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UNIFORM DENSITIES ON HYPERBOLIC RIEMANN SURFACES

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We are interested in the question how the spaces of solutions of elliptic equations vary according to the variations of underlying regions and coefficients of the equation. We will discuss this question for the case of equations $\Delta u = Pu$ considered on noncompact Riemann surfaces R. Typically we ask the properties of mappings $\tau_X : (R, P) \to \dim PX(R)$ from the space Φ of pairs (R, P) of noncompact Riemann surfaces R and densities P on R, i.e. $P(z)dxdy$ are 2-forms on R such that $P(z)dxdy \not\equiv 0$ and $P(z) \geq 0$ are Hölder continuous with respect to local parameters $z = x + iy$, into cardinals, where $PX(R)$ are the linear spaces of solutions of $\Delta u = Pu$ on R with certain boundedness properties X. The possibilities for X that we consider are B meaning the boundedness, D the finiteness of Dirichlet integrals $D(u) = \int_R |\operatorname{grad} u(z)|^2 dxdy$, E the finiteness of energy integrals $E^P(u) = \int_R (|\operatorname{grad} u(z)|^2 + u^2(z)P(z))dxdy$, and their combinations BD and BE. Particularly interesting are the subspaces of degenerate character of Φ:

$$O_X = \tau_X^{-1}(0) = \{(R, P) \in \Phi\, ;\, PX(R) = \{0\}\}\, .$$

If we denote by O_G the subspace of Φ consisting of those pairs (R, P) such that R are harmonically parabolic, i.e. there do not exist harmonic Green's function on R, then we have the following strict inclusion relations established by many authors listed in the references at the end of this paper:

$$(1) \qquad O_G < O_B < O_D = O_{BD} < O_E = O_{BE}\, .$$

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An important question in this context is: Which one of R and P is more decisive for the degeneracies $(R, P) \in O_B - O_G$, $O_D - O_B$, and $O_E - O_D$? By the nature of the question R must be postulated to be hyperbolic, i.e. nonparabolic. The purpose of this paper is to show that it is P that determines the degeneracies of the pair (R, P), i.e. we will prove

THE MAIN THEOREM. *On an arbitrarily fixed hyperbolic Riemann surface R there always exist densities P_B, P_D, and P_E such that the pairs (R, P_B), (R, P_D), and (R, P_E) belong to $O_B - O_G$, $O_D - O_B$, and $O_E - O_D$, respectively.*

To prove the theorem we will study the equation $\Delta u = Pu$ with densities P uniformly distributed on the hyperbolic Riemann surface R in the following sense. Let $(dr(z), d\theta(z))$ be the polar coordinate differentials on R with center z_0, i.e. $dr(z)$ is the differential of the global function $r(z) = e^{-G(z,z_0)}$ on R and $d\theta(z) = -{}^*dG(z, z_0)$, where $G(z, z_0)$ is the harmonic Green's function on R with pole z_0. If $P(z)dxdy$ is an arbitrary density on R, then

$$\tilde{P}(z) = \frac{P(z)dxdy}{r(z)dr(z) \wedge d\theta(z)} \geq 0$$

is a global function on R less the Green's singular set S, i.e. the set of isolated points in R where $|\operatorname{grad} G(\cdot, z_0)| = 0$. If there exists a continuous function $\varphi \geq 0$ defined on the interval $[0, 1)$ such that $\tilde{P}(z) = \varphi(r(z))$ on R, then we say that the density P is rotation free. More generally if there exists a constant $c \geq 1$ such that

$$c^{-1}\varphi(r(z)) \leq \tilde{P}(z) \leq c\varphi(r(z))$$

on R except for a compact subset of R, then we say that P is almost rotation free. We also call such a density (rotation free or almost rotation free) as a uniform density and denote it by P_φ indicating the dependence on φ. Set

$$b(\varphi) = \int_0^1 (1 - \tau)\varphi(\tau)d\tau\,;$$

$$d(\varphi) = \int_0^1\int_0^1 (1 - \max(\tau, \sigma))\varphi(\tau)\varphi(\sigma)d\tau d\sigma\,;$$

$$e(\varphi) = \int_0^1 \varphi(\tau)d\tau\,.$$

We will show that $b(\varphi) < \infty$ ($d(\varphi) < \infty$, $e(\varphi) < \infty$, resp.) is equivalent to that $P_\varphi B(R)(P_\varphi D(R)$, $P_\varphi E(R)$, resp.) is canonically isomorphic to $HB(R)$ ($HD(R)$, $HD(R)$, resp.), where $HX(R)$ denotes the class of harmonic functions on R with the boundedness property $X = B$ or D, and also that $b(\varphi) = \infty (d(\varphi) = \infty$, $e(\varphi) = \infty$, resp.) is equivalent to $P_\varphi B(R) = \{0\}$ ($P_\varphi D(R) = \{0\}$, $P_\varphi E(R) = \{0\}$, resp.), i.e. $(R, P_\varphi) \in O_B(O_D, O_E$, resp.). Therefore the required densities P_B, P_D, and P_E in our main theorem can be chosen, for example, among densities $P_\alpha = P_{\varphi_\alpha}$ with $\varphi_\alpha(\tau) = (1-\tau)^{-\alpha}$ for $\alpha \in (-\infty, \infty)$:

$$(2)\qquad \begin{cases} (R, P_\alpha) \in O_B - O_G & (\alpha \in [2, \infty)); \\ (R, P_\alpha) \in O_D - O_B & (\alpha \in [\frac{3}{2}, 2)); \\ (R, P_\alpha) \in O_E - O_D & (\alpha \in [1, \frac{3}{2})); \\ (R, P_\alpha) \notin O_E & (\alpha \in (-\infty, 1)). \end{cases}$$

In nos. 1–3, a mean formula on level lines of Green's function is discussed for a certain class of functions. In particular the circle mean formula of Green's function will prove to be useful. We relate vector space structures of subspaces of $P(R)$ to those of subspaces of $H(R)$ by what we call canonical isomorphisms in no. 4. The determination of classes $P_\varphi X(R)$ in terms of $x(\varphi)(X = B, D, E;\ x = b, d, e)$ will be carried over for rotation free densities first in nos. 5–9, which are also the main body of this paper, and then for almost rotation free ones in nos. 10–14.

Harmonic Green's Function

1. The hyperbolicity of a Riemann surface R is characterized by the existence of the harmonic *Green's function* $G(z, \zeta) = G_R(z, \zeta)$ with pole ζ situated at any point of R. (We use the same letters z, ζ, etc. to denote the generic point of R and also a local parameter around the point.) It is the smallest positive solution of R of the Poisson equation $\Delta G(\cdot, \zeta) = -2\pi\delta_\zeta$ with the Dirac measure δ_ζ whose unit mass is distributed at ζ. By the aid of Green's function we consider the *polar coordinate differentials* $(dr(z), d\theta(z))$ with center $z_0 \in R$ given by

$$(3)\qquad \begin{cases} \dfrac{dr(z)}{r(z)} = -dG(z, z_0), \\ d\theta(z) = -{}^*dG(z, z_0). \end{cases}$$

The differential $dr(z)$ is the differential of the global function

$$r(z) = e^{-G(z,z_0)} \tag{4}$$

on R whose range of values is the interval $[0, 1)$. The set $S = \{z \in R;\ dr(z) \wedge d\theta(z) = 0\}$ is referred to as the singular set, which is an isolated subset of R. We denote by Ω_ρ the set $\{z \in R;\ r(z) < \rho\}$ and by C_ρ the level lines $\{z \in R;\ r(z) = \rho\}$ for each $\rho \in (0, 1)$. The set Ω_ρ is a subregion of R and its relative boundary $\partial\Omega_\rho = C_\rho$. If ρ is sufficiently small, then Ω_ρ is relatively compact, and if ρ is sufficiently close to 0, then $\bar{\Omega}_\rho$ is homeomorphic to the closed unit disk. The differential $d\theta(z)$ considered on C_ρ is a positive regular measure with total mass 2π(cf. e.g. [23]):

$$\int_{C_\rho} d\theta(z) = 2\pi\,. \tag{5}$$

An open arc γ is said to be a regular Green line issuing from z_0 if $z_0 \in \bar{\gamma}$, $\gamma \cap S = \phi$, $d\theta(z) = 0$ along γ, and $\sup_{z \in \gamma} r(z) = 1$. The Brelot-Choquet theorem [3] says that C_ρ is pierced almost everywhere by regular Green lines issuing from z_0(cf. [23]). This is the basis for the justification of the following iterated integration:

$$\begin{aligned} &\int_R \varphi(z) r(z) dr(z) \wedge d\theta(z) \\ &= \int_{\bigcup_{\gamma \in \Gamma} \gamma} \varphi(z) r(z) dr(z) \wedge d\theta(z) \\ &= \int_0^1 \left(\int_{C_\tau} \varphi(z) d\theta(z) \right) \tau d\tau\,, \end{aligned} \tag{6}$$

where Γ is the set of regular Green lines issuing from z_0 and φ is a function on R.

Consider a 2-form $P(z)dxdy$ on R. We say that it is nonnegative if $P(z) \geq 0$ for every local parameter $z = x + iy$. Consequently $P(z)dxdy \geq Q(z)dxdy$ can be defined by the nonnegativeness of $(P(z) - Q(z))dxdy$. If $P(z)$ is Hölder continuous for every $z = x + iy$, then we say that the 2-form $P(z)dxdy$ is Hölder continuous. In this case the potential $\int_R G(\cdot, \zeta) P(\zeta) d\xi d\eta (\zeta = \xi + i\eta)$ is of class C^2 and

$$\Delta_z \int_R G(z, \zeta) P(\zeta) d\xi d\eta = -2\pi P(z) \tag{7}$$

if $\int_R G(z, \zeta) |P(\zeta)| d\xi d\eta < \infty$ for one and hence by the Harnack inequality for every $z \in R$(cf. e.g. Miranda [10]). The *Dirichlet integral* $D(u) =$

$D_R(u)$ over R of a function u with weak differential du is $\int_R du \wedge {}^*du = \int_R |\text{grad}\, u(z)|^2 dxdy \leq \infty$. If $P(z)dxdy$ is Hölder continuous, then

$$(8) \qquad D\left(\int_R G(\cdot, \zeta)P(\zeta)d\xi d\eta\right) = 2\pi\int_{R\times R} G(z, \zeta)P(z)P(\zeta)dxdyd\xi d\eta \leq \infty$$

if the right hand side has a definite meaning (cf. e.g. Constantinescu-Cornea [4]).

2. An exhaustion $\{R_n\}_0^\infty$ of R is a sequence of regular subregions R_n such that $\bar{R}_n \subset R_{n+1}$ and $\bigcup_0^\infty R_n = R$. We denote by $H(F)$ the class of harmonic functions on an open subset F of R. Let $z_0 \in R_0 \subset \bar{R}_0 \subset \Omega_\rho$. Consider a function $w_{\rho,n} = w_n \in H(\Omega_\rho \cap R_n - \bar{R}_0) \cap C(\bar{\Omega}_\rho \cap \bar{R}_n - R_0)$ for each $n \geq 1$ such that $w_n | \partial R_0 = 0$, $w_n | \Omega_\rho \cap \partial R_n = 1$, and ${}^*dw_n | \partial\Omega_\rho \cap R_n = 0$. We set $w_n = 0$ on R_0 and $w_n = 1$ on $\Omega_\rho - \bar{R}_n$. Then $\{w_n\}_1^\infty$ is decreasing and converges to a function $w_\rho = w$ on Ω_ρ and

$$(9) \qquad \lim_{n\to\infty} D_{\Omega_\rho}(w_\rho - w_{\rho,n}) = 0\,.$$

The Kuramochi theorem [7] (cf. Kusunoki-Mori [8]; [13], [23]) says that

$$(10) \qquad w_\rho(z) \equiv 0\,.$$

This is equivalent to that the double $\hat{\Omega}_\rho$ of Ω_ρ along C_ρ is parabolic. Here we append a very simple proof of (10). Let $G_n(z, z_0)$ be the Green's function on R_n and set $G_n(z, z_0) = 0$ on $R - R_n$. Take a $\rho_0 \in (0, \rho)$ such that Ω_{ρ_0} is relatively compact. Consider functions

$$E_n(z) = \frac{\min\,(G_n(z, z_0), -\log \rho_0)}{\min\,(G(z, z_0), -\log \rho_0)}$$

on $\bar{\Omega}_\rho (n = 1, 2, \cdots)$. Observe that E_n converges to 1 uniformly on each compact subset of $\bar{\Omega}_\rho$ as $n \to \infty$ and

$$\lim_{n\to\infty} D_{\Omega_\rho}(E_n) = 0\,.$$

By the Green formula

$$\begin{aligned} D_{\Omega_\rho}(w_n) &= D_{\Omega_\rho \cap R_n}(1 - w_n) \\ &= \int_{\partial R_0} {}^*dw_n = -\int_{\partial(\Omega_\rho \cap R_n - \bar{R}_0)} E_n {}^*dw_n \\ &= -D_{\Omega_\rho}(E_n, w_n) \end{aligned}$$

and therefore, by the Schwarz inequality, $D_{\Omega_\rho}(w_n) \leq D_{\Omega_\rho}(E_n)$, proving (10).

As an application of (10) we shall prove the following useful identity:

$$\frac{1}{2\pi}\int_{C_\rho} u(z)d\theta(z) = u(z_0) + \sum_{k=1}^{m} \log r(z_k) - m\log\rho + \frac{1}{2\pi}\int_{\Omega_\rho} \log\frac{\rho}{r(z)}\cdot \Delta u(z)dxdy \tag{11}$$

for any nonnegative function $u \in C^2(\bar{\Omega}_\rho - \{z_1, \cdots, z_m\})$, where $z_1, \cdots, z_m$ are a finite number $m \geq 0$ of points in $\bar{\Omega}_\rho$, such that $\Delta u \geq 0$, $u(z) + \log|z - z_k|$ has a C^1-extension to $z_k (k = 1, \cdots, m)$, and u is bounded on $\Omega_\rho - V$ or $D_{\Omega_\rho - V}(u) < \infty$, where V is a relatively compact open neighborhood of $\{z_1, \cdots, z_m\}$ with $\bar{V} \subset \bar{\Omega}_\rho$. To prove (11) let $g_n(z)$ be the Green's function on $\Omega_\rho \cap R_n$ with pole z_0 such that $n \geq 1$ and $\bar{V} \subset R_n$. The standard application of the Green formula to g_n and u on $\Omega_\rho \cap R_n$ less small disks at $z_k (k = 1, \cdots, m)$ and the limiting process, making disks at z_k shrink to z_k, yield

$$\frac{1}{2\pi}\int_{C_\rho \cap R_n} u(z)(-{}^*dg_n(z)) + \frac{1}{2\pi}\int_{\Omega_\rho \cap \partial R_n} u(z)(-{}^*dg_n(z)) = u(z_0) - \sum_{k=1}^{m} g_n(z_k) + \frac{1}{2\pi}\int_{\Omega_\rho \cap R_n} g_n(z)\Delta u(z)dxdy\,.$$

Since g_n and $-{}^*dg_n$ converge to $G(\cdot, z_0) + \log\rho = -\log r + \log\rho$ and $d\theta$ on $\bar{\Omega}_\rho$ and C_ρ respectively, on letting $n \to \infty$ we deduce (11) if we can show

$$\lim_{n\to\infty}\int_{\Omega_\rho \cap \partial R_n} u(z)(-{}^*dg_n(z)) = 0\,.$$

Take exhaustion $\{R_n\}_0^\infty$ with $R_0 = \Omega_{\rho_0}$ and corresponding $w_{\rho,n} = w_n$. Clearly $\rho_0(1 - w_n) \geq g_n$ on $\bar{\Omega}_\rho \cap \bar{R}_n - \Omega_{\rho_0}$ and thus $\rho_0{}^*dw_n \geq -{}^*dg_n$. If $D(u) < \infty$,

$$\left|\int_{\Omega_\rho \cap \partial R_n} u(z)(-{}^*dg_n(z))\right| \leq \rho_0\int_{\Omega_\rho \cap \partial R_n} u(z){}^*dw_n(z) \leq D_{\Omega_\rho \cap R_n - \bar{\Omega}_{\rho_0}}(u, w_n) + \int_{C_{\rho_0}} |u(z){}^*dw_n(z)|\,.$$

If $k = \sup_{\Omega_\rho - V}|u| < \infty$, then

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$$\left|\int_{\Omega_\rho \cap \partial R_n} u(z)(-{}^*dg_n(z))\right| \le \rho_0 \int_{\Omega_\rho \cap \partial R_n} u(z){}^*dw_n(z)$$
$$\le k\rho_0 \int_{\Omega_\rho \cap \partial R_n} {}^*dw_n(z)$$
$$= k\rho_0 \int_{\partial(\Omega_\rho \cap R_n)} w_n(z){}^*dw_n(z) = k\rho_0 D_{\Omega_\rho \cap R_n}(w_n)\,.$$

In either case the required conclusion now follows from (10).

3. As direct consequences of (11) we first obtain the circle mean formula of Green's function ([18]) which will be convenient for our later calculations:

$$\int_{C_\rho} G(z,\zeta)d\theta(\zeta) = -2\pi \max(\log\rho,\ \log r(\zeta))\,. \tag{12}$$

Another consequence we need is

$$\int_{C_{\rho_1}-C_{\rho_2}} u^2(z)d\theta(z) \le 2\log\frac{\rho_1}{\rho_2} D_{\Omega_{\rho_1}-\bar\Omega_{\rho_2}}(u) \tag{13}$$

for any $1 > \rho_1 > \rho_2 > 0$ and any $u \in HD(\Omega_\rho) \cap C^2(\bar\Omega_\rho)$, where we denote by $HD(F)(HBD(F))$ the class of harmonic (bounded harmonic) functions on an open subset F of R with finite Dirichlet integrals over F. Since we can find a sequence $\{u_n\}$ in $HBD(\Omega_\rho) \cap C^2(\bar\Omega_\rho)$ for any $u \in HD(\Omega_\rho) \cap C^2(\bar\Omega_\rho)$ such that $\{u_n\}$ converges to u, $\{D_{\Omega_\rho}(u_n - u)\}$ converges to 0, and $u_n = ((-n) \vee u) \wedge n$, where $\wedge$ and $\vee$ are lattice operations in $H(\Omega_\rho)$ (cf. e.g. [23]), we may suppose $u \in HBD(\Omega_\rho)$. Then u^2 is admissible for the validity of (11) and $\Delta u^2 = 2|\operatorname{grad} u|^2$. Therefore

$$\frac{1}{2\pi}\int_{C_\tau} u^2(z)d\theta(z) = u^2(z_0) + \frac{1}{2\pi}\int_{\Omega_\tau} \log\frac{\tau}{r(z)} 2|\operatorname{grad} u(z)|^2 dxdy\,.$$

The difference of these identities for $\tau = \rho_1$ and ρ_2 yields (13).

Canonical Isomorphisms

4. A nonnegative Hölder continuous 2-form $P(z)dxdy$ which is not identically zero on a Riemann surface R will be referred to as a *density* on R. Given a density $P(z)dxdy$ on R, we can consider the self-adjoint elliptic partial differential equation

$$\Delta u(z) = P(z)u(z) \tag{14}$$

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invariantly defined on R. We denote by $P(F)$ the linear space of C^2 solutions of (14) on an open set F of R. The presheaf $\mathfrak{P} = \{P(F);\ F \subset R\}$ defines a harmonic structure on R in the sense of Brelot [2]. A superharmonic (subharmonic) function with respect to $\mathfrak{P}$ will be referred to as a supersolution (subsolution) of (14). For local properties of $\mathfrak{P}$ and related structures we refer to e.g. Royden [21]; [12], etc. Our main concern is the global properties of $P(R)$. We denote by $P^+(R)$ the subspace of nonnegative solutions in $P(R)$. The linear subspace of $P(R)$ generated by $P^+(R)$ will be denoted by $P'(R)$. The notations H^+ and H' are understood in the same sense. The Myrberg theorem [11] says that $\dim P'(R) \geq 1$ for every noncompact R. This is the reason we do not consider the null class O_P in the scheme (1). We will always assume that R is noncompact. We will make constant use of the following (cf. e.g. [23]).

$$HX(R) = H'X(R) \quad (X = B, BD, D) \tag{15}$$

i.e. any function in the class $HX(R)$ can be represented as a difference of two nonnegative functions in $HX(R)$ $(X = B, BD, D)$, and similarly (Royden [21], Glasner-Katz [5], [12])

$$PX(R) = P'X(R) \quad (X = B, BD, D, BE, E)\,, \tag{16}$$

i.e. any solution of (14) with the property X can be represented as a difference of two nonnegative solutions of (14) with the property $X (X = B, BD, D, BE, E)$.

A linear mapping τ of a subspace of $P'(R)$ into $H'(R)$ is said to be *canonical* if $\tau u - u$ is a difference of two potentials, i.e. nonnegative superharmonic functions whose greatest harmonic minorants are zero. The intuitive meaning of this is that τu and u have the same ideal boundary values. If there exists a nontrivial canonical mapping, then R must be hyperbolic, and there is a unique maximal canonical mapping T_P. By the Riesz decomposition of positive superharmonic functions and (7), $T = T_P$ is seen to have the representation

$$T_P u = u + \frac{1}{2\pi}\int_R G(\cdot\,,\zeta)u(\zeta)P(\zeta)d\xi d\eta \ \ (\zeta = \xi + i\eta)\,. \tag{17}$$

We denote by $\mathfrak{D}(T_P)$ the domain of the operator T_P. Clearly a $u \in P'(R)$ belongs to $\mathfrak{D}(T_P)$ if and only if $\int_R G(z,\zeta)|u(\zeta)|P(\zeta)d\xi d\eta < \infty$ for some and hence, by the Harnack inequality, for all $z \in R$. Take an exhaustion $\{R_n\}_1^\infty$ of R and denote by $G_n(z, \zeta)$ the Green's function on R_n. Consider auxiliary operators

$$T_{P,n}u = T_n u = u + \frac{1}{2\pi}\int_{R_n} G_n(\cdot, \zeta)u(\zeta)P(\zeta)d\xi d\eta$$

which maps $P(R_n) \cap C(\bar{R}_n)$ bijectively to $H(R_n) \cap C(\bar{R}_n)$ and $\sup_{R_n}|T_n u| = \sup_{R_n}|u|$, which follows from $T_n u|\partial R_n = u|\partial R_n$. Clearly

$$Tu = \lim_{n\to\infty} T_n u$$

uniformly on each compact subset of R for any $u \in \mathfrak{D}(T_P) \cap P'(R)$. Therefore we see at once that $\sup_R|Tu| \leq \sup_R|u|$. If $T_P u = 0$, then the subharmonic function $2\pi|u|$ is dominated by the potential $\int_R G(\cdot, \zeta)|u(\zeta)|P(\zeta)d\xi d\eta$ and therefore $u = 0$. Thus T_P is injective and for this reason we call T_P the *canonical isomorphism*. Clearly T_P is a positive operator, i.e. $u \geq 0$ implies $T_P u \geq 0$, and in fact $T_P u \geq u \geq 0$. In this case $\sup_R Tu = \sup_R u$. We remark that

$$PB(R),\quad PD(R),\quad PE(R) \subset \mathfrak{D}(T_P) \tag{18}$$

and

$$T_P(PB(R)) \subset HB(R),\ T_P(PD(R)) \subset HD(R),\ T_P(PE(R)) \subset HD(R)\,. \tag{19}$$

To prove these take a $u \in P^+(R)$ and an exhaustion $\{R_n\}$ of R. Since

$$T_n u(z) = u(z) + \frac{1}{2\pi}\int_{R_n} G_n(z, \zeta)u(\zeta)P(\zeta)d\xi d\eta$$

belongs to $H^+(R_n)$ and increasing with n, by the Lebesgue-Fatou theorem,

$$\lim_{n\to\infty} T_n u(z) = u(z) + \frac{1}{2\pi}\int_R G(z, \zeta)u(\zeta)P(\zeta)d\xi d\eta\,.$$

If u is bounded, then the left converges to a function $Tu \in H^+(R)$ and bounded, i.e. $Tu \in HB(R)$ and $u \in \mathfrak{D}(T)$. If $D(u) < \infty$, then by the Dirichlet principle, $D_{R_n}(T_n u) \leq D(u)$, and since R is hyperbolic, $\lim_{n\to\infty} T_n u(z) < \infty$ and $D(Tu) < \infty$ (cf. [23]). Therefore $u \in \mathfrak{D}(T)$ and $Tu \in HD(R)$. In view of (16) we deduce (18) and (19).

To determine when $T = T_P$ is surjective in (19) is a very difficult question to settle. We only have partial results for the class PB and its subclasses PBD and PBE (cf. Royden [21], Glasner-Katz [5], Glasner-Nakai [6], Maeda [9]; [12], [16], [17]). However for the special case of uniform densities we will have the complete answer in this paper. We denote by $T^X = T_P^X$ the restriction of T_P on $PX(R)$ $(X = B, D, E, BD, BE)$. Let $Q(z)dxdy$ be another density on R. The mapping $T_{Q,P}^X$ from $PX(R)$ onto $QX(R)$ $(X = B, D, E)$ such that $T_P^X = T_Q^X \circ T_{Q,P}^X$, if exists, is also referred to as a *canonical isomorphism* of $PX(R)$ onto $QX(R)$. If T_P^X

$$\begin{array}{ccc} PX(R) & \xrightarrow{\;T_P^X\;} & H'(R) \\ \Big\downarrow T_{Q,P}^X & \nearrow T_Q^X & \\ QX(R) & & \end{array}$$

and T_Q^X are surjective, then clearly $T_{Q,P}^X$ exists $(X = B, D, E)$. If P and Q differ only on a compact subset of R, then $T_{Q,P}^X$ exists for every $X = B, D, E$. This follows from the fact that

$$T_{Q,P}u = u + \frac{1}{2\pi}\int_{R_0} G^Q(\cdot,\zeta)u(\zeta)(P(\zeta) - Q(\zeta))d\xi d\eta$$

is a bijective mapping from $P'(R)$ to $Q'(R)$ and $T_{Q,P}|PX(R) = T_{Q,P}^X$ (cf. [16]), where R_0 is a relatively compact subset of R such that $(P(z) - Q(z))dxdy \equiv 0$ on $R - R_0$, and $G^P(z,\zeta)$ is the Green's function of the equation (14), i.e. the smallest positive solusion of the Poisson equation $(\Delta - P)u = -2\pi\delta_\zeta$, whose existence is always assured by the Myrberg theorem [11] (cf. [23]). To determine the existence of $T_{Q,P}^X$ is also a difficult question (cf. [16]). For later use we only state the following simple observation: If $T_{cP,P}^X$ exists for every $c > 0$ and

$$k^{-1}P \leq Q \leq kP \tag{20}$$

for some constant $k \geq 1$ on R except for a compact subset of R, then $T_{Q,P}^X$ exists $(X = B, D, E)$. That the sole condition (20) is sufficient for the existence of $T_{Q,P}^B$ is shown by Royden [21]. The same is also true for the existence of $T_{Q,P}^E$ by the energy principle. It is likely* that only (20) implies the existence of $T_{Q,P}^D$, but since the above assertion is sufficient for our later purpose, we prove it here under the additional conditions on P. We denote by U_X the common ranges of T_{cP}^X for all $c > 0$ and by V_X the range of T_Q^X. Set $U_X^+ = U_X \cap H^+$ and $V_X^+ = V_X \cap H^+$.

*) That this is certainly the case is shown in the present author's recent paper: *Order comparisons on canonical isomorphisms*, Nagoya Math. J., **50** (1973), 67–87.

By (15) we only have to show that $U_X^+ = V_X^+$. Let $h \in U_X^+$ and $T_{k^{-1}P}^X v = h$ with $v \in k^{-1}P^+X(R)$. Let $u_n \in Q^+(R_n) \cap C(\bar{R}_n)$ such that

$$T_{k^{-1}Q,n}v = T_{Q,n}u_n .$$

Since we can assume that the exceptional compact set in (20) is empty, we deduce

$$0 \leq u_n \leq v$$

on R_n and $\{u_n\}$ is decreasingly convergent to a $u \in Q^+(R)$. Since

$$\int_R G(z,\zeta)u(\zeta)Q(\zeta)d\xi d\eta \leq k^2\int_R G(z,\zeta)v(\zeta)k^{-1}P(\zeta)d\xi d\eta < \infty$$

and $T_{Q,n}u_n = T_{k^{-1}P,n}v$ converges to h as $n \to \infty$, we conclude, by the Lebesgue convergence theorem, that

$$T_Q u = h, \quad 0 \leq u \leq v .$$

If $X = B$, then $u \in Q^+B(R)$. If $X = D$, then by (8)

$$D(v) = D(h) + \frac{1}{2\pi}\int_{R\times R} G(z,\zeta)v(z)v(\zeta)k^{-1}P(z)k^{-1}P(\zeta)dxdyd\xi d\eta < \infty$$

and therefore

$$\begin{aligned} D(u) &= D(h) + \frac{1}{2\pi}\int_{R\times R} G(z,\zeta)u(z)u(\zeta)Q(z)Q(\zeta)dxdyd\xi d\eta \\ &\leq k^4 D(v) < \infty \end{aligned}$$

and $u \in Q^+D(R)$. Finally suppose $X = E$. By the above we deduce

$$\begin{aligned} E^Q(u) &= D(u) + \int_R u^2(z)Q(z)dxdy \\ &\leq k^4 D(v) + \int_R v^2(z)kP(z)dxdy \leq k^4 E^{k^{-1}P}(v) < \infty \end{aligned}$$

and hence $u \in Q^+E(R)$. We have seen that $U_X^+ \subset V_X^+$. The inclusion $U_X^+ \supset V_X^+$ can be shown in the similar fashion.

Rotation Free Densities

5. A density $P(z)dxdy$ on a hyperbolic Riemann surface R will be referred to as being *rotation free* with respect to the point z_0 if there exists a function φ on the unit interval $[0,1)$ such that

(21) $$P(z)dxdz = P_\varphi(z)dxdy = \varphi(r(z))r(z)dr(z) \wedge d\theta(z)\,.$$

The function φ is automatically nonnegative and (locally) Hölder continuous. To the density P_φ we associate the following quantities:

(22) $$\begin{cases} b(\varphi) = \int_0^1 (1-\tau)\varphi(\tau)d\tau\,; \\ d(\varphi) = \int_0^1\int_0^1 (1-\max(\tau,\sigma))\varphi(\tau)\varphi(\sigma)d\tau d\sigma\,; \\ e(\varphi) = \int_0^1 \varphi(\tau)d\tau\,. \end{cases}$$

These are finite or infinite nonnegative numbers. Observe that $e(\varphi) < \infty$ implies $d(\varphi) < \infty$, and $d(\varphi) < \infty$ in turn implies $b(\varphi) < \infty$.

6. First we study the class $P_\varphi B(R)$. We will denote by O_B the class of pairs (R, P) of Riemann surfaces R and densities P on R such that $\dim PB(R) = 0$. In general the linear space $L = T_P^B(PB(R))$ isomorphic to $PB(R)$ is a subspace of $HB(R)$: $\{0\} \subset L \subset HB(R)$. For rotation free densities we now show that only extreme cases occur, i.e. either $L = \{0\}$ or $HB(R)$:

PROPOSITION. *The following three conditions are equivalent by pairs for rotation free densities P_φ on hyperbolic Riemann surfaces R*:

(α) $b(\varphi) < \infty$;

(β) *$P_\varphi B(R)$ is canonically isomorphic to $HB(R)$*;

(γ) $(R, P_\varphi) \notin O_B$.

Proof. Assume (α) and fix an arbitrary $h \in H^+B(R)$. Let $\{R_n\}_1^\infty$ be an exhaustion of R and $G_n(z, \zeta)$ be the harmonic Green's function on R_n. Take $u_n \in P_\varphi^+(R_n) \cap C(\bar{R}_n)$ such that $u_n | \partial R_n = h$. Then $\{u_n\}$ is decreasing and converges to a $u \in P_\varphi^+B(R)$. Observe that

$$h(z_0) = T_{P_\varphi, n}u_n(z_0) = u_n(z_0) + \frac{1}{2\pi}\int_{R_n} G_n(z_0, \zeta)u_n(\zeta)P_\varphi(\zeta)d\xi d\eta$$
$$= u_n(z_0) + \frac{1}{2\pi}\int_{R_n} G_n(z_0, \zeta)u_n(\zeta)\varphi(r(\zeta))r(\zeta)dr(\zeta) \wedge d\theta(\zeta)\,.$$

The integrand is dominated by $G(z_0, \zeta)h(\zeta)\varphi(r(\zeta))r(\zeta)dr(\zeta) \wedge d\theta(\zeta)$. Here we remark that, by (11),

(23) $$\int_{C_\tau} h(z)d\theta(z) = 2\pi h(z_0)\,.$$

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This is true not only for $h \in HB(R)$ but also for $h \in HD(R)$. Since $\tau \log \tau^{-1} \leq (1-\tau)$ on $(0,1)$, we deduce

$$\begin{aligned}\int_R G(z_0,\zeta)h(\zeta)\varphi(r(\zeta))r(\zeta)dr(\zeta) \wedge d\theta(\zeta) \\ = \int_0^1\left(\int_{C_\tau} h(\zeta)d\theta(\zeta)\right)\log \tau^{-1}\cdot\varphi(\tau)\tau d\tau \\ \leq \int_0^1 2\pi h(z_0)(1-\tau)\varphi(\tau)d\tau = 2\pi h(z_0)b(\varphi) < \infty .\end{aligned}$$

Therefore the Lebesgue convergence theorem can be applied to deduce

$$h(z_0) = u(z_0) + \frac{1}{2\pi}\int_R G(z_0,\zeta)u(\zeta)P_\varphi(\zeta)d\xi d\eta = T_{P_\varphi}u(z_0) .$$

Since $T_{P_{\varphi,n}}u \leq T_{P_{\varphi,n}}u_n = h$, we have $T_{P_\varphi}u \leq h$ and hence $T_{P_\varphi}u = h$, i.e. $T^B_{P_\varphi}$ is surjective and we obtain (β). The implication from (β) to (γ) is trivially true.

Suppose that (γ) is true. By (16) there exists a nonzero $u \in P^+_\varphi B(R)$. From (11) it follows that

$$\int_{C_\rho} u(z)d\theta(z) \geq 2\pi u(z_0) . \tag{24}$$

This is also true for $u \in P^+_\varphi D(R)$. This comes from the fact that $\Delta u = P_\varphi u \geq 0$. On the other hand, since $\log \tau^{-1} \geq (1-\tau)$ on $(0,1)$,

$$\begin{aligned}h(z_0) &= T_{P_\varphi}u(z_0) \\ &= u(z_0) + \frac{1}{2\pi}\int_R G(z_0,\zeta)u(\zeta)\varphi(r(\zeta))r(\zeta)dr(\zeta) \wedge d\theta(\zeta) \\ &= u(z_0) + \int_0^1\left(\frac{1}{2\pi}\int_{C_\tau} u(\zeta)d\theta(\zeta)\right)\log \tau^{-1}\cdot\varphi(\tau)\tau d\tau \\ &\geq u(z_0)\int_0^1(1-\tau)\varphi(\tau)\tau d\tau .\end{aligned}$$

The nonnegative nonzero solution cannot vanish and thus $u(z_0) > 0$. Therefore

$$\int_0^1(1-\tau)\varphi(\tau)\tau d\tau < \infty$$

and this in turn implies $b(\varphi) < \infty$. Q.E.D.

7. From the class $P_\varphi B(R)$ we turn to the class $P_\varphi E(R)$ of solutions

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with finite P_φ*-energy integrals* $E_R^{P_\varphi}(u)=\int_R(|\operatorname{grad} u(z)|^2+u^2(z)P_\varphi(z))dxdy$. We use the notation O_E for the class of pairs (R, P) of Riemann surfaces R and densities P on R such that $\dim PE(R)=0$. The meaning of O_{BE} should be clear. We know that $O_E=O_{BE}$ (cf. Ozawa [20], Glasner-Katz [5]). The linear space $L=T_P^E(PE(R))$ isomorphic to $PE(R)$ is a subspace of $HD(R)$: $\{0\}\subset L\subset HD(R)$. For rotation free densities we also show that either $L=\{0\}$ or $HD(R)$.

PROPOSITION. *The following four conditions are equivalent by pairs for rotation free densities P_φ on hyperbolic Riemann surfaces R:*

(α) $e(\varphi)<\infty$;

(β) $P_\varphi E(R)$ *is canonically isomorphic to* $HD(R)$;

(γ) $P_\varphi BE(R)$ *is canonically isomorphic to* $HBD(R)$;

(δ) $(R, P_\varphi)\notin O_E=O_{BE}$.

Proof. Assume (α) and fix an arbitrary $h\in H^+D(R)$. Take u_n as in the proof of Proposition 6. Observe that

$$\begin{aligned}E_R^{P_\varphi}(h)&=D_R(h)+\int_R h^2(z)P_\varphi(z)dxdy\\&=D_R(h)+\int_R h^2(z)\varphi(r(z))dr(z)\wedge d\theta(z)\\&=D_R(h)+\int_0^1\left(\int_{C_\tau}h^2(z)d\theta(z)\right)\varphi(\tau)\tau d\tau\,.\end{aligned}$$

By (13), $\sup_{\tau\in(0,1)}\int_{C_\tau}h^2(z)d\theta(z)=k<\infty$, and therefore

$$E_R^{P_\varphi}(h)\le D(h)+k\int_0^1\varphi(\tau)d\tau\le D(h)+ke(\varphi)<\infty\,.$$

Since $E_{R_n}^{P_\varphi}$ is the variation whose Euler-Lagrange equation is $\Delta u=P_\varphi u$,

$$E_{R_n}^{P_\varphi}(u_n)\le E_{R_n}^{P_\varphi}(h)\le E_R^{P_\varphi}(h)<\infty\,.$$

Therefore the decreasing sequence $\{u_n\}$ converges to a $u\in P^+E(R)$. Since $e(\varphi)<\infty$ implies $b(\varphi)<\infty$, by the same proof as in no. 6, we see that $T_{P_\varphi}u=h$ and (β) follows. By (19), $T_{P_\varphi}^{BE}$ maps $P_\varphi BE(R)$ injectively to $HBD(R)$. For an arbitrary $h\in H^+BD(R)$ there exists a $u\in P_\varphi^+E(R)$ such that $T_{P_\varphi}u=h$ if (β) is true. But $0\le u\le h$ implies that $u\in P_\varphi^+B(R)$ and therefore $u\in P_\varphi^+B(R)\cap P_\varphi^+E(R)=P_\varphi^+BE(R)$, i.e. ($\beta$) implies ($\gamma$). The implication from (γ) to (δ) is trivial.

If (δ) is true, then, by (16), there exists $u \in P_\varphi BE(R)$ such that $u > 0$ on R. By (11), $\int_{C_\tau} u^2(z)d\theta(z) \geq u^2(z_0)$ and

$$\infty > E_R^{P_\varphi}(u) > \int_R u^2(z)P_\varphi(z)dxdy$$
$$= \int_0^1 \left(\int_{C_\tau} u^2(z)d\theta(z) \right) \varphi(\tau)\tau d\tau \geq u^2(z_0) \int_0^1 \varphi(\tau)\tau d\tau\ ,$$

i.e. $\int_0^1 \varphi(\tau)\tau d\tau < \infty$ and this implies $e(\varphi) < \infty$. Q.E.D.

8. For the study of $P_\varphi D(R)$, a little more sophisticated considerations than those in nos. 3 and 4 are in order. We consider the mean operator $*: u \to u^*$ defined by

$$u^*(z) = \frac{1}{2\pi} \int_{C_{r(z)}} u(\zeta)d\theta(\zeta) \tag{25}$$

whose domain $\mathscr{D}_* = \mathscr{D}_*(R)$ is the class of measurable function u on R such that

$$\int_{C_\rho} |u(\zeta)|\, d\theta(\zeta) < \infty$$

for every $\rho \in (0, 1)$. The function u^* is rotation free in the sense that $u^*|C_\rho$ is a constant for each $\rho \in (0, 1)$. We associate the function

$$u^{**}(\tau) = u^*|C_\tau$$

on $(0, 1)$ with u^*. Then

$$u^*(z) = u^{**}(r(z))\ . \tag{26}$$

Among properties of $*$ we state the following which will be needed for the study of $P_\varphi D(R)$. Let $u \in P_\varphi^+ D(R)$. Then $u \in \mathscr{D}_*$ and

$$D(u^*) \leq D(u)\,; \tag{27}$$

$$\Delta u^*(z) = P_\varphi(z) u^*(z)\,; \tag{28}$$

$$\inf_{z \in R} u^*(z) > 0 \quad (for\ u \not\equiv 0)\,; \tag{29}$$

$$\sup_{z \in R} u^*(z) < \infty\ . \tag{30}$$

Let $h = T_{P_\varphi}^D u \in H^+ D(R)$. Then $u \leq h$ and, by (23),

$$0 \leq u^*(z) = \frac{1}{2\pi}\int_{C_{r(z)}} u(z)d\theta(z)$$
$$\leq \frac{1}{2\pi}\int_{C_{r(z)}} h(z)d\theta(z) = h(z_0)\,,$$

i.e. $u \in \mathscr{D}_*$ and (30) is valid. We apply (11) to u to deduce

$$\frac{1}{2\pi}\int_{C_{r(z)}} u(z)d\theta(z) = u(z_0) + \frac{1}{2\pi}\int_{\Omega_{r(z)}} \log\frac{r(z)}{r(\zeta)}u(\zeta)\varphi(r(\zeta))r(\zeta)dr(\zeta) \wedge d\theta(\zeta)$$
$$= u(z_0) + \int_0^{r(z)}\left(\frac{1}{2\pi}\int_{C_\tau} u(\zeta)d\theta(\zeta)\right)\log\frac{r(z)}{\tau}\varphi(\tau)\tau d\tau\,.$$

Therefore we have

$$u^*(z) = u(z_0) + \int_0^{r(z)} u^{**}(\tau)(\log r(z) - \log\tau)\varphi(\tau)\tau d\tau\,.$$

First of all, by the above integral representation of u^*, the boundedness of u^* (or u^{**}) implies the continuity of u^* in R. Then u^* is of class C^1 with respect to $r(z) > 0$ and

(31) $$\frac{\partial u^*(z)}{\partial r(z)} = \frac{1}{r(z)}\int_0^{r(z)} u^{**}(\tau)\varphi(\tau)\tau d\tau\,.$$

Clearly $\partial u^*(z)/\partial\theta(z) = 0$. These show that u^* is of class C^2 with respect to $r(z) > 0$ and

(32) $$\frac{\partial^2 u^*(z)}{\partial r(z)^2} = -\frac{1}{r^2(z)}\int_0^{r(z)} u^{**}(\tau)\varphi(\tau)\tau d\tau + u^*(z)\varphi(r(z))\,.$$

If we choose a branch of $\theta(z) = \int^z d\theta(z)$ at $z \in R$ which is not in the Green's singular set S and if we take $re^{i\theta} = r(z)e^{i\theta(z)}$ as a local parameter at z, then $\partial u^*/\partial\theta = 0$ and, by (31) and (32), we have

$$\Delta u^* r dr d\theta = \left(\frac{\partial^2}{\partial r^2}u^* + \frac{1}{r}\,\frac{\partial}{\partial r}u^* + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}u^*\right) r dr d\theta$$
$$= u^*\varphi(r) r dr d\theta\,,$$

i.e. we conclude that (28) is valid on $R - S$ and hence on R since S is removable for solutions with finite Dirichlet integrals on $R - S$. Again by (31),

$$\begin{aligned}\frac{\partial u^*}{\partial\rho} &= \frac{1}{\rho}\int_{0\le r(z)\le\rho}\left(\frac{1}{2\pi}\int_{C_{r(z)}} u(z)\varphi(r(z))d\theta(z)\right)r(z)dr(z)\\ &= \frac{1}{2\pi\rho}\int_0^\rho\left(\int_{C_\tau}\Delta u(z)d\theta(z)\right)\tau d\tau\\ &= \frac{1}{2\pi\rho}\int_{\Omega_\rho}\Delta u(z)dxdy\,.\end{aligned}$$

Take $E_n(z)$ considered in no. 2. By the Green formula

$$\int_{C_\rho}E_n(z)^*du(z) = \int_{\Omega_\rho}E_n(z)\Delta u(z)dxdy + D_{\Omega_\rho}(E_n, u)\,.$$

Observe that $^*du = (\partial u/\partial\rho)\rho d\theta$ on C_ρ, and thus

$$\int_{\Omega_\rho}E_n(z)\Delta u(z)dxdy \le (D_{\Omega_\rho}(E_n)D(u))^{1/2} + \left(\int_{C_\rho}E_n^2(z)\rho d\theta(z)\int_{C_\rho}\left(\frac{\partial u}{\partial\rho}\right)^2\rho d\theta\right)^{1/2}.$$

On letting $n\to\infty$, we have

$$\left(\frac{\partial u^*}{\partial\rho}\right)^2 \le \frac{1}{2\pi\rho}\int_{C_\rho}\left(\frac{\partial u}{\partial\rho}\right)^2\rho d\theta\,.$$

On integrating both sides with respect to $d\theta$ on C_ρ, we obtain

$$\int_{C_\rho}\left(\frac{\partial u^*}{\partial\rho}\right)^2\rho d\theta \le \int_{C_\rho}\left(\left(\frac{\partial u}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial\theta}\right)^2\right)\rho d\theta\,.$$

Again the integration of both sides with respect to $d\rho$ on $(0,1)$ yields $D(u^*)\le D(u)$, proving (27). Finally, if $\inf_R u^* = 0$, then there exists an increasing sequence $\{r_k\}$ converging to 1 such that $u^*|C_{r_k}\le 1/\mathrm{k}$. Take $w_{1/k,n}$ in (9) in no. 2 and let $c = \max_{\bar R_0} u^*$ and $c' = \max_R u^*$. Then

$$u^*(z) \le \frac{1}{k} + c + c'w_{1/k,n}$$

on $\Omega_{1/k}\cap R_n - \bar R_0$, and on letting $n\to\infty$, we conclude, by (10), that

$$u^*(z) \le \frac{1}{k} + c$$

on $\Omega_{1/k} - \bar R_0$. Again by making $k\to\infty$, we obtain $u^*(z)\le c$ on $R - \bar R_0$ and hence on R. This means that u^* takes its maximum at some point of R, a contradiction, and (29) follows.

9. We can now complete the study of $P_\varphi D(R)$. We will denote by O_D the class of pairs (R, P) of Riemann surfaces R and densities P on

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R such that $\dim PD(R) = 0$. The meaning of the class O_{BD} should be clear. We know that $O_D = O_{BD}$ ([14], [6]). The linear space $L = T_P^D(PD(R))$ isomorphic to $PD(R)$ is a subspace of $HD(R)$: $\{0\} \subset L \subset HD(R)$. For rotation free densities we only have two cases: $L = \{0\}$ or $HD(R)$:

PROPOSITION. *The following four conditions are equivalent by pairs for rotation free densities P_φ on hyperbolic Riemann surfaces R:*

(α) $d(\varphi) < \infty$;

(β) *$P_\varphi D(R)$ is canonically isomorphic to $HD(R)$;*

(γ) *$P_\varphi BD(R)$ is canonically isomorphic to $HBD(R)$;*

(δ) $(R, P_\varphi) \notin O_D = O_{BD}$.

Proof. Assume (α) and fix an arbitrary $h \in H^+D(R)$. Take u_n as in the proof of Proposition in no. 6. Observe that $u_n \leq h$ and

$$u_n = h + \frac{1}{2\pi}\int_{R_n} G_n(\cdot\,, \zeta)u_n(\zeta)P_\varphi(\zeta)d\xi d\eta\,,$$

and by (8)

$$\begin{aligned} D_{R_n}(u_n) &= D_{R_n}(h) + \frac{1}{2\pi}\int_{R_n \times R_n} G_n(z, \zeta)u_n(z)u_n(\zeta)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\eta \\ &\leq D(h) + \frac{1}{2\pi}\int_{R \times R} G(z, \zeta)h(z)h(\zeta)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\eta\,. \end{aligned}$$

By the symmetry of the measure $G(z, \zeta)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\eta$ and by the Schwarz inequality we deduce

$$\begin{aligned} &\left(\int_{R \times R} G(z, \zeta)h(z)h(\zeta)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\zeta\right)^2 \\ &\qquad \leq \left(\int_{R \times R} G(z, \zeta)h^2(z)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\eta\right)^2. \end{aligned}$$

If we denote by a the term on the right, then

$$\begin{aligned} a &= \int_R \left(\int_R G(z, \zeta)P_\varphi(\zeta)d\xi d\zeta\right) h^2(z)P_\varphi(z)dxdy \\ &= \int_R \left(\int_0^1 \left(\int_{C_\tau} G(z, \zeta)d\theta(\zeta)\right)\varphi(\tau)\tau d\tau\right) h^2(z)P_\varphi(z)dxdy \\ &= \int_R \left(\int_0^1 (-2\pi \max(\log \tau, \log r(z)))\varphi(\tau)\tau d\tau\right) h^2(z)\varphi(r(z))r(z)dr(z) \wedge d\theta(z) \\ &= \int_0^1 \left(\int_{C_\sigma}\left(\int_0^1 (-2\pi \max(\log \tau, \log \sigma))\varphi(\tau)\tau d\tau\right) h^2(z)\,d\theta(z)\right)\varphi(\sigma)\sigma d\sigma\,. \end{aligned}$$

By (13), there exists a constant K such that

$$\int_{C_\sigma} h^2(z)d\theta(z) \leq K$$

for every $\sigma \in (0, 1)$. Therefore

$$a \leq 2\pi K \int_0^1 \int_0^1 (-\max(\tau\sigma \log \tau, \tau\sigma \log \sigma))\varphi(\sigma)d\tau d\sigma$$

Since $-\tau\sigma \log \tau \leq -\tau \log \tau \leq 1 - \tau$ and similarly $-\tau\sigma \log \sigma \leq 1 - \sigma$, we have

$$\begin{aligned} -\max(\tau\sigma \log \tau, \tau\sigma \log \sigma) &= \min(-\tau\sigma \log \tau, -\tau\sigma \log \sigma) \\ &\leq \min(1 - \tau, 1 - \sigma) = 1 - \max(\tau, \sigma) \end{aligned}$$

and therefore

$$a \leq 2\pi K d(\varphi) .$$

This implies that

$$D_{R_n}(u_n) \leq D(h) + 2\pi K d(\varphi) < \infty .$$

Since $u_n \leq u_{n+1} \leq h$ for every n, $\{u_n\}$ converges to a $u \in P^+D(R)$. Observe that $d(\varphi) < \infty$ implies $b(\varphi) < \infty$. By the same proof as in no. 6 we see that $T_{P_\varphi} u = h$ and (β) follows. By (19) T_{P_φ} maps $P_\varphi BD(R)$ injectively to $HBD(R)$. For an arbitrary $h \in H^+BD(R)$ we can find $u \in P_\varphi^+ D(R)$ with $T_{P_\varphi} u = h$ if (β) is true. But $0 \leq u \leq h$ implies that $u \in P_\varphi^+ B(R)$ and therefore $u \in P_\varphi^+ BD(R)$, i.e. (β) implies (γ). The implication from (γ) to (δ) is trivially true.

The essential part of the proof now appears. Here the preparation in no. 8 plays a central role. Suppose that (δ) is the case. By (16) we can find a u in $P_\varphi D(R)$ such that $u > 0$ on R. Consider its mean $u^* \in P_\varphi^+ D(R)$ in no. 8. Since

$$D(u^*) = D(T_{P_\varphi} u^*) + D\left(\frac{1}{2\pi}\int_R G(\cdot, \zeta)u^*(\zeta)P_\varphi(\zeta)d\xi d\eta\right) ,$$

we deduce by (8) that

$$\begin{aligned} \infty &> \int_{R\times R} G(z, \zeta)u^*(z)u^*(\zeta)P_\varphi(z)P_\varphi(\zeta)dxdyd\xi d\eta \\ &\geq k^2 \int_0^1 \left(\int_{C_\sigma}\left(\int_0^1\left(\int_{C_\tau} G(z, \zeta)d\theta(\zeta)\right)\varphi(\tau)\tau d\tau\right)d\theta(z)\right)\varphi(\sigma)\sigma d\sigma . \end{aligned}$$

Here $k=\inf_R u^*>0$. By (12) we obtain

$$\infty > \int_0^1\left(\int_{C_\sigma}\left(\int_0^1(-2\pi\max(\log\tau,\log r(\zeta)))\varphi(\tau)\tau d\tau\right)d\theta(z)\right)\varphi(\sigma)\sigma d\sigma$$
$$= \int_0^1\int_0^1(-2\pi\max(\log\tau,\log\sigma))\varphi(\tau)\varphi(\sigma)\tau\sigma d\tau d\sigma .$$

Since $-\log t \geq (1-t)$,

$$-\max(\log\tau,\log\sigma)=\min(-\log\tau,-\log\sigma)$$
$$\geq \min(1-\tau,1-\sigma)=1-\max(\tau,\sigma)$$

and therefore

$$\int_0^1\int_0^1(1-\max(\tau,\sigma))\varphi(\tau)\varphi(\sigma)\tau\sigma d\tau d\sigma<\infty .$$

This implies $d(\varphi)<\infty$. Q.E.D.

Uniform Densities

10. To a density $P(z)dxdy$ on a hyperbolic Riemann surface R we associate the global function

$$\tilde{P}(z)=\frac{P(z)dxdy}{r(z)dr(z)\wedge d\theta(z)} \tag{33}$$

on R, which is nonnegative and Hölder continuous (locally) on R except possibly for the Green's singular set S (cf. no. 1) on which $\tilde{P}$ may take the infinite values continuously. Consider the equivalence class $[\varphi]$ of locally Hölder continuous nonnegative functions φ on $[0,1)$, where two functions φ_1 and φ_2 are equivalent if there exists a constant $c\geq 1$ and $\sigma\in(0,1)$ such that

$$c^{-1}\varphi_1\leq\varphi_2\leq c\varphi_1$$

on $(\sigma,1)$. Functions φ with

$$c^{-1}\varphi(r(z))\leq\tilde{P}(z)\leq c\varphi(r(z)) \tag{34}$$

for some constant $c=c_\varphi\geq 1$ on R except for a compact subset K_φ of R, if exist, constitute an equivalence class $[\varphi]$. In such a case we write

$$P(z)dxdy=P_{[\varphi]}(z)dxdy$$

and the density $P_{[\varphi]}(z)dxdy$ will be referred to as being *almost rotation free*. Rotation free densities are of course almost rotation free. We also call such a density as a *uniform density*. Observe that there exists a constant $c \geq 1$ and a compact subset K of R such that

$$c^{-1}P_\varphi(z) \leq P_{[\varphi]}(z) \leq cP_\varphi(z) \tag{35}$$

on $R - K$ for each $\varphi \in [\varphi]$. Set

$$\begin{cases} b[\varphi] = \inf_{\varphi \in [\varphi]} b(\varphi) = \inf_{\varphi \in [\varphi]} \int_0^1 (1-\tau)\varphi(\tau)d\tau\,; \\ d[\varphi] = \inf_{\varphi \in [\varphi]} d(\varphi) = \inf_{\varphi \in [\varphi]} \int_0^1\int_0^1 (1 - \max(\tau, \sigma))\varphi(\tau)\varphi(\sigma)d\tau d\sigma\,; \\ e[\varphi] = \inf_{\varphi \in [\varphi]} e(\varphi) = \inf_{\varphi \in [\varphi]} \int_0^1 \varphi(\tau)d\tau\,. \end{cases} \tag{36}$$

These are either 0 or ∞, and

$$0 \leq b[\varphi] \leq d[\varphi] \leq e[\varphi] \leq \infty\,. \tag{37}$$

11. First we prove that either $P_{[\varphi]}B(R)$ is $\{0\}$ or isomorphic to $HB(R)$ according as $b[\varphi] = \infty$ or $b[\varphi] = 0$ (Recall that $b[\varphi]$ is either 0 or ∞).

THEOREM. *The following three conditions are equivalent by pairs for uniform densities $P_{[\varphi]}$ on hyperbolic Riemann surfaces R:*

(α) $b[\varphi] = 0$;

(β) *$P_{[\varphi]}B(R)$ is canonically isomorphic to $HB(R)$;*

(γ) $(R, P_{[\varphi]}) \notin O_B$.

Proof. Let $\varphi \in [\varphi]$. Then $b[\varphi] = 0$ is equivalent to $b(c\varphi) < \infty$ for every $c > 0$. By Proposition 6, $T^B_{cP,P}$ exists for every $c > 0$ and therefore, by (20) in no. 4, (35) implies that $P_{[\varphi]}B(R)$ is canonically isomorphic to $P_\varphi B(R)$. Here observe that $cP_\varphi = P_{c\varphi}$. In particular $(R, P_{[\varphi]})$ and (R, P_φ) simultaneously belong to or do not belong to O_B. Again by Proposition 6 we deduce the required conclusion. Q.E.D.

12. We turn to the proof for that either $P_{[\varphi]}D(R)$ is $\{0\}$ or isomorphic to $HD(R)$ according as $d[\varphi] = \infty$ or $d[\varphi] = 0$ (Recall that $d[\varphi]$ is either 0 or ∞). Namely,

THEOREM. *The following four conditions are equivalent by pairs for uniform densities $P_{[\varphi]}$ on hyperbolic Riemann surfaces R:*

(α) $d[\varphi] = 0$;

(β) *$P_{[\varphi]}D(R)$ is canonically isomorphic to $HD(R)$;*

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(γ) *$P_{[\varphi]}BD(R)$ is canonically isomorphic to $HBD(R)$;*

(δ) $(R, P_{[\varphi]}) \notin O_D = O_{BD}$.

Proof. Fix a $\varphi \in [\varphi]$. Then $d[\varphi] = 0$ is equivalent to $d(c\varphi) < \infty$ for every $c > 0$. By Proposition 9, $T^{D}_{cP,P}$ and $T^{BD}_{cP,P}$ exist for every $c > 0$ and therefore, by (20) in no. 4, (35) implies that $P_{[\varphi]}D(R)$ ($P_{[\varphi]}BD(R)$, resp.) is canonically isomorphic to $P_\varphi D(R)(P_\varphi BD(R)$, resp.). In particular $(R, P_{[\varphi]})$ belongs to $O_D = O_{BD}$ if and only if (R, P_φ) belongs to $O_D = O_{BD}$. Once again by Proposition 9, we see that pairwise equivalences of (α)–(δ). Q.E.D.

13. Finally we prove that either $P_{[\varphi]}E(R)$ is $\{0\}$ or isomorphic to $HD(R)$ according as $e[\varphi] = \infty$ or 0 (Recall that $e[\varphi]$ is either 0 or ∞). We claim:

THEOREM. *The following four conditions are equivalent by pairs for uniform densities $P_{[\varphi]}$ on hyperbolic Riemann surfaces R:*

(α) $e[\varphi] = 0$;

(β) *$P_{[\varphi]}E(R)$ is canonically isomorphic to $HD(R)$;*

(γ) *$P_{[\varphi]}BE(R)$ is canonically isomorphic to $HBD(R)$;*

(δ) $(R, P_{[\varphi]}) \notin O_E = O_{BE}$.

Proof. Choose a $\varphi \in [\varphi]$. Then $e[\varphi] = 0$ is equivalent to $e(c\varphi) < \infty$ for every $c > 0$. By Proposition 7, $T^{E}_{cP,P}$ and $T^{BE}_{cP,P}$ exist for every $c > 0$ and therefore, by (20) in no. 4, (35) implies that $P_{[\varphi]}E(R)$ ($P_{[\varphi]}BE(R)$, resp.) is canonically isomorphic to $P_\varphi E(R)$ ($P_\varphi BE(R)$, resp.). In particular $(R, P_{[\varphi]})$ and (R, P_φ) together belong to or do not belong to $O_E = O_{BE}$. Therefore, by Proposition 7, we deduce the required equivalences. Q.E.D.

14. We denote by $P^{(\alpha)}(z)dxdy$ the density $P_{[\varphi]}(z)dxdy$ such that $\varphi_\alpha(\tau) = (1-\tau)^{-\alpha} \in [\varphi](\alpha \in (-\infty, \infty))$. Observe that

$$b(\varphi_\alpha) = \int_0^1 (1-\tau)^{1-\alpha} d\tau = \infty$$

if and only if $\alpha \geq 2$; if $\alpha \geq 2$, then $d(\varphi_\alpha) = \infty$, and if $\alpha < 2$, then

$$d(\varphi_\alpha) = 2\int_{\tau > \sigma} (1 - \max(\tau, \sigma))\varphi_\alpha(\tau)\varphi_\alpha(\sigma) d\tau d\sigma$$
$$= 2\int_0^1 \left(\int_\sigma^1 (1-\tau)\varphi_\alpha(\tau) d\tau\right) \varphi_\alpha(\sigma) d\sigma$$

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$$= 2\int_0^1 \left(\int_\sigma^1 (1-\tau)^{1-\alpha} d\tau\right)(1-\sigma)^{-\alpha} d\sigma$$
$$= \frac{2}{2-\alpha}\int_0^1 (1-\sigma)^{2-2\alpha} d\sigma = \infty$$

if and only if $\alpha \geq 3/2$: $d(\varphi_\alpha) = \infty$ if and only if $\alpha \geq 3/2$;

$$e(\varphi_\alpha) = \int_0^1 (1-\tau)^{-\alpha} d\tau = \infty$$

if and only if $\alpha \geq 1$. Hence we obtain

$$(38)\qquad \begin{cases} b[\varphi_\alpha] = 0 \quad (\alpha \in (-\infty, 2)), & b[\varphi_\alpha] = \infty \quad (\alpha \in [2, \infty)); \\ d[\varphi_\alpha] = 0 \quad (\alpha \in (-\infty, 3/2)), & d[\varphi_\alpha] = \infty \quad (\alpha \in [3/2, \infty)); \\ e[\varphi_\alpha] = 0 \quad (\alpha \in (-\infty, 1)), & e[\varphi_\alpha] = \infty \quad (\alpha \in [1, \infty)). \end{cases}$$

Therefore, for any fixed hyperbolic Riemann surface R, every degree of degeneracies in the classification scheme (cf. e.g. [14])

$$(39)\qquad O_G < O_B < O_D = O_{BD} < O_E = O_{BE}$$

can occur with respect to the equation $\Delta u(z) = P^{(\alpha)}(z)u(z)$ on R:

THEOREM. *The pair $(R, P^{(\alpha)})$ of an arbitrary hyperbolic Riemann surface R and a special uniform density $P^{(\alpha)} = P_{[(1-\tau)^{-\alpha}]}(\alpha \in (-\infty, \infty))$ satisfies the following relations*:

$$(40)\qquad \begin{cases} (R, P^{(\alpha)}) \in O_B - O_G & (\alpha \in [2, \infty)); \\ (R, P^{(\alpha)}) \in O_D - O_B & (\alpha \in [3/2, 2)); \\ (R, P^{(\alpha)}) \in O_E - O_D & (\alpha \in [1, 3/2)); \\ (R, P^{(\alpha)}) \notin O_E & (\alpha \in (-\infty, 1)). \end{cases}$$

The proof is immediate if we use Propositions in nos. 11–13 and (38). This also proves our main theorem stated in the introduction. This theorem was originally obtained in the special case of R of the unit disk $|z| < 1$ by the present author [17]. The first relation in (40) can also be regarded as a generalization of a classical result of Brelot [1] obtained for plane regions R. Uniform densities we have discussed in the present paper are those distributed almost uniformly in the θ-direction with respect to r. Another possible uniform densities are those distributed almost uniformly in the r-direction with respect to θ. An analogous result as in this paper can be expected for the latter kind of uniform densities, which we shall discuss later elsewhere.

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