SINGULAR SETS OF SOME KLEINIAN GROUPS (II)

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Dedicated to Professor K. Noshiro on his sixtieth birthday

Introduction

In the theory of automorphic functions it is important to investigate the properties of the singular sets of the properly discontinuous groups. But we seem to know nothing about the size or structure of the singular sets of Kleinian groups except the results due to Myrberg and Akaza [1], which state that the singular set has positive capacity and there exist Kleinian groups whose singular sets have positive 1-dimensional measure. In our recent paper [2], we proved the existence of Kleinian groups with fundamental domains bounded by five circles whose singular sets have positive 1-dimensional measure and presented the problem whether there exist or not such groups in the case of four circles. The purpose of this paper is to solve this problem. Here we note that, by Schottky's condition [4], the 1-dimensional measure of the singular set is always zero in the case of three circles.

In § § 1-3 we shall give the more extensive criterion than that of the former paper [2] for the singular sets of the Kleinian groups to have positive 1-dimensional measure and define the general computing functions of order ν on a Kleinian group. In § 4, using these computing functions we shall give the example which solves the problem.

§ 1. Kleinian groups and isometric circles of linear transformations

1. Consider the properly discontinuous groups G of the linear transformations which have the fundamental domain B_0 bounded by N mutually disjoint circles $\{K_i\}_{i=1}^{N}$. Then there exist two different kinds of generators. A generator S_{i_0} of the first kind transforms the outside of a boundary circle K_{i_0} onto the inside of a boundary circle K'_{i_0} different from K_{i_0} and a generator S_{j_0} of the second kind transforms the outside of K_{j_0} onto the inside of K_{j_0} itself. The former

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is the hyperbolic or loxodromic transformation and the latter is the elliptic transformation with period 2.

Let us start from B_0 and form a properly discontinuous group of linear transformations with the fundamental domain B_0. Take $2p$ $(N \geqq 2p)$ boundary circles $\{H_i, H_i'\}_{i=1}^{p}$ from $\{K_i\}_{i=1}^{N}$. Let S_i be a hyperbolic or loxodromic generator which transforms the outside of H_i onto the inside of H_i'. We denote by S_i^{-1} the inverse transformation of S_i. Then $\{S_i\}_{i=1}^{p}$ generate a Schottky group G_1 whose fundamental domain $B_1 \supset B_0$ is bounded by $\{H_i, H_i'\}_{i=1}^{p}$. Let $\{T_j\}_{j=1}^{q}$ be the elliptic transformations with period 2 corresponding to the remaining boundary circles $\{K_j\}_{j=1}^{q}$, where $N-2p=q$. Then $\{T_j\}_{j=1}^{q}$ generate a properly discontinuous group G_2 whose fundamental domain $B_2 \supset B_0$ is the outside of the boundary circles $\{K_j\}_{j=1}^{q}$. By combining two groups G_1 and G_2, a new group $G = G_1 \cdot G_2$, which is generated by $\{S_i\}_{i=1}^{p}$ and $\{T_j\}_{j=1}^{q}$, is obtained and is called a Kleinian group. It is easily seen that the fundamental domain of G coincides with $B_0 = B_1 \cap B_2$ and G is properly discontinuous.

2. We denote by ST the transformation obtained by composition of transformations S and T contained in G, that is,

$$ST(z) = S(T(z)).$$

We put $SS = S^2$ and $S^{\lambda} = S \cdot S^{\lambda-1}$ inductively for any integer λ (>1). For a negative integer λ, S^{λ} denotes $(S^{-1})^{|\lambda|}$. Then any element S of G has the form

$$S = S_{(\nu_k)} T_{j_k} \cdots S_{(\nu_1)} T_{j_1} S_{(\nu_0)}, \text{ viz.,} \tag{1}$$
$$S(z) = S_{(\nu_k)}(T_{j_k}(\cdots(T_{j_1}(S_{(\nu_0)}(z)) \cdots)),$$

where ν_i $(i = 0, \ldots, k)$ are integers and $S_{(\nu_i)}$ denotes the $|\nu_i|$ product of generators of G_1 or their inverses and T_{j_i} $(T_{j_i}^2 = \text{identity})$ denotes the generator of G_2. We call the sum

$$m = \sum_{i=0}^{k} |\nu_i| + k$$

the grade of S. The image $S(B_0)$ of the fundamental domain B_0 by S $(\in G)$ with grade m $(\geqq 1)$ is bounded by N circles $S(H_i)$, $S(H_i')$ and $S(K_j)$, $(i = 1, \ldots, p,$ $j = 1, \ldots, q,$ $N = 2p + q)$. For simplicity, we call the outer boundary circle $C^{(m)}$ of $S(B_0)$, which is contained in the boundary of the image of B_0 under some T $(\in G)$ with grade $m-1$, a circle of grade m. Circles $\{H_i, H_i'\}_{i=1}^{p} \cup \{K_j\}_{j=1}^{q}$;

which bound B_0, are of grade 1. The number of circles of grade m is obviously equal to $N(N-1)^{m-1}$.

Denote by D_m the $N(N-1)^{m-1}$-ply connected domain bounded by the whole circles of grade m. Evidently $\{D_m\}$ $(m=0, 1, \ldots)$ is a monotone increasing sequence of domains. The complementary set D_m^c of D_m with respect to the extended z-plane consists of $N(N-1)^{m-1}$ mutually disjoint closed discs. The set $E = \bigcap_{m=1}^{\infty} D_m^c$ is perfect and nowhere dense. We call E the singular set of G. The group G is properly discontinuous in the complementary set of E.

3. For a linear transformation of the form

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1, \ c \neq 0,$$

the circle $I: |cz+d|=1$ is called the isometric circle of the transformation (See Ford [3]). The radius of I equals $1/|c|$.

By a transformation lengths and areas inside its isometric circle are increased in magnitude and lengths and areas outside the isometric circle are decreased in magnitude. A transformation carries its isometric circle into the isometric circle of the inverse transformation. The radii of the isometric circles of a transformation and its inverse are equal.

Let G denote a properly discontinuous group of linear transformations. We suppose that, if an element of G transforms the point at infinity into itself, then the element is the identity of G. Consider two arbitrary transformations of G

$$T: T(z) = \frac{az+c}{cz+d}, \quad ad-bc=1, \ c \neq 0,$$

and

$$S: S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}, \quad \alpha\delta-\beta\gamma=1, \ \gamma \neq 0.$$

We assume that $S \neq T^{-1}$. The isometric circle of $ST = S(T(z))$ is the circle

$$|(\gamma a+\delta c)z+\gamma b+\delta d| = 1.$$

Denote by I_S, I'_S, I_T, I'_T and I_{ST} isometric circles of S, S^{-1}, T, T^{-1} and ST, respectively. Let g_S, g'_S, g_T, g'_T and g_{ST} be their centers, and let R_S, R_T and R_{ST} be radii of I_S, I_T and I_{ST}.

As to these values, the relation

$$(2) \qquad R_{ST} = \frac{1}{|\gamma a + \delta c|} = \frac{R_S \cdot R_T}{|g'_T - g_S|}$$

holds.

If the grade of a transformation in G is m, its isometric circle is called an isometric circle of grade m. The number of the isometric circles with grade m is obviously equal to $N(N-1)^{m-1}$.

§ 2. Measure of the singular sets of Kleinian groups

4. Given a set ε of points in the z-plane and a positive number δ, we denote by $I(\delta, \varepsilon)$ a family of a countable number of closed discs U of diameter $\ell_U \leqq \delta$ such that every point of ε is an interior point of at least one U.

We call the quantity

$$\Lambda^{\eta}\varepsilon = \lim_{\delta\to 0}[\inf_{\{I(\delta,\varepsilon)\}} \sum_{U\in I(\delta,\varepsilon)} \ell_U^{\eta}]$$

the η-dimensional measure of ε.

In [2] we obtained the important criterion for the singular set E of a Kleinian group G to have the positive η-dimensional measure. But we need a more extensive one to get a deeper result about the property of the singular set E.

5. Denoting by $r_j^{(m)}$ and by $r_i^{(m+1)}$ $(i=1, \ldots, N-1)$ the radius of the outer boundary circle $C_j^{(m)}$, that is, a circle of grade m and the radii of $N-1$ inner boundary circles $C_i^{(m+1)}$ $(i=1, \ldots, N-1)$ of the image B_m of the fundamental domain B_0 by a transformation $S^{(m)}$ $(\in G)$ with grade m, we have the following (See [1]).

Proposition 1. *There exist positive constants K_0 (<1) and k_0 depending only on B_0 such that*

$$(3) \qquad k_0 r_j^{(m)} \leqq r_i^{(m+1)} \leqq K_0 r_j^{(m)}, \qquad (i=1, \ldots, N-1).$$

Denote by F_{n_0} the family of all closed discs bounded by circles of grade n $(\geqq n_0)$. It is easy to see that F_{n_0} is a covering of the singular set of our Kleinian group G and by Proposition 1 that the diameter of any discs of F_{n_0} is less than a given δ (>0) for sufficiently large n_0.

For such covering F_{n_0} we have the following important

Proposition 2 ([1]). *Let $F_{n_0}^{\delta/k_0}$ be a covering of E constructed by discs in F_{n_0}*

whose radii are not greater than $\delta/2k_0$ *and let* r_C *be the radius of a disc* C *in* $F_{n_0}^{\delta/k_0}$, *where* k_0 *is a positive constant in Proposition* 1. *Then it holds*

$$(4)\qquad L^{\eta}E = \lim_{\delta\to 0} \inf_{\{F_{n_0}^{\delta/k_0}\}} \sum_{C\in F_{n_0}^{\delta/k_0}} (2r_C)^{\eta} \leqq \kappa\left(\frac{k_0}{2}\right)^{-\eta} \Lambda^{\eta}E,$$

where κ *is an absolute constant.*

6. Now we shall give a sufficient condition for the singular set of G to be positive. For this purpose we need a following lemma.

LEMMA 1. *Let*

$$S^{(m)} : z' = S^{(m)}(z) = \frac{a^{(m)}z + b^{(m)}}{c^{(m)}z + d^{(m)}}$$

be a transformation of grade m *in* G *and denote by* $r_i^{(m)}$ *the radius of a boundary circle* $C_i^{(m)}$ *of* $S^{(m)}(B_0)$. *Then there exist positive constants* $k(G)$ *and* $K(G)$ *depending only on* G *such that*

$$(5)\qquad k(G)(R^{(m)})^{\mu} \leqq (r_i^{(m)})^{\mu/2} \leqq K(G)(R^{(m)})^{\mu}, \qquad (i = 1, 2, \ldots, N),$$

where $R^{(m)} = 1/|c^{(m)}|$ *is the radius of isometric circle of* $S^{(m)}$.

Proof. The radius $r_i^{(m)}$ of a circle $C_i^{(m)}$ of grade m by $S^{(m)}(z)$ is given by

$$2\pi r_i^{(m)} = \int_H \left|\frac{dS^{(m)}(z)}{dz}\right| |dz| = \int_H \frac{|dz|}{|c^{(m)}z + d^{(m)}|^2},$$

where H is a suitable one in $\{H_i, H_i'\}_{i=1}^{p} \cup \{K_j\}_{j=1}^{q}$ which $S^{(m)}$ carries into $C_i^{(m)}$.

Hence, we have

$$2\pi r_i^{(m)} = \frac{1}{|c^{(m)}|^2}\int_H \frac{|dz|}{|z + (d^{(m)}/c^{(m)})|^2}.$$

Again we note that the point $-d^{(m)}/c^{(m)}$ is outside of B_0. If we put

$$\Delta = \max_{z\in H} |z + (d^{(m)}/c^{(m)})| \qquad \text{and} \qquad \delta = \min_{z\in H} |z + (d^{(m)}/c^{(m)})|,$$

then

$$\frac{r}{\Delta^2}\cdot\frac{1}{|c^{m}|^2} \leqq r_i^{(m)} \leqq \frac{r}{\delta^2}\cdot\frac{1}{|c^{(m)}|^2}.$$

where r is the radius of H.

Such inequalities hold for all circles of grade m. Hence, there exist positive constants $k(G)$ and $K(G)$ such that

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$$k(G)(R^{(m)})^{\mu} \leqq (r_i^{(m)})^{\mu/2} \leqq K(G)(R^{(m)})^{\mu}, \qquad (\mu>0).$$

In fact, we may take $k(G)$ as the minimum of $(r/\Delta^2)^{\mu/2}$ and $K(G)$ as the maximum of $(r/\delta^2)^{\mu/2}$, when H runs in $\{H_i, H_i'\}_{i=1}^{p} \cup \{K_j\}_{j=1}^{q}$ and $S^{(m)}$ $(m>1)$ varies in G. q.e.d.

By using this lemma, we can prove the following

THEOREM 1. *Let G be a Kleinian group defined in* §1. *If there exists a positive integer* ν *such that*

$$(6) \qquad \sum_{S^{(\nu)}} (R_{S^{(m+\nu)}})^{\mu} \geqq (R_{S^{(m)}})^{\mu}, \qquad (0<\mu<4,\ S^{(m+\nu)} = S^{(m)} \cdot S^{(\nu)})$$

for radius $R_{S^{(m)}}$ *of any isometric circle* $I_{S^{(m)}}$ *of grade m and radii* $R_{S^{(m+\nu)}} = R_{S^{(m)}S^{(\nu)}}$ *of* $(N-1)^{\nu}$ *isometric circles* $I_{S^{(m)}S^{(\nu)}}$ *of grade* $m+\nu$, *where the right element* T_1 *of* $S^{(m)}$ *does not equal to the inverse of the left element* T^* *of* $S^{(\nu)}$ *in* $S^{(m+\nu)} = S^{(m)}S^{(\nu)}$, *then the* $(\mu/2)$*-dimensional measure of the singular set E of G is positive.*

Proof. Take a covering $F_{n_0}^{\delta/k_0}$ of E constructed by a finite number of closed discs $D_{S^{(m_1)}}, \ldots, D_{S^{(m_Q)}}$, which are bounded by circles

$$(7) \qquad C_{S^{(m_1)}}, \ldots, C_{S^{(m_Q)}},$$

respectively, where $C_{S^{(m_j)}}$ $(1 \leqq j \leqq Q)$ is a circle of grade m_j, that is, an outer boundary circle of the image $S^{(m_j)}(B_0)$.

Denote $\min_{1\leqq j\leqq Q} (m_j)$ by m^*. We amend the covering $F_{n_0}^{\delta/k_0}$ in the following manner: (i) if $m_j - m^*$ is a integral multiple of ν, we leave the circle $C_{S^{(m_j)}}$ untouched, and (ii) if $m_j - m^* = \nu \cdot p + \tau$, $(0<\tau<\nu)$, where p is positive integer, we replace the circle $C_{S^{(m_j)}}$ with the $(N-1)^{\nu-\tau}$ circles $C_{S_1^{(m_j')}}, C_{S_2^{(m_j')}}, \ldots, C_{S^{(m_j')}_{(N-1)^{\nu-\tau}}}$ of grade m_j' contained in $C_{S^{(m_j)}}$, where $m_j' - m^* = \nu(p+1)$. After such amendment we get a new covering ${}^*F_{n_0}^{\delta/k_0}$ whose elements are all the discs bounded by the circles of grade $m^* + \nu \cdot p$. Denote such circles by

$$(8) \qquad C_{S^{(m_1')}}, C_{S^{(m_2')}}, \ldots, C_{S^{(m_R')}}, \qquad (Q \leqq R).$$

Then we get from (3) of Proposition 1 the following inequality

$$(9) \qquad \sum_{j=1}^{Q} (r_{S^{(m_j)}})^{\mu/2} \geqq K(\nu) \sum_{j=1}^{R} (r_{S^{(m_j')}})^{\mu/2}, \qquad (Q \leqq R),$$

where $r_{S^{(m_j)}}$ and $r_{S^{(m_j')}}$ are the radii of the circles (7) and (8), respectively and $K(\nu)$ is the constant depending only on ν and B_0. By using (5) of Lemma 1, we obtain

(10) $$\sum_{j=1}^{R}(r_{S(m_j')})^{\mu/2}\geqq k(G)\sum_{j=1}^{R}(R_{S(m_j')})^{\mu},$$

where $R_{S(m_j')}$ is the radius of the isometric circle of the transformation $S^{(m_j')}$.

From the construction of ${}^*F_{n_0}^{\delta/k_0}$, there exist in (8) some systems $\{W_{m_k^*}\}$, each of which consists of $(N-1)^{\nu}$ boundary circles with the following properties: (i) $(N-1)^{\nu}$ circles of $W_{m_k^*}$ have same grade number m_k^*, while the grade of circles of different systems are not necessarily equal, (ii) $(N-1)^{\nu}$ circles of each system $W_{m_k^*}$ are totality of inner boundary circles of $S^{(m_k^*-\nu)}(B_0)$ for a transformation $S^{(m_k^*-\nu)}$ of grade $m_k^*-\nu$ so that they are bounded by a circle of grade $m_k^*-\nu$.

These $(N-1)^{\nu}$ circles in $W_{m_k^*}$ are arranged $N-1$ by $N-1$ and are replaced by circles of grade m_k^*-1 and after that, we repeat also such procedure and so on. After ν time procedure, we reach to the circle of grade $m_k^*-\nu$, that is, the outer boundary circle of $S^{(m_k^*-\nu)}(B_0)$. By the assumption (6), it holds, for each system,

$$\sum_{S^{(\nu)}}(R_{S^{(m_k^*-\nu)}S^{(\nu)}})^{\mu} \geqq R^{\mu}_{S^{(m_k^*-\nu)}},$$

where $\sum_{S^{(\nu)}}$ denotes the sum when $S^{(\nu)}$ runs over all the transformations of grade ν whose left elements are not equal to the inverse of the right element of $S^{(m_k^*-\nu)}$. After replacing $(N-1)^{\nu}$ circles of each system $W_{m_k^*}$ by a circle $C_{S^{(m_k^*-\nu)}}$ of grade $m_k^*-\nu$ surrounding them, that is, an outer boundary circle of $S^{(m_k^*-\nu)}(B_0)$, we have also a new covering of E consisting of closed discs which are denoted by $D_{S(m_1'')}, D_{S(m_2'')}, \dots, D_{S(m_U'')}$. They are bounded by circles

(11) $$C_{S(m_1'')},\ C_{S(m_2'')},\ \dots,\ C_{S(m_U'')},\qquad (U<R).$$

Then there exist in (11) some systems $\{W_{m_{k'}^*}\}$ which satisfy the above conditions (i) and (ii) and hence, for each system of $\{W_{m_{k'}^*}\}$, it holds also

$$\sum_{S_1^{(\nu)}}(R_{S^{(m_{k'}^*-\nu)}S_1^{(\nu)}})^{\mu} \geqq R^{\mu}_{S^{(m_{k'}^*-\nu)}}.$$

Repeating this procedure, we obtain the following

(12) $$\sum_{j=1}^{R}(R_{S(m_j')})^{\mu}\geqq \sum_{S(m^*)}(R_{S(m^*)})^{\mu},$$

where $m^* = \min_{1 \leqq j \leqq Q}(m_j)$ and the summation in the right hand side is taken over all transformations in G with grade m^*. By a similar argument, if we put $m^* = \nu \cdot p_0 + \tau_0$, $(1 \leqq \tau_0 < \nu)$, where p_0 is a positive integer, we see easily

$$\sum_{S(m^*)} (R_{S(m^*)})^\mu \geqq \sum_{S(\tau_0)} (R_{S(\tau_0)})^\mu \geqq \min_{1 \leqq \tau_0 < \nu} \Big(\sum_{S(\tau_0)} (R_{S(\tau_0)})^\mu \Big), \tag{13}$$

where $\sum_{S(\tau_0)}$ denotes the sum with respect to all elements and their inverses of grade τ_0 in G and in particular $S^{(1)}$, $(\tau_0 = 1)$, denotes a generator or its inverse. Here the quantity in the right hand side of (13) is a positive constant. Thus, for any covering $F_{n_0}^{\delta/k_0}$ of E, we have from (9), (10), (12) and (13)

$$\sum_{j=1}^{Q} (r_{S(m_j)})^{\mu/2} \geqq K^* \min_{1 \leqq \tau_0 < \nu} \Big(\sum_{S(\tau_0)} (R_{S(\tau_0)})^\mu \Big) > 0, \tag{14}$$

where $K^* = K(\nu) k(G)$. Putting $\eta = \frac{\mu}{2}$ in (4), we can prove our Theorem from (14) and Proposition 2. q.e.d.

§ 3. General computing function of a Kleinian group

7. Let us consider a transformation

$$S^{(m+\nu)} = S^{(m)} S^{(\nu)} = S^{(m)} T_\nu T_{\nu-1} \cdots T_2 T_1, \quad (S^{(m)} = S^{(m-1)} T_k,\ T_1^{-j} \neq T_{j+1} \ (1 \leqq j \leqq \nu - 1) \text{ and } T_\nu^{-1} \neq T_k)$$

of a Kleinian group, where T_k and T_j $(1 \leqq j \leqq \nu - 1)$ are generators or their inverses. Let $R_{S(k)}$ be the radius of the isometric circle of $S^{(k)}$. Then we have from (2)

$$R_{S(m+\nu)} = \frac{R_{T_1} R_{S(m+\nu-1)}}{|g'_{T_1} - g_{S(m+\nu-1)}|} = \frac{R_{T_1} R_{T_2} R_{S(m+\nu-2)}}{|g'_{T_1} - g_{S(m+\nu-1)}|\,|g'_{T_2} - g_{S(m+\nu-2)}|} = \cdots$$
$$= \frac{R_{T_1}}{|g'_{T_1} - g_{S(m+\nu-1)}|} \cdot \frac{R_{T_2}}{|g'_{T_2} - g_{S(m+\nu-2)}|} \cdots \frac{R_{T_\nu}}{|g'_{T_\nu} - g_{S(m)}|} \cdot R_{S(m)},$$

and therefore

$$\Big(\frac{R_{S(m+\nu)}}{R_{S(m)}} \Big)^\mu = \prod_{i=1}^{\nu} \Big\{ \frac{R_{T_i}}{|g'_{T_i} - g_{S(m+\nu-i)}|} \Big\}^\mu, \quad (0 < \mu < 4). \tag{15}$$

Noting that $g_{S(m)} = S^{-(m)}(\infty)$, $g_{S(m)T_\nu} = T_\nu^{-1} S^{-(m)}(\infty)$, $\ldots$, $g_{S(m)T_\nu T_{\nu-1} \ldots T_{i+1}} = g_{S(m+\nu-i)} = T_{i+1}^{-1} \cdots T_{\nu-1}^{-1} T_\nu^{-1} S^{-(m)}(\infty)$, (15) is also written in the form

$$\Big(\frac{R_{S(m+\nu)}}{R_{S(m)}} \Big)^\mu = \prod_{i=1}^{\nu} \Big\{ \frac{R_{T_i}}{|g_{T_i^{-1}} - T_{i+1}^{-1} \cdots T_{\nu-1}^{-1} T_\nu^{-1} S^{-(m)}(\infty)|} \Big\}^\mu. \tag{16}$$

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Since $g_{T_i^{-1}} = T_i(\infty)$, $g_{S^{(m+\nu-i)}} = T_{i+1}^{-1}\cdots T_{\nu-1}^{-1}T_\nu^{-1}S^{-(m)}(\infty)$ and $T_i \neq T_{i+1}^{-1}$, $g_{T_i^{-1}}$ and $g_{S^{(m+\nu-i)}}$ are contained in the different boundary circles of B_0 and hence each denominator of the product in the right hand side of (16) does not vanish.

If we replace $g_{S^{(m)}} = S^{-(m)}(\infty)$ by z in the denominator of (16) and form the summation with respect to all $S^{(\nu)}(T_k \neq T_\nu^{-1})$ of grade ν in G, we obtain the following function

$$f_{T_k}^{(\mu)\nu}(z) = \sum_{S^{(\nu)}}\left[\prod_{i=1}^{\nu}\left\{\frac{R_{T_i}}{|g_{T_i^{-1}} - T_{i+1}^{-1}\cdots T_{\nu-1}^{-1}T_\nu^{-1}(z)|}\right\}^{\mu}\right], \tag{17}$$

$$(T_{\nu+1}:\text{ identity, } T_\nu^{-1} \neq T_k),$$

where z varies on the closed disc bounded by H_{i_1}, the boundary circle of B_0 mapped onto the boundary circle H'_{i_1} of B_0 by T_k. Since the $(N-1)^\nu$ denominators of (17) don't vanish, $f_{T_k}^{(\mu)\nu}(z)$ is continuous in the closed disc D_{i_1} bounded by H_{i_1} and hence uniformly continuous. It is obvious that

$$f_{T_k}^{(\mu)\nu}(g_{S^{(m)}}) = \sum_{S^{(\nu)}}\left(\frac{R_{S^{(m+\nu)}}}{R_{S^{(m)}}}\right)^{\mu}.$$

We call $f_{T_k}^{(\mu)\nu}(z)$ the μ-dimensional computing function of order ν on T_k and there exist N computing functions $f_{T_k}^{(\mu)\nu}(z)$ $(k=1,\ldots,N)$ in all, since the last element T_k of $S^{(m)}$ is any generator or its inverse of G. Such functions $\{f_{T_k}^{(\mu)\nu}(z)\}$, $(k=1,\ldots,N)$ are called the μ-dimensional computing functions of order ν on a Kleinian group G.

8. We take a generator or its inverse T_i and consider the μ dimensional computing function $f_{T_i}^{(\mu)\nu}(z)$ of order ν on T_i. Then $f_{T_i}^{(\mu)\nu}(z)$ is defined in the closed disc $D_{T_i}: |z-a_{T_i}| \leqq r_{T_i}$ bounded by H_{T_i} which is a boundary circle of B_0 mapped onto H'_{T_i} by T_i. Since $f_{T_i}^{(\mu)\nu}(z)$ is uniformly continuous in D_{T_i}, we can choose δ depending only on any small ε, so that it holds $|f_{T_i}^{(\mu)\nu}(z) - f_{T_i}^{(\mu)\nu}(z')| < \varepsilon$ for z and z' satisfying $|z-z'| < \delta$ in D_{T_i}.

Denote by E_i the subset of E contained in D_{T_i}. Since, from Proposition 1, any radius $r^{(m)}$ of circles of grade m is equal or less than $K_0^{m-1}r^{(1)}$ $(K_0 < 1)$, which tends to zero for $m \to \infty$, there exists a grade number m_0 depending only on δ so that for any $S^{(m)} = S^{(m-1)}T_i$ $(m \geqq m_0)$ there is $z_0 \in E_i$ such that $g_{S^{(m)}} \in D_\delta(z_0)$, where $D_\delta(z_0)$ denotes the disc with center z_0 and with radius δ. Hence it can be seen that

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(18) $$|f_{T_i}^{(\mu)\nu}(z_0) - f_{T_i}^{(\mu)\nu}(g_{S^{(m)}})| < \varepsilon, \qquad (m \geqq m_0).$$

Suppose that

(19) $$f_{T_i}^{(\mu)\nu}(z) > \lambda_i, \qquad \text{for any } z \in E_i.$$

Then we have from (18) and (19)

$$|f_{T_i}^{(\mu)\nu}(g_{S^{(m)}})| \geqq |f_{T_i}^{(\mu)\nu}(z_0)| - |f_{T_i}^{(\mu)\nu}(g_{S^{(m)}}) - f_{T_i}^{(\mu)\nu}(z_0)| > \lambda_i - \varepsilon.$$

Now we prove the following

THEOREM 2. *Let G be a Kleinian group whose fundamental domain is bounded by N boundary circles as in* § 1. *If*

(20) $$f_{T_i}^{(\mu)\nu}(z) > \lambda_i > 1, \qquad (i = 1, \ldots, N)$$

on the singular subset E_i of E contained in the boundary circle H_{T_i} $(i = 1, \ldots, N)$ of B_0 respectively, then the singular set E of G has the positive $\left(\frac{\mu}{2}\right)$-dimensional measure.

Proof. For any i, take ε so small that it may hold $\lambda_i - \varepsilon > 1$ $(i = 1, \ldots, N)$. Then we can determine the grade number m_0 such that the inequalities

$$f_{T_i}^{(\mu)\nu}(g_{S^{(m)}}) > \lambda_i - \varepsilon > 1, \quad (S^{(m)} = S^{(m-1)}T_i, \ m \geqq m_0 \ ; \ i = 1, \ldots, N)$$

hold. Hence we have the following inequalities

$$\sum_{S^{(\nu)}} (R_{S^{(m+\nu)}})^{\mu} \geqq (R_{S^{(m)}})^{\mu}, \quad (S^{(m+\nu)} = S^{(m)}S^{(\nu)}, \ 0 < \mu < 4)$$

for radius $R_{S^{(m)}}$ of any isometric circle $I_{S^{(m)}}$ of grade m and radii $R_{S^{(m)}S^{(\nu)}}$ of the $(N-1)^{\nu}$ isometric circles $I_{S^{(m)}S^{(\nu)}}$ of grade $(m+\nu)$. Thus, by Theorem 1, we get the theorem. q.e.d.

9. In order to determine the positiveness of the μ-dimensional measure of E, it is important to seek for the values λ_i $(i = 1, \ldots, N)$ as sharp as possible one can.

If we put $z_{i+1}(z) = T_{i+1}^{-1} \cdots T_{\nu-1}^{-1} T_1^{-\nu}(z)$ $(i = 1, \ldots, \nu - 1)$ in (17), we obtain

(21) $$f_{T_k}^{(\mu)\nu}(z) = \sum_{S^{(\nu)}} \left[\frac{R_{T_1}}{|g_{T_1^{-1}} - z_2(z)|} \cdots \frac{R_{T_{i-1}}}{|g_{T_{i-1}^{-1}} - z_i(z)|} \cdots \frac{R_{T_\nu}}{|g_{T_\nu^{-1}} - z|} \right]^{\mu}.$$

Denote by D_S^* the minimum closed subdomain, which is contained in D_S bounded by the boundary circle H_S mapped onto H'_S by S and further contains the singular subset of E contained in H_8. We put $z_{i-1}(D^*_{T_k}) = \tilde{D}^*_{i+1}$ and note that $\tilde{D}^*_{i+1} \subset D^*_{T_{i+1}}$.

Let

$$(22)\qquad \{z_2(z), \ldots, z_j(z), \ldots, z_\nu(z), z\}$$

be a coordinate of ν complex numbers, where $z \in D^*_{T_k}$ and $z_j(z) \in \tilde{D}^*_j$, $(2 \leqq j \leqq \nu)$. Since the number of $S^{-(\nu)} = T_1^{-1} S^{-(\nu-1)} = T_1^{-1} T_2^{-1} \cdots T_\nu^{-1}$ with $T_\nu^{-1} \neq T_k$ is $(N-1)^{\nu-1}$, there are $(N-1)^{\nu-1}$ number of coordinates in all.

Let only the first component z_2 in (22) move freely in $\tilde{D}^*_2$ for fixed $z_j(z)$ $(3 \leqq j \leqq \nu)$. Then we have

$$(23)\qquad f^{(\mu)\nu}_{T_k}(z) \geqq \min_{(z_2) \in (\tilde{D}^*_2)} \sum_{S^{(\nu)}} \left[\frac{R_{T_1}}{|g_{T_1^{-1}} - z_2(z)|} \cdot \frac{R_{T_2}}{|g_{T_2^{-1}} - z_3(z)|} \cdots \frac{R_{T_\nu}}{|g_{T_\nu^{-1}} - z|} \right]^\mu$$

on $D^*_{T_k}$, where $(z_2) \in (\tilde{D}^*_2)$ denotes that the first component z_2 of each coordinate with the form (22) moves in each minimum closed subdomain $\tilde{D}^*_2$. We note that there are such $(N-1)^{\nu-1}$ closed subdomains.

After this procedure, let only the second component z_3 of each coordinate with the form (22) move freely in $\tilde{D}^*_3$ for fixed $z_j(z)$ $(4 \leqq j \leqq \nu)$. Then we have

$$f^{(\mu)\nu}_{T_k}(z) \geqq \min_{(z_3) \in (\tilde{D}^*_3)} \{ \min_{(z_2) \in (\tilde{D}^*_2)} \sum_{S^{(\nu)}} [\quad '' \quad]^\mu \} \text{ on } D^*_{T_k},$$

where $(z_3) \in (\tilde{D}^*_3)$ denotes that the second component z_3 of each coordinate with the form (22) moves in each minimum closed subdomain $\tilde{D}^*_3$. There are such $(N-1)^{\nu-2}$ closed subdomains. Repeating this procedure, we obtain the following inequality:

$$(24)\qquad f^{(\mu)\nu}_{T_k}(z) \geqq \min_{z \in D^*_{T_k}} \{ \min_{(z_\nu) \in (\tilde{D}^*_\nu)} \{ \cdots \{ \min_{(z_2) \in (\tilde{D}^*_2)} \sum_{S^{(\nu)}} [\quad '' \quad]^\mu \} \cdots \}$$

on $D^*_{T_k}$, where $(z_i) \in (\tilde{D}^*_i)$, $(i = 2, \ldots, \nu)$ denote that z moves in a minimum closed subdomain $\tilde{D}^*_i$. If we denote the right hand side of (24) by λ_k, we get the following

$$(25)\qquad f^{(\mu)\nu}_{T_k}(z) \geqq \lambda_k, \qquad (k = 1, \ldots, N).$$

§ 4. Examples of Kleinian groups whose singular sets have positive 1-dimensional measure

10. In our recent paper [2], we proved the existence of Kleinian groups with fundamental domains bounded by five circles whose singular sets have

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positive 1-dimensional measure and presented the problem whether there exist or not such groups in the case of four circles. Here we note that, by Schottky's condition [4], the 1-dimensional measure of the singular set is always zero in the case of three circles.

In this chapter, by using the conditions (for $\mu=2$) of Theorem 2 and the method of No. 9, we shall show the existence of Kleinian groups with fundamental domains bounded by four circles whose singular sets have positive 1-dimensional measure.

As the preliminary to give our example, at first we shall show how to construct a transformation T which maps the outside of a circle H onto the inside of another circle H', where H and H' have equal radii, though in generally we can set up infinitely many such transformations.

Denote two circles by

$$H : |z-q|=r, \qquad H' : |z-q'|=r.$$

If T is restricted by the conditions: $q'=T(\infty)$ and $q=T^{-1}(\infty)$, it is easily seen that T has the following form

$$z'=T(z)=\frac{q'z-(qq'+r^2e^{i\theta})}{z-q}, \tag{26}$$

where θ is any real number and the isometric circles I_T and $I_{T'}$ are H and H' respectively.

11. Secondly we shall give two lemmas which we shall need later.

LEMMA 2. *Let $P_1=P(R, 0)$ and $P_2=P(R, \pi)$ be fixed on real axis in the complex z-plane and $P=P(r, \theta)$ move on the fixed circle $C_r : |z|=r$. Then the function $f(P)=\sum_{i=1}^{2}\frac{1}{\overline{PP_i}^2}$ of P attains its minimum at the points on the imaginary axis, where $\overline{PP_i}$ denote the distances between P and P_i.*

Proof. By using the polar coordinates, we obtain

$$f(r, \theta)=\frac{1}{R^2+r^2-2Rr\cos\theta}+\frac{1}{R^2+r^2+2Rr\cos\theta}$$
$$=\frac{2(R^2+r^2)}{(R^2+r^2)^2-4R^2r^2\cos^2\theta}, \qquad (r \text{ ; fixed}).$$

Hence the minimum is attained at $\theta=\frac{\pi}{2}$ or $\frac{3}{2}\pi$. q.e.d.

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LEMMA 3. *Let* $P_1 = P\left(R, -\frac{\pi}{3}\right)$, $P_2 = P\left(R, \frac{\pi}{3}\right)$ *and* $P_3 = P(R, \pi)$ *be fixed in the complex z-plane, and* $P = P(r, \theta)$ *move in the fixed closed disc* $U: |z| \leqq \frac{R}{2}$. *Then the function* $f(P) = \sum_{i=1}^{3} \frac{1}{\overline{PP_i}^2}$ *attains its minimum at the origin.*

Proof. As in Lemma 2, we obtain the following representation of $f(P)$:

$$f(r, \theta) = \frac{1}{R^2 + r^2 - 2Rr\cos\left(\frac{\pi}{3} - \theta\right)} + \frac{1}{R^2 + r^2 - 2Rr\cos\left(\frac{\pi}{3} + \theta\right)} + \frac{1}{R^2 + r^2 + 2Rr\cos\theta}.$$

If we differentiate with respect to θ for fixed r, we have

$$\frac{\partial f}{\partial \theta} = \frac{2Rr}{(R^2+r^2)^2}\left[\frac{\sin\left(\frac{\pi}{3} - \theta\right)}{\left\{1 - \frac{2Rr}{R^2+r^2}\cos\left(\frac{\pi}{3} - \theta\right)\right\}^2} - \frac{\sin\left(\frac{\pi}{3} + \theta\right)}{\left\{1 - \frac{2Rr}{R^2+r^2}\cos\left(\frac{\pi}{3} + \theta\right)\right\}^2} + \frac{\sin\theta}{\left\{1 + \frac{2Rr}{R^2+r^2}\cos\theta\right\}^2}\right] = \frac{9}{16}a^2(2+a)(2-a)\sin 3\theta,$$

where $a = \frac{2Rr}{R^2+r^2}$ $(\leqq 1)$. Hence the values which satisfy the equation $\frac{\partial f}{\partial \theta} = 0$ in $0 \leqq \theta \leqq \frac{\pi}{3}$, are 0 and $\frac{\pi}{3}$.

Since

$$\left[\frac{\partial f}{\partial \theta}\right]_{\theta = \pi/6} = \frac{1}{2}\left\{\frac{1}{\left(1 - \frac{\sqrt{3}}{2}a\right)^2} + \frac{1}{\left(1 + \frac{\sqrt{3}}{2}a\right)^2} - 1\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\left(1 - \frac{3}{4}a^2\right)^2} - 1\right\} > 0, \quad \left(a = \frac{2Rr}{R^2+r^2} \leqq 1\right),$$

$P(r, 0)$ $\left(\text{or } P\left(r, \frac{\pi}{3}\right)\right)$ is the point at which $f(r, \theta)$ attains the minimum (or the maximum) for any fixed r $\left(0 \leqq r \leqq \frac{R}{2}\right)$.

We differentiate $f(r, 0)$ with respect to r, and see that $\frac{df(r, 0)}{dr}$ has only one zero point in $0 < r < \frac{R}{2}$, at which $f(r, 0)$ takes the maximum value in $0 \leqq r \leqq \frac{R}{2}$. Hence the inequality

$$f(0, 0) = \frac{3}{R^2} < f\left(\frac{R}{2}, \ 0\right) = \frac{28}{9R^2},$$

implies that $f(0, 0)$ is the minimum value of $f(r, 0)$ in $0 \leqq r \leqq \frac{R}{2}$ and hence that of $f(r, \theta)$ in U. q.e.d.

12. Example. The case of $N=4$.

Consider the three circles H_j $(j=1, 2, 3)$ with centers $a_j = 2e^{i\frac{(4j-1)}{6}\pi}$ $(j=1, 2, 3\ ;\ i^2=-1)$ and equal radii $\sqrt{3}-\varepsilon$, respectively. We let these three circles H_j $(j=1, 2, 3)$ correspond to the elliptic transformations S_j $(j=1, 2, 3)$ with period 2.

Then we obtain a Fuchsian group G_1 of the second kind with the fixed circle $|z|=1+\varepsilon_1$. The singular set of G_1 is on the circle $|z|=1+\varepsilon_1$ and is nowhere dense. Next we describe a circle H_4 with center at the origin and the radius $2-\sqrt{3}$ and let it correspond to the elliptic transformation S_4 with period 2.

Combining the Fuhcsian group G_1 with G_2 generated by S_4 only, we obtain a Kleinian group G, that is, a combination group $G_1 \cdot G_2$, whose fundamental domain B_0 is connected and bounded by four circles H_j $(j=1, 2, 3, 4)$.

For convenience of the calculation, we consider the limit case $\varepsilon=0$. Then B_0 is no more connected and the fixed circle of G_1 is $|z|=1$.

Denote by D_j $(j=1, 2, 3, 4)$ the closed discs bounded by H_j $(j=1, 2, 3, 4)$ and by V the closed unit disc. Then the singular set E of G lies in the inside of $V \cap \left\{\bigcup_{j=1}^{4} D_j\right\}$. The generating transformations of G have the following forms (see (26)):

(27)
$$S_j(z) = \frac{\frac{2i}{\sqrt{3}}z + \frac{1}{\sqrt{3}}e^{i\frac{2(j-1)}{3}\pi}}{(-1)^{j-1}\frac{e^{i\frac{(j-1)}{3}\pi}}{\sqrt{3}}z + \frac{2i}{\sqrt{3}}}, \qquad (j=1, 2, 3)$$

$$S_4(z) = \frac{(2-\sqrt{3})^2 e^{i\theta}}{z}, \qquad (\theta\ ;\ \forall \text{ real number}).$$

By the symmetricity of the figure, it is sufficient to calculate the values of the computing functions $f_{S_1}^{(2)\nu}(z)$ and $f_{S_4}^{(2)\nu}(z)$ of order ν in $V \cap D_1$ and D_4, respectively.

(I) Case of order $\nu=1$.

(a) It holds that in $V \cap D_1$

$$f_{S_1}^{(2)_1}(z) = \sum_{j=2}^{3} \frac{(\sqrt{3})^2}{|z-a_j|^2} + \frac{(2-\sqrt{3})^2}{|z|^2}.$$

We see from Lemma 2 that it attains the minimum at $z=i$ in $V \cap D_1$. Hence

$$f_{S_1}^{(2)_1}(z) > \frac{6}{7} + (2-\sqrt{3})^2 > 0.928 \quad \text{on } V \cap D_1.$$

The condition (20) of Theorem 2 is not satisfied.

(b) It holds that in D_4

$$f_{S_4}^{(2)_1}(z) = \sum_{j=1}^{3} \frac{(\sqrt{3})^2}{|z-a_j|^2}.$$

Since $f_{S_4}^{(2)_1}(z)$ attains the minimum in D_4 at the origin from Lemma 3, it holds

$$f_{S_4}^{(2)_1}(z) \geqq \left(\frac{\sqrt{3}}{2}\right)^2 \times 3 = \frac{9}{4} = 2.25.$$

In the cases of order $\nu = 2, 3, 4$, we can not obtain the desired results. But in the case of order $\nu = 5$, we do succeed as shown below.

(II) Case of order $\nu = 5$.

The 2-dimensional computing function of order 5 is as follows:

$$f_{T_6}^{(2)_5}(z) = \sum_{S^{(5)}} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - T_2^{-1}T_3^{-1}T_4^{-1}T_5^{-1}(z)|^2} \cdot \frac{R_{T_2}^2}{|g_{T_2^{-1}} - T_3^{-1}T_4^{-1}T_5^{-1}(z)|^2}$$

(28)
$$\cdot \frac{R_{T_3}^2}{|g_{T_3^{-1}} - T_4^{-1}T_5^{-1}(z)|^2} \cdot \frac{R_{T_4}^2}{|g_{T_4^{-1}} - T_5^{-1}(z)|^2} \cdot \frac{R_{T_5}^2}{|g_{T_5^{-1}} - z|^2},$$
$$(S^{(5)} = T_5T_4T_3T_2T_1,\ T_j^{-1} \neq T_{j+1}\ (1 \leqq j \leqq 5)),$$

where $z \in D_{T_6}^*$ $(T_6 \neq T_5^{-1})$.

By the symmetricity of the figure, it is sufficient to calculate the values of $f_{S_1}^{(2)_5}(z)$ and $f_{S_4}^{(2)_5}(z)$ in $V \cap D_1$ and D_4, respectively, according as T_6 is S_1 or S_4.

Now we shall give some preliminary appreciations.

(a) We denote by $\sum^{(1)}$ the sum taken over all the $S^{(5)} = S^{(4)}T_1$ with the same $S^{(4)} = T_5T_4T_3T_2$. By (I) we have

$$\sum{}^{(1)} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - (S^{(4)})^{-1}(z)|^2} \geqq 0.928 \text{ or } 2.25$$

according as T_2 is one of S_1, S_2 and S_3 or $T_2 = S_4$.

(b) We denote by $\sum^{(2)}$ the sum taken over all the $S^{(5)} = S^{(5)}T_2T_1$ with the same $S^{(3)} = T_5T_4T_3$. See the table from above.

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(b. 1) The case of $T_3 = S_1$. By (a) and Lemma 2, we have

$$\sum^{(2)} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - (S^{(4)})^{-1}(z)|^2} \cdot \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2} \geqq 0.928 \sum_{T_2 = S_2, S_3} \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2}$$

$$+ 2.25 \times \frac{R_{S_4}^2}{|g_{S_4^{-1}} - (S^{(3)})^{-1}(z)|^2} \geqq 0.928 \times \frac{6}{7} + 2.25 \times (2 - \sqrt{3})^2.$$

In the cases of $T_3 = S_2$ or S_3 we have the same.

(b. 2) The case of $T_3 = S_4$. From (a) and (I) we have

$$\sum^{(2)} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - (S^{(4)})^{-1}(z)|^2} \cdot \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2} \geqq 0.928 \sum_{T_2 = S_1, S_2, S_3} \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2}$$

$$\geqq 0.928 \times 2.25.$$

Thus we obtain the last column of the table.

TABLE

T_3	T_2				T_1				
	T_2	$R_{T_2}^2$	$g_{T_1^{-1}}$	$(S^{(3)})^{-1}(z)$ moves in	T_1	$R_{T_1}^2$	$g_{T_1^{-1}}$	$(S^{(4)})^{-1}(z)$ moves in	
S_1	S_2	3	a_2	$D_1 \cap V$	S_1	3	a_1	$D_2 \cap V$	c_1
					S_3	3	a_3		c_1
					S_4	$(2-\sqrt{3})^2$	a_4		c_2
	S_3	3	a_3		S_1	3	a_1	$D_3 \cap V$	c_1
					S_2	3	a_2		c_1
					S_4	$(2-\sqrt{3})^2$	a_4		c_2
	S_4	$(2-\sqrt{3})^2$	a_4		S_1	3	a_1	D_4	c_1
					S_2	3	a_2		c_1
					S_3	3	a_3		c_1
S_4	S_1	3	a_1	D_4	S_2	3	a_2	$D_1 \cap V$	c_1
					S_3	3	a_3		c_1
					S_4	$(2-\sqrt{3})^2$	a_4		c_1
	S_2	3	a_2		S_1	3	a_1	$D_2 \cap V$	c_1
					S_3	3	a_3		c_1
					S_4	$(2-\sqrt{3})^2$	a_4		c_2
	S_3	3	a_3		S_1	3	a_1	$D_3 \cap V$	c_1
					S_2	3	a_2		c_1
					S_4	$(2-\sqrt{3})^2$	a_4		c_2
T_5	T_4	$R_{T_4}^2$	$g_{T_4^{-1}}$	$T_5^{-1}(z)$ moves in	T_3	$R_{T_3}^2$	$g_{T_3^{-1}}$	$T_4^{-1}T_5^{-1}(z)$ moves in	J is not less than
	T_4				T_3				

$$J = \sum^{(2)} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - (S^{(4)})^{-1}(z)|^2} \cdot \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2}.$$

$c_1 = 0.928 \times \frac{6}{7} + 2.25 \times (2 - \sqrt{3})^2.$

$c_2 = 0.928 \times 2.25.$

(c) We denote by $\sum^{(4)}$ the sum taken over all the $S^{(5)}$ with the same T_5. See the table from below.

(c. 1) The case of $T_5 = S_1$. Using (a), (b), Lemma 2 and (I), we have

$$I_{T_5} = \sum^{(4)} \frac{R_{T_1}^2}{|g_{T_1^{-1}} - (S^{(4)})^{-1}(z)|^2} \cdot \frac{R_{T_2}^2}{|g_{T_2^{-1}} - (S^{(3)})^{-1}(z)|^2} \cdot \frac{R_{T_4}^2}{|g_{T_3^{-1}} - T_4^{-1}T_5^{-1}(z)|^2} \cdot \frac{R_{T_4}^2}{|g_{T_4^{-1}} - T_5^{-1}(z)|^2}$$

$$\geqq \left(c_1 \times \sum_{T_3=S_1, S_3} \frac{R_{T_3}^2}{|g_{T_3^{-1}} - T_4^{-1}T_5^{-1}(z)|^2} + c_2 \times \frac{R_{S_4}^2}{|g_{S_4^{-1}} - T_4^{-1}T_5^{-1}(z)|^2}\right) \times \frac{R_{S_2}^2}{|g_{S_2^{-1}} - T_5^{-1}(z)|^2}$$

$$+ \left(c_1 \times \sum_{T_3=S_1, S_2} \frac{R_{T_3}^2}{|g_{T_3^{-1}} - T_4^{-1}T_5^{-1}(z)|^2} + c_2 \times \frac{R_{S_4}^2}{|g_{S_4^{-1}} - T_4^{-1}T_5^{-1}(z)|^2}\right) \times \frac{R_{S_3}^2}{|g_{S_3^{-1}} - T_5^{-1}(z)|^2}$$

$$+ c_1 \times \sum_{T_3=S_1, S_2, S_3} \frac{R_{T_3}^2}{|g_{T_3^{-1}} - T_4^{-1}T_5^{-1}(z)|^2} \times \frac{R_{S_4}^2}{|g_{S_4^{-1}} - T_5^{-1}(z)|^2}$$

$$\geqq \left\{\frac{6}{7} \times c_1 + (2-\sqrt{3})^2 \times c_2\right\} \sum_{T_4=S_2, S_3} \frac{R_{T_4}^2}{|g_{T_4^{-1}} - T_5^{-1}(z)|^2} + 2.25(2-\sqrt{3})^2 \times c_1$$

$$\geqq \left\{\left(\frac{6}{7}\right)^2 + 2.25(2-\sqrt{3})^2\right\} c_1 + \frac{6}{7}(2-\sqrt{3})^2 c_2.$$

In the cases of $T_5 = S_2$ or S_3 we have the same.

(c. 2) The case of $T_5 = S_4$. Similarly we have

$$I_{T_5} \geqq \left\{\frac{6}{7} \times c_1 + (2-\sqrt{3})^2 c_2\right\} \sum_{T_4=S_1, S_2, S_3} \frac{R_{T_4}^2}{|g_{T_4^{-1}} - T_5^{-1}(z)|^2}$$

$$\geqq 2.25 \times \frac{6}{7} \times c_1 + 2.25 \times (2-\sqrt{3})^2 \times c_2.$$

Now we can show that $f_{T_6}^{(2)_5}(z) > 1$ for any T_6.

(A) The case of $T_6 = S_1$. In this case T_5 takes on S_2, S_3 and S_4. Hence, by (c) and Lemma 2, we have

$$f_{S_1}^{(2)_5}(z) \geqq I_{S_2} \times \frac{R_{S_2}^2}{|g_{S_2^{-1}} - z|^2} + I_{S_3} \times \frac{R_{S_3}^2}{|g_{S_3^{-1}} - z|^2} + I_{S_4} \times \frac{R_{S_4}^2}{|g_{S_4^{-1}} - z|^2}$$

$$\geqq \left[\left\{\left(\frac{6}{7}\right)^2 + 2.25 \times (2-\sqrt{3})^2\right\} \times c_1 + \frac{6}{7}(2-\sqrt{3})^2 \times c_2\right] \times \sum_{T_5=S_2, S_3} \frac{R_{T_5}^2}{|g_{T_5^{-1}} - z|^2}$$

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$$+(2-\sqrt{3})^2\times\left\{2.25\times\frac{6}{7}\times c_1+2.25\times(2-\sqrt{3})^2\times c_2\right\}$$
$$\geqq\frac{6}{7}\left[\left\{\left(\frac{6}{7}\right)^2+2.25\times(2-\sqrt{3})^2\right\}\times c_1+\frac{6}{7}(2-\sqrt{3})^2\times c_2\right]+(2-\sqrt{3})^2\times$$
$$\times\left\{2.25\times\frac{6}{7}\times c_1+2.25\times(2-\sqrt{3})^2\times c_2\right\}>1.002004.$$

(B) The case of $T_6=S_4$. T_5 takes on S_1, S_2 and S_3, so that we have

$$f_{S_4}^{(2)'6}(z)\geqq\left[\left\{\left(\frac{6}{7}\right)^3+2.25\times(2-\sqrt{3})^2\right\}\times c_1+\frac{6}{7}(2-\sqrt{3})^2c_2\right]\times 2.25>2.218873.$$

Thus we see that the condition of Theorem 2 is satisfied and have

THEOREM 3. *Under Kleinian groups whose fundamental domains are bounded by mutually disjoint N ($\geqq 4$) circles, there exist ones whose singular sets have positive* 1*-dimensional measure.*

Recalling our result about Poincaré theta-series [1], we have the following

COROLLARY. *Under Kleinian groups whose fundamental domains are bounded by mutually disjoint N ($\geqq 4$) circles, there exist one, the (-2)-dimensional Poincaré theta-series $\Theta_2(z)$ with respect to which does not converge in D^*, where D^* denotes the compact subdomain of E^c given by deleting the suitable neighbourhoods of the poles of $\Theta_2(z)$ and their transforms on G from any compact subdomain $D\subset E^c$.*

13. Considering the Schottky subgroups G^* of G given by inversion method (see [2]), we have the following

THEOREM 4. *There exist Schottky groups whose fundamental domains are bounded by* 6 *boundary circles and whose singular sets have positive* 1*-dimensional measure. The (-2)-dimensional Poincaré theta-series $\Theta_2(z)$ with respect to such Schottky group does not converge in D^*.*

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