)/R(y)^2.$$

Since $x^2-cy^2-d=0$, the v-residue Z^* of Z satisfies the polynomial $X^2-\eta^*$ over k'. In view of (8) and the fundamental inequality referred to above, v is the only extension to $K=K_0(y,Z)$ of the valuation v' defined on $K_0(y)$. Recall that $\operatorname{char} k'\neq 2$; it now follows from Lemma 2.4 applied to the extension $K/K_0(y)$ that $Z^*=\sqrt{\eta^*}$ is not in k'. Since k' contains Δ' which equals the algebraic closure of Δ' in k, we conclude that Z^* and hence η^* is transcendental over Δ'. Therefore $k=k'(\sqrt{\eta^*})$ is proved to be a function field and hence a regular function field of a conic over $\Delta'=\Delta$, as soon as we show that there exists a generator u of the simple transcendental extension k'/Δ' such that η^* is a polynomial in u of degree $\leqq 2$ with coefficients from Δ'. By Lemma 2.3, η^* is itself a generator, say u, of the simple transcendental extension k'/Δ', if $\deg(cy^2+d)\leqq E$; in fact in this situation $k=\Delta'(\sqrt{\eta^*})$ is a simple transcendental extension of Δ'. The remaining case is when $E=1$, i.e., when there exist $a,b\in K_0$ such that $((y-a)/b)^*=u$ (say) is transcendental over k_0. In this case the polynomial $R(y)$ being of degree less than $E=1$, must be a constant say R. Therefore on writing $\eta=(cy^2+d)/R^2$ as a polynomial in $(y-a)/b$, we conclude that η^* is a polynomial of degree $\leqq 2$ in u over k_0. The theorem is now completely proved.

Proof of Theorem 1.2. We retain the notation v', k', Δ', G' and Δ of the above proof. It is clear from this proof that the situation when the transcendental extension k/Δ is not a simple transcendental extension can arise only when $[k:k']=2$, $\Delta'=\Delta$ and $E(v'/v_0)=1$. As remarked in the proof, $G=G'$ in this case. The desired assertion now follows from the well-known inequality (cf. [**10**, p. 586, § 1.2])

$$E(v'/v_0) \geqq [G':G_0][\Delta':k_0]=[G:G_0][\Delta:k_0].$$

Before constructing an example to show that the result of Theorem 1.1 may not hold even if K/K_0 is regular in the case char $k_0=2$, we prove a small lemma which occurs essentially in [**11**].

Lemma B. *Let c and d be elements of a field* F_0.

(i) *If* $cd \neq 0$, *then the polynomial* X^2-cy^2-d *is irreducible over* F_0 *provided* char $F_0 \neq 2$.

(ii) *If* $\sqrt{c} \notin F_0$ *or* $\sqrt{d} \notin F_0$, *then the polynomial* X^2-cY^2-d *is irreducible over* F_0 *when* char $F_0=2$.

(iii) *Suppose that* char F_0 *is* 2 *and that* $[F_0(\sqrt{c},\sqrt{d}):F_0]=4$. *If* $F=F_0(x,y)$ *is the function field of a conic over* F_0 where (x,y) *satisfies* $x^2-cy^2-d=0$, *then* F_0 *is algebraically closed in F.*

Proof. The proof of (i) and (ii) is a routine calculation and is omitted. To prove (iii), let α be an element of F which is algebraic over F_0. Since $F(\sqrt{c},\sqrt{d})=F_0(\sqrt{c},\sqrt{d},y)$ is a simple transcendental extension of $F_0(\sqrt{c},\sqrt{d})$, so α must be in $F_0(\sqrt{c},\sqrt{d})$, say $\alpha=r+s\sqrt{c}+t\sqrt{d}+u\sqrt{cd}$ where $r,s,t,u \in F_0$. On the other hand, we can write $\alpha=\beta+\gamma x$ where $\beta,\gamma \in F_0(y)$ and hence $\alpha=\beta+\gamma y\sqrt{c}+\gamma\sqrt{d}$ as $x=y\sqrt{c}+\sqrt{d}$. Since $[F_0(y)(\sqrt{c},\sqrt{d}):F_0(y)]=4$, we can equate the coefficients of 1, $\sqrt{c}$, $\sqrt{d}$, $\sqrt{cd}$ in these formulae for α and deduce that $\gamma=0$, so $\alpha=\beta=r \in F_0$.

Example 3.1. Let K_0 be a field of char 0 and v_0 be a valuation of K_0 having residue field k_0 with char $k_0=2$ and suppose there exist $u,t \in k_0$ such that $[k_0(\sqrt{u},\sqrt{t}):k_0]=4$ (e.g. one can take $K_0=Q(x_1,x_2)$, a purely transcendental extension of tr. degree 2 of the field of rationals; define v_0 on $Q[x_1,x_2]$ by $v_0(\sum a_{ij}x_1^i x_2^j)=\min(w_2(a_{ij}))$ where w_2 is the 2-adic valuation of Q, and choose t,u to be the v_0-residues of x_1 and x_1+x_2). Pick any $c,d \in K_0$ having v_0-residues u,t respectively. Let $K=K_0(x,y)$ where x,y satisfy the relation $X^2-cY^2-d=0$; observe that the polynomial on the left hand side is irreducible over the algebraic closure of K_0 by Lemma B(i) and hence defines a regular function field of a conic over K_0 in view of [**12**, p. 18, Theorem 5]. Let v' denote the valuation of the field $K_0(y)$ which is defined for any polynomial $F(y)=\sum_{i=0}^r f_i y^i$, $f_i \in K_0$, by

$$v'(F(y))=\min_i(v_0(f_i)).$$

The residue field k' of v' is a simple transcendental extension $k_0(y^*)$ of k_0 where y^*

denotes the v'-residue of y (cf. [2, Chapter 6, § 10.1, Proposition 2]). Let v be a valuation of K which extends v' and k denote its residue field. It is easily verified that $k=k_0(x^*, y^*)$ where the v-residues x^*, y^* of x and y satisfy the irreducible (by Lemma B(ii)) polynomial relation $X^2-uY^2-t=0$. Since $[k_0(\sqrt{u},\sqrt{t}):k_0]=4$, k/k_0 is not separable; further in view of Lemma B(iii), k_0 is algebraically closed in k. Hence k cannot be regular function field of a conic over any subfield of k containing k_0.

Remark 3.2. Let K_0, K, v_0, v, k_0, k be as in Theorem 1.1 with k/k_0 non-algebraic. It follows from the genus inequality of Matignon ([4]) that k is a function field of a conicover a finite extension of k_0 even if char. $k_0=2$.

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