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Eventually positive matrices with rational eigenvectors†

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Abstract. Let A be an $n \times n$ real matrix; sufficient conditions were previously worked out, assuming non-commensurability of eigenvectors, for A to be $\mathrm{SL}(n, \mathbb{Z})$-conjugate to a matrix all sufficiently large powers of which have strictly positive entries. We show that when the 'large' eigenvectors are commensurable and satisfy the obvious necessary conditions, then A is also going to be so conjugate. In particular, we deduce, if A is a rational matrix with large eigenvalue exceeding 1 and of multiplicity one, then A is algebraically shift equivalent to an eventually positive matrix, using only integer rectangular matrices.

Let B be an $n \times n$ matrix with real entries. We will consider the following question, much of which was solved in [**H**, theorem 2.2]: Decide when there exists P in $\mathrm{SL}(n, \mathbb{Z})$ so that all sufficiently large powers of PBP^{-1} have all of their entries strictly positive. By the Perron theorem (applied to powers of PBP^{-1}), there must be a real eigenvalue $\lambda_B > 0$ of multiplicity one such that $\lambda_B > |\lambda|$ for all other eigenvalues λ of B. If this is the case, λ_B is called the weak Perron eigenvalue of B. Let v_B, w_B be non-zero choices for the left, right eigenvectors, respectively, of B corresponding to λ_B. In [**H**, 2.2], it was shown that if either w_B or v_B contains an irrational ratio among its entries, then the desired P exists. This leaves the case that all the entries of w_B and v_B be commensurable. Then another condition intervenes, as was mentioned briefly in [**H**, p. 61].

Assume B has a weak Perron eigenvalue, and v_B, w_B have no irrational ratios. Then by multiplying each one by a suitable non-zero real number, we may assume v_B, w_B have only integer entries; by dividing by the appropriate integers we may assume each is unimodular (i.e. the greatest common divisor of the entries is 1). Noticing that the scalar product $v_B \cdot w_B$ is invariant with respect to $v_B \to v_B P^{-1}$, $w_B \to P w_B$ (the corresponding eigenvectors for PBP^{-1}, hence for its powers), a necessary additional condition is $|v_B \cdot w_B| \geq n$.

We prove as conjectured in [**H**] that this is sufficient for the desired P to exist. Moreover, if $|v_B w_B| < n$ and $\lambda_B > 1$, we show how to enlarge B to an $(n+k) \times (n+k)$

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matrix B' algebraically shift equivalent to B so that B' is conjugate (via $\mathrm{SL}(n+k, \mathbb{Z})$) to an ultimately positive matrix. Here k is the smallest integer such that $|vw|\lambda_B^k \geq n+k$.

In particular, if B is in $M_n\mathbb{Z}$ (integer entries), then the only case not covered by [**H**, 2.2] occurs when λ_B is an integer (as a rational algebraic integer is an integer).

Returning to our first result, let v_B, w_B be a unimodular row and column respectively, with integer entries, such that $|v_B \cdot w_B| \geq n$. By replacing v_B by $-v_B$ if necessary, we may assume $v_B \cdot w_B \geq n$. By [**H**, 2.1] it is sufficient to find P in $\mathrm{SL}(n, \mathbb{Z})$ such that $v_B P^{-1}$, Pw_B are strictly positive.

(1.1) Lemma. *Let $v \in \mathbb{Z}^{1\times n}$, $w \in \mathbb{Z}^{n\times 1}$ be unimodular with $vw \geq n$. Then there exists P in* $\mathrm{SL}(n, \mathbb{Z})$ *with*

$$\left.\begin{aligned} vP^{-1} &= (1, 1, \dots, 1) \\ Pw &= (m', 1, \dots, 1)^T \end{aligned}\right\} \quad \text{if } n \geq 3$$

and

$$\left.\begin{aligned} vP^{-1} &= (1, 1) \\ Pw &> (0, 0)^T \end{aligned}\right\} \quad \text{if } n = 2,$$

where $m' + n - 1 = vw$.

Proof. We repeatedly use the following result and its transpose: if

$$\begin{matrix} x = (z_1\ z_2 \cdots z_r) \\ x' = (z'_1\ z'_2 \cdots z'_r) \end{matrix} \in \mathbb{Z}^{1\times r} \quad \text{and} \quad \gcd\{z_i\} = \gcd\{z'_i\},$$

then there exists Q in $\mathrm{GL}(r, \mathbb{Z})$ with $xQ = x'$.

There thus exists A_1 in $\mathrm{GL}(n, \mathbb{Z})$ so that $vA_1 = (1\ 0 \cdots 0)$ and then $A_1^{-1}w = (m\ r_1\ r_2 \cdots r_{n-1})^T$, with $vw = m \geq n$. Set $k = \gcd\{r_i\}$; then $(m, k) = 1$, so there exist positive integers a, b with $ak - bm = \pm 1$. Suppose $n \geq 3$. Then there exists $B \in \mathrm{GL}(n-1, \mathbb{Z})$ such that

$$B(r_1\ r_2 \cdots r_{n-1})^T = (k\ ak\ 0\ 0 \cdots 0)^T \in \mathbb{Z}^{(n-1)\times 1}.$$

Set $A_2 = I_1 \oplus B^{-1}$, so that $vA_1A_2 = (1\ 0 \cdots 0)$ and

$$(A_1A_2)^{-1}w = (m\quad k\quad ak\quad 0\quad \cdots\quad 0)^T.$$

The operation of subtracting multiples of the first entry (of the column) from the third is elementary and its inverse leaves vA_1A_2 fixed; hence we obtain

$$\begin{aligned} v &\to (1, 0, 0, \dots, 0), \\ w &\to (m, k, \pm 1, 0, \dots, 0)^T. \end{aligned}$$

There exists C in $\mathrm{GL}(n-1, \mathbb{Z})$ such that $C(k, \pm 1, 0, \dots, 0)^T = (1, 1, \dots, 1)^T$; this yields a transformation to

$$\begin{matrix} (1, 0, \dots, 0, 0) \\ (m, 1, 1, \dots, 1)^T; \end{matrix}$$

subtract each column entry after the first, from the first entry; the inverse operations add one to each of the zeros; this final transformation yields

$$\begin{aligned} v &\to (1, 1, 1, \dots, 1) \\ w &\to (m - n + 1, 1, 1, \dots, 1)^T \end{aligned}$$

Transposing a pair of positions having only ones, has the effect of multiplying the determinant of the implementing matrix by -1; so we can find a matrix in $\mathrm{SL}(n, \mathbb{Z})$ to implement the transformation.

Now consider the case $n=2$; we have

$$v \to (1 \;\; 0)$$
$$w \to (m \;\; k)^T$$

If $m=k$, w being unimodular entails $m=1$, a contradiction to $|vw| \geq 2$. If $m<k$, subtract as many copies of m from k as will leave a positive remainder; this yields $(1 \;\; 0), (m \;\; k')^T$ with $k'<m$ (since $(m, k)=1$). So we may assume $k<m$. Now subtract the k term from m; the inverse operation adds the 1 of the row to the second entry. We thus obtain

$$v \to (1 \;\; 1)$$
$$w \to (m-k, k)^T. \qquad \square$$

(1.2) Theorem. *Let B in $M_n\mathbb{R}$ have a weak Perron eigenvalue $\lambda>0$. Then there exists P in* $\mathrm{SL}(n, \mathbb{Z})$ *such that all sufficiently large powers of PBP^{-1} are strictly positive if and only if either*:

(i) *one of the ratios of entries in either v_B or w_B is irrational*;

(ii) *all of the entries in each of v_B, w_B are commensurable and when made into unimodular elements of $\mathbb{Z}^{1\times n}$, $\mathbb{Z}^{n\times 1}$ respectively, satisfy $|v_B w_B| \geq n$.*

Proof. Everything, except the 'if (ii)' result is in [**H**, 2.2]. Assume (ii); by the previous result, there exists P in $\mathrm{SL}(n, \mathbb{Z})$ so that vP^{-1} and Pw are simultaneously strictly positive. By [**H**, 2.1] and its proof, all sufficiently high powers of $P^{-1}AP$ are strictly positive. $\square$

(1.3) Corollary. *Let A belong to $M_n\mathbb{Z}$. Then there exists P in* $\mathrm{SL}(n, \mathbb{Z})$ (*equivalently in* $\mathrm{GL}(n, \mathbb{Z})$) *so that for all sufficiently large k, $P^{-1}A^kP$ consists of strictly positive entries if and only if A admits a positive real eigenvalue λ of multiplicity one, exceeding $|\alpha|$ for all other eigenvalues α <u>and</u> either*:

(i) $\lambda \notin \mathbb{Z}$;

(ii) $\lambda \in \mathbb{Z}$ *and if v, w are left, right eigenvectors of A corresponding to λ, normalized so that both are unimodular, then $|vw| \geq n$.*

An algebraic (strong) shift equivalence between A in $M_T\mathbb{R}$ and B in $M_t\mathbb{R}$ is a sequence of rectangular matrices X_i, Y_j of the appropriate dimensions so that

$$A = X_1Y_1, \quad Y_1X_1 = X_2Y_2, \quad Y_2X_2 = X_3Y_3, \ldots, Y_sX_s = B.$$

In the case where the X_i's and Y_i's admit only integer entries, we say that the algebraic shift equivalence is implementable over the integers.

(1.4) Theorem. *Let A in $M_n\mathbb{R}$ have a weak Perron eigenvalue $\lambda \equiv \lambda_A$ whose corresponding left and right eigenvectors v_A, w_A have no irrational ratios among their entries, and when put in unimodular form, $|v_Aw_A| = m$ in $\mathbb{N}^+$. Suppose $\lambda>1$. Then there exists A' in $M_{n+k}\mathbb{R}$ that is eventually strictly positive and algebraically shift equivalent to A over the integers, and with k any integer such that $\lambda^k \geq n+k$. In particular, there exists* *a power of A which is algebraically strongly shift equivalent over* $\mathbb{Z}$ *(with lag 1) to an* A'' *in* $M_{n+1}\mathbb{R}$ *that is eventually strictly positive.*

Remark. If $\lambda \leq 1$, A may be replaced by a scalar multiple of itself.

Proof. First we observe that $vw \neq 0$; the proof of this in [**H**, 2.2] remains valid here. We may assume $vw > 0$, by multiplying v or w by -1 if necessary.

Write $w^T = (w_1, w_2, \ldots, w_n)$; as w is unimodular, we may find integers $r_1, \ldots, r_n$ so that $\sum r_i w_i = 1$. Define a square matrix, A_1, of size $n+1$ by adjoining at the bottom of A, a new row $k = (r_1, \ldots, r_n, 0)$ (set $k' = (r_1, \ldots, r_n)$), and the column $(0, \ldots, 0)^T$ to the right; this yields a matrix A_1. Setting

$$X = (I_n \quad 0) \in \mathbb{Z}^{n\times(n+1)}, \qquad Y = \begin{pmatrix} A \\ k' \end{pmatrix} \in \mathbb{Z}^{(n+1)\times n},$$

we have $XY = A$, $YX = A_1$. In particular, A is algebraically shift equivalent to A_1, and the latter has λ as its large eigenvalue.

Define the row of size $n+1$, $v' = (v \ \ 0)$, and the column $w' = (w \ \ w_{n+1})^T$, where w_{n+1} is a real number to be determined so that w' is a right eigenvector for A_1. Clearly $v'A_1 = \lambda v'$; on the other hand, $A_1 w' = \lambda w'$ occurs precisely when $\lambda w_{n+1} = 1$. To make w' integral, we must multiply by λ; then $w'' = \lambda w'$ will be unimodular.

As $v'w' = vw$, we see that $v'w'' = \lambda vw$. This process, of adding a row and column, may be repeated to a total of k times until $\lambda^k vw \geq n+k$. Then A_k (in $M_{n+k}\mathbb{Z}$) satisfies the conditions of (1.1) and the first result follows.

Replacing A by A^r (r chosen so that $\lambda^r m \geq n+1$) allows us to require only one application of this process. □

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