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THE ρ-VARIATION OF THE HEAT SEMIGROUP IN THE HERMITIAN SETTING: BEHAVIOUR IN L^∞

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Abstract Let $\mathcal{V}_\rho(\mathrm{e}^{-tH})$, $\rho > 2$, be the ρ-variation of the heat semigroup associated to the harmonic oscillator $H = \frac{1}{2}(-\Delta + |x|^2)$. We show that if $f \in L^\infty(\mathbb{R})$, then $\mathcal{V}_\rho(\mathrm{e}^{-tH})(f)(x) < \infty$, a.e. $x \in \mathbb{R}$. However, we find a function $G \in L^\infty(\mathbb{R})$, such that $\mathcal{V}_\rho(\mathrm{e}^{-tH})(G)(x) \notin L^\infty(\mathbb{R})$. We also analyse the local behaviour in L^∞ of the operator $\mathcal{V}_\rho(\mathrm{e}^{-tH})$. We find that its growth is smaller than that of a standard singular integral operator. As a by-product of our work we obtain an $L^\infty(\mathbb{R})$ function F, such that the square function

$$\left(\int_0^\infty t\left|\frac{\partial}{\partial t}P_t^\Delta F(x)\right|^2 \mathrm{d}t\right)^{1/2} = \infty,$$

a.e. $x \in \mathbb{R}$, where P_t^Δ is the classical Poisson kernel in $\mathbb{R}$.

Keywords: Hermite operator; ρ-variation; heat semigroup; Poisson semigroup

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1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{T_t\}_{t>0}$ be a uniparameter family of linear operators bounded in $L^p(X)$ for some $p \in [1, \infty]$. In recent years, a relatively large number of papers have been devoted to analysing the boundedness properties of the 'ρ-variation operator', $\mathcal{V}_\rho(T_t)$, $\rho > 2$, defined by

$$\mathcal{V}_\rho(T_t)(f)(x) = \sup_{t_j \searrow 0}\left(\sum_{j=1}^\infty |T_{t_j} f(x) - T_{t_{j+1}} f(x)|^\rho\right)^{1/\rho},$$

where the supremum is taken over all the sequences of real numbers $\{t_j\}_{j=1}^\infty$ that decrease to zero. The ρ-variation can be considered a kind of measure of the speed of convergence

$\lim_{t_j \searrow 0} T_{t_j} f(x)$. It is also related with some estimates of the jumps of the family $\{T_t f(x)\}$ [**5**]. Moreover, as it is clearly bigger than the maximal operator, $\sup_{t_j} |T_{t_j} f(x)|$, any boundedness of the ρ-variation can be considered an improvement of the corresponding boundedness for the maximal operator. Thus, [**1**] can be considered an improvement of the boundedness of the maximal Hilbert transform, while [**6**] is a sharpening of the Carleson maximal theorem.

Consider the Hermite differential operator

$$H = \tfrac{1}{2}(-\Delta + |x|^2), \quad x \in \mathbb{R}^n. \tag{1.1}$$

Let $W_t^H := \mathrm{e}^{-tH}$ be its heat semigroup. It is known that the operator $\mathcal{V}_\rho(W_t^H)$, $\rho > 2$, maps $L^p(\mathbb{R}^n)$, $1 < p < \infty$, into itself; it is of weak type $(1,1)$ and maps $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$ [**2**]. The aim of this paper is to discuss the behaviour of $\mathcal{V}_\rho(W_t^H)$ in L^∞.

The behaviour of $\mathcal{V}_\rho(W_t^H)$ appears rather interesting at this point, as parts (i) and (ii) of Theorem 1.1 will show. In fact, for any L^∞-function f, $\mathcal{V}_\rho(W_t^H)(f)(x)$ is finite a.e. $x \in \mathbb{R}$. However, there exists an L^∞-function G such that $\mathcal{V}_\rho(W_t^H)(G) \notin L^\infty(\mathbb{R})$. This is in contrast with the behaviour of the operator $\mathcal{V}_\rho(\mathrm{e}^{t\Delta})$, since it is known that there exists $f \in L^\infty(\mathbb{R})$ such that $\mathcal{V}_\rho(\mathrm{e}^{t\Delta})(f)(x) = \infty$ a.e. x (see [**2**] for details).

We recall a classical result about singular integrals, namely that the growth of the image by a singular integral (Riesz transform) of a general L^∞-function is of logarithmic order. We present an example (see Proposition 3.1) that shows that this is also true in the Hermitian case. The 'Riesz transform' in our case is the operator $\partial(H)^{-1/2}$. However, the growth of the image of $L^\infty(\mathbb{R})$ by the operator $\mathcal{V}_\rho(W_t^H)$ will be of a $1/\rho$ power of the logarithm (see Theorem 1.1 (iii)).

As a by-product of our research, we found the following result of independent interest. Let

$$P_t^\Delta(x) = \frac{1}{\pi}\frac{t}{t^2 + x^2}, \quad t > 0,\ x \in \mathbb{R},$$

be the classical Poisson semigroup. Consider the square function

$$g_1(f)(x) = \biggl(\int_0^\infty t\biggl|\frac{\partial}{\partial t} P_t^\Delta f(x)\biggr|^2 \,\mathrm{d}t\biggr)^{1/2}. \tag{1.2}$$

There exists a bounded function f such that $g_1(f)(x) = \infty$ for almost all $x \in \mathbb{R}$. Moreover, there exists a bounded function with compact support F such that $g_1(F)(x) < \infty$ but $g_1(F) \notin L^\infty(\mathbb{R})$. Wang [**9**] proved a similar result for the square function

$$\biggl(\int_0^\infty t\biggl|\frac{\partial}{\partial x} P_t^\Delta f(x)\biggr|^2 \,\mathrm{d}t\biggr)^{1/2}.$$

Wang [**10**] also proved that if $f \in \mathrm{BMO}$ and $g_1(f)(x) < \infty$ a.e. x, then $g_1(f) \in \mathrm{BMO}$.

Some results for the case of the heat semigroup associated to the Euclidean Laplacian can be found in [**1**, **3**]. On the other hand, for Markovian semigroups, some results of boundedness in L^p can be found in [**4**].

We now present our results.

Theorem 1.1. *Let $\rho > 2$.*

(i) $\mathcal{V}_\rho(W_t^H)(f)(x) < \infty$ *a.e.* $x \in \mathbb{R}$ *for every* $f \in L^\infty(\mathbb{R})$.

(ii) *There exists a function* $G \in L^\infty(\mathbb{R})$ *such that* $\mathcal{V}_\rho(W_t^H)(G) \notin L^\infty(\mathbb{R})$.

(iii) *There exists a constant* $C > 0$ *such that, for every function* $f \in L^\infty(\mathbb{R})$ *with support contained in* $[-1, 1]$*, and each ball* $B_r = B(x_0, r) \subset [-1, 1]$*, we have*

$$\frac{1}{r}\int_{B_r} \mathcal{V}_\rho(W_t^H)(f)(x)\,\mathrm{d}x \leqslant C\biggl(\log\frac{1}{2r}\biggr)^{1/\rho}\|f\|_\infty.$$

Theorem 1.2. *Let g_1 the square function defined in (1.2). Then*

(i) *there exists a function* $f \in L^\infty(\mathbb{R})$ *such that* $g_1(f)(x) = \infty$ *a.e.* $x \in \mathbb{R}$,

(ii) *there exists a compactly supported function* $F \in L^\infty(\mathbb{R})$ *such that* $g_1(F)(x) < \infty$ *a.e.* x*, but* $g_1(F) \notin L^\infty(\mathbb{R})$.

Throughout this paper, by C we always denote a positive constant that can change from line to line. We will make frequent use, sometimes without explicitly mentioning it, of the fact that, for every positive constant C and non-negative constant c,

$$\sup_{t>0} t^c \mathrm{e}^{-Ct} < \infty. \tag{1.3}$$

2. L^∞ results

According to Mehler's Formula [**8**, (1.1.47)], the heat semigroup $\{W_t^H\}_{t>0}$ associated with H admits, for every $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, and $t > 0$, the following integral representation:

$$W_t^H(f)(x) = \int_{-\infty}^{\infty} W_t^H(x, y) f(y)\,\mathrm{d}y,$$

where

$$W_t^H(x, y) = \biggl(\frac{\mathrm{e}^{-t}}{\sqrt{\pi}(1 - \mathrm{e}^{-2t})}\biggr)^{1/2} \exp\biggl(-\frac{1}{2}\frac{1 + \mathrm{e}^{-2t}}{1 - \mathrm{e}^{-2t}}(x^2 + y^2) + \frac{2\mathrm{e}^{-t}}{1 - \mathrm{e}^{-2t}}xy\biggr), \qquad t > 0,\ x, y \in \mathbb{R}.$$

By making the change of variable (due to Meda)

$$t = t(s) = \log\frac{1 + s}{1 - s}, \quad 0 < s < 1,\ t > 0,$$

we obtain

$$W_{t(s)}^H(x, y) = \biggl(\frac{1 - s^2}{\sqrt{\pi}4s}\biggr)^{1/2} \exp\biggl(-\frac{1}{4}\biggl(\frac{|x - y|^2}{s} + s|x + y|^2\biggr)\biggr), \quad s \in (0, 1). \tag{2.1}$$

The semigroup $\{W_t^H\}_{t>0}$ is contractive in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and self-adjoint in $L^2(\mathbb{R})$.

Remark 2.1. If we perform the last change of parameter $t \to s$, then for every function f and every x we have

$$\sup_{t_j<\infty, t_j \searrow 0} \left(\sum_{j=1}^{\infty} |W_{t_j}^H(f)(x) - W_{t_{j+1}}^H(f)(x)|^\rho \right)^{1/\rho} = \sup_{s_j<1, s_j \searrow 0} \left(\sum_{j=1}^{\infty} |W_{t(s_j)}^H(f)(x) - W_{t(s_{j+1})}^H(f)(x)|^\rho \right)^{1/\rho}.$$

In other words, dealing with $\mathcal{V}_\rho(W_t^H)$, we can use the expression of the kernel given by (2.1).

In [**2**] Crescimbeni *et al.* studied the kernel of $\mathcal{V}_\rho(W_t^H)$. They showed that the singularities at 0 and at ∞ are at most as in the case of singular integral kernels. Nevertheless, the following result shows that the behaviour of the kernel is better than in the case of singular integral operators.

Lemma 2.2. *There exists a positive constant C such that for all $x, y \in \mathbb{R}$ we have*

$$\sup_{s_j<1, s_j \searrow 0} \sum_{j=1}^{\infty} |W_{t(s_j)}^H(x,y) - W_{t(s_{j+1})}^H(x,y)| \leqslant C \frac{\mathrm{e}^{-|x-y|^2/8}}{|x-y|} \mathrm{e}^{-|y|\,|x-y|/8}.$$

Proof. We have

$$\begin{aligned}
\left| \frac{\partial W_{t(s)}^H}{\partial s}(x,y) \right| &\leqslant C \left\{ \frac{1}{1-s} + \frac{1}{s} + \frac{|x-y|^2}{s^2} + \frac{|x+y|^2}{4} \right\} \\
&\quad \times \left(\frac{1-s^2}{4s} \right)^{1/2} \exp\left(-\frac{1}{4} \left(\frac{|x-y|^2}{s} + s|x+y|^2 \right) \right) \\
&\leqslant C \left\{ \frac{1}{1-s} + \frac{1}{s} \right\} \left(\frac{1-s^2}{4s} \right)^{1/2} \exp\left(-\frac{1}{8} \left(\frac{|x-y|^2}{s} + s|x+y|^2 \right) \right) \\
&\leqslant C \left\{ \frac{1}{(1-s)^{1/2}} + \frac{1}{s^{3/2}} \right\} \exp\left(\frac{-|x-y|^2}{8s} \right) \exp\left(-\frac{|y|\,|x-y|}{8} \right), \\
&\qquad x, y \in \mathbb{R},\ s \in (0,1),
\end{aligned}$$

where in the last inequality we have used the fact that for $x, y \in \mathbb{R}$, $xy \geqslant 0$ and $s \in (0,1)$ we have

$$\begin{aligned}
\exp\left(-\frac{1}{4} \left[\frac{1}{s}|x-y|^2 + s|x+y|^2 \right] \right) &\leqslant \exp\left(-\frac{|x-y|^2}{8s} \right) \exp\left(-\frac{1}{8} \left[\frac{1}{s}|x-y|^2 + s|x+y|^2 \right] \right) \\
&\leqslant \exp\left(-\frac{|x-y|^2}{8s} \right) \exp(-\tfrac{1}{8}|x-y|\,|x+y|) \\
&\leqslant \exp\left(-\frac{|x-y|^2}{8s} \right) \exp(-\tfrac{1}{8}|y|\,|x-y|),
\end{aligned}$$

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while for $x, y \in \mathbb{R}$, $xy \leqslant 0$ and $s \in (0,1)$ we have

$$\begin{aligned}\exp\left(-\frac{1}{4}\left[\frac{1}{s}|x-y|^2 + s|x+y|^2\right]\right) &\leqslant \exp\left(-\frac{|x-y|^2}{8s}\right)\exp\left(-\frac{|x-y|^2}{8s}\right)\\ &\leqslant \exp\left(-\frac{|x-y|^2}{8s}\right)\exp\left(-\frac{|x-y|^2}{8}\right)\\ &\leqslant \exp\left(-\frac{|x-y|^2}{8s}\right)\exp\left(-\frac{|y|\,|x-y|}{8}\right).\end{aligned}$$

Hence, if $\{s_j\}_{j\in\mathbb{N}} \subset (0,1)$ is decreasing and $\lim_{j\to\infty} s_j = 0$, then

$$\begin{aligned}&\sum_{j=1}^{\infty} |W^H_{t(s_j)}(x,y) - W^H_{t(s_{j+1})}(x,y)|\\ &\leqslant C\mathrm{e}^{-(|y|\,|x-y|)/8}\int_0^1 \left((1-s)^{-1/2} + \frac{1}{s^{3/2}}\right)\exp\left(-\frac{1}{8}\frac{|x-y|^2}{s}\right)\mathrm{d}s\\ &\leqslant C\mathrm{e}^{-(|y|\,|x-y|)/8}\left(\int_0^{1/2}\frac{1}{s^{1/2}}\exp\left(-\frac{1}{8}\frac{|x-y|^2}{s}\right)\frac{\mathrm{d}s}{s} + \int_{1/2}^1 (1-s)^{-1/2}\mathrm{e}^{-(|x-y|^2)/8}\,\mathrm{d}s\right)\\ &\leqslant C\mathrm{e}^{-(|y|\,|x-y|)/8}\left(\frac{1}{|x-y|}\int_{2|x-y|^2}^{\infty} u^{1/2}\mathrm{e}^{-u}\frac{\mathrm{d}u}{u} + \mathrm{e}^{-(|x-y|^2)/8}\right), \quad x, y \in \mathbb{R}.\end{aligned}$$

This gives the lemma. □

Proof of Theorem 1.1 (i). Let $x \in [-1,1]$. Then by using Lemma 2.2 we have

$$\begin{aligned}\mathcal{V}_\rho(W^H_t)(f)(x) &\leqslant \mathcal{V}_\rho(W^H_t)(f\chi_{[-2,2]})(x) + \mathcal{V}_\rho(W^H_t)(f\chi_{[-2,2]^c})(x)\\ &\leqslant \mathcal{V}_\rho(W^H_t)(f\chi_{[-2,2]})(x) + C\int_{|x-y|>1} \mathrm{e}^{-(|x-y|^2)/8}|f(y)|\,\mathrm{d}y.\end{aligned}$$

$\mathcal{V}_\rho(W^H_t)$ is an operator bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$ [**2**]. Hence, as $f\chi_{[-2,2]}$ belongs to $L^p(\mathbb{R})$ for all $p > 1$, the first term of the last sum has to be finite a.e. $x \in \mathbb{R}$. While the second term is finite for all $x \in \mathbb{R}$. The obvious scaling argument gives the result for a.e. x, and then we have proved (i) in Theorem 1.1. □

Remark (notation). In this paper, we make some comparisons with the kernel of the classical heat semigroup $\mathrm{e}^{t\Delta}$. We shall denote the kernel by W^Δ_t and the corresponding ρ-variation by $\mathcal{V}_\rho(W^\Delta_t)$.

Lemma 2.3. *There exists a function $G \in L^\infty(\mathbb{R})$ with support in $[0,1]$ and a decreasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset (0,1)$ convergent to zero such that*

$$\sum_{j=1}^{\infty} |W^\Delta_{s_{j+1}}(G)(x) - W^\Delta_{s_j}(G)(x)|^r \notin L^\infty([0,1]) \quad \textit{for every } r > 1.$$

Proof. Let $a > 1$. We define the functions

$$g = \sum_{k\in\mathbb{Z}} (-1)^{k+1} \chi_{[a^k, a^{k+1})} \quad \text{and} \quad G = \sum_{k=-\infty}^{-1} (-1)^{k+1} \chi_{[a^k, a^{k+1})}.$$

It is clear that $G = g\chi_{[0,1)}$. Moreover, for every $j \in \mathbb{Z}$ and $y \in \mathbb{R}$, we have $G(a^j y) = (-1)^j g(y)\chi_{[0,a^{-j})}(y)$. Therefore, for each $j \in \mathbb{Z}$, we can write

$$\begin{aligned}\frac{1}{a^j}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{2j}} G(y)\,\mathrm{d}y &= \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} G(a^j u)\,\mathrm{d}u \\ &= (-1)^j \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j})}(u)\,\mathrm{d}u.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{a^j}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{2j}} G(y)\,\mathrm{d}y - \frac{1}{a^{j+1}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{2(j+1)}} G(y)\,\mathrm{d}y \\ &\quad = (-1)^j \biggl(\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j})}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j-1})}(u)\,\mathrm{d}u\biggr) \\ &\quad = (-1)^j \biggl(2\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j-1})}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[a^{-j-1},a^{-j})}(u)\,\mathrm{d}u\biggr)\end{aligned}$$

for $j \in \mathbb{Z}$. Observe that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j-1})}(u)\,\mathrm{d}u \to \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\,\mathrm{d}u \quad \text{as } j \to -\infty$$

and

$$\biggl|\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[a^{-j-1},a^{-j})}(u)\,\mathrm{d}u\biggr| \leqslant \mathrm{e}^{-1/(8a^{2(j+1)})} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/8}\,\mathrm{d}u \to 0 \quad \text{as } j \to -\infty.$$

Hence, if we choose a such that [**2**]

$$\biggl|\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\,\mathrm{d}u\biggr| > 0,$$

there exist $C > 0$ and $j_0 \in \mathbb{N}$ such that

$$\biggl|\frac{1}{a^j}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{2j}} G(y)\,\mathrm{d}y - \frac{1}{a^{j+1}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{2(j+1)}} G(y)\,\mathrm{d}y\biggr| \geqslant C, \quad j \leqslant -j_0. \tag{2.2}$$

Consequently, we can ensure that, given $r > 1$ and any positive constant $M > 0$, there exists a finite subset J of $\mathbb{N}$ such that

$$\begin{aligned}&\sum_{j\in J}\biggl|\frac{1}{a^{-(j+1)}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/(4a^{-2(j+1)})} G(y)\,\mathrm{d}y - \frac{1}{a^{-j}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/4a^{-2j}} G(y)\,\mathrm{d}y\biggr|^r \\ &\qquad = \sum_{j\in J}\biggl|\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-(j+1)})}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{-j})}(u)\,\mathrm{d}u\biggr|^r \\ &\qquad \geqslant M.\end{aligned}$$

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On the other hand, the obvious changes of variables give, for every $x \in \mathbb{R}$,

$$\begin{aligned}\left|\frac{1}{a^{-(j+1)}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|^2/4a^{-2(j+1)}} G(y)\,\mathrm{d}y - \frac{1}{a^{-j}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|^2/4a^{-2j}} G(y)\,\mathrm{d}y\right| \\ = \left|\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u + a^{-(j+1)}x)\chi_{[0,a^{-(j+1)})}(u + a^{-(j+1)}x)\,\mathrm{d}u \right. \\ \left. + \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u + a^{-j}x)\chi_{[0,a^{-j})}(u + a^{-j}x)\,\mathrm{d}u\right|.\end{aligned}$$

We observe that

$$\lim_{h\to 0}\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u+h)\chi_{[0,A)}(u+h)\,\mathrm{d}u = \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,A)}(u)\,\mathrm{d}u,$$

where the convergence is uniform in $A \in (0,\infty)$. Hence, there exists $\eta > 0$ such that if $|x| < \eta a^{-(j+1)}$, $j \in J$,

$$\begin{aligned}\sum_{j\in J}\left|\frac{1}{a^{-(j+1)}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|^2/4a^{-2(j+1)}} G(y)\,\mathrm{d}y - \frac{1}{a^{-j}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|^2/4a^{-2j}} G(y)\,\mathrm{d}y\right|^r \\ \geqslant \frac{1}{2}\sum_{j\in J}\left|\int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{j+1})}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/4} g(u)\chi_{[0,a^{j})}(u)\,\mathrm{d}u\right|^r \\ \geqslant \tfrac{1}{2}M.\end{aligned}$$

As $M > 0$ is arbitrary we obtain the lemma. □

The following corollary is an obvious consequence of the above lemma.

Corollary 2.4. *There exists a function $f \in L^\infty(\mathbb{R})$ with support in $[0,1]$ such that $\mathcal{V}_\rho(W_t^\Delta)(f) \notin L^\infty([0,1])$, for every $\rho > 2$.*

The next lemma shows a pointwise relation between the variations of the heat semigroups of the classical Laplace operator and Hermite operator.

Lemma 2.5. *Let $\rho > 2$ and let f be an $L^\infty(\mathbb{R})$ function with support in $[-1,1]$. There exists $C > 0$ such that, for every decreasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset (0,1]$ that converges to zero,*

$$\left(\sum_{j=1}^{\infty} |W_{s_j}^\Delta(f)(x) - W_{t(s_j)}^H(f)(x) - (W_{s_{j+1}}^\Delta(f)(x) - W_{t(s_{j+1})}^H(f)(x))|^\rho\right)^{1/\rho} \leqslant C\|f\|_\infty,$$

for $x \in [-1,1]$.

Proof. Assume that $\{s_j\}_{j\in\mathbb{N}} \subset (0,1]$ is a decreasing sequence that converges to zero. Then

$$\begin{aligned}
&W^{\Delta}_{s_j}(f)(x) - W^{H}_{t(s_j)}(f)(x) - (W^{\Delta}_{s_{j+1}}(f)(x) - W^{H}_{t(s_{j+1})}(f)(x)) \\
&= \frac{1}{2\sqrt{\pi}}\biggl[\biggl(\frac{1}{\sqrt{s_j}}\int_{-\infty}^{\infty}\biggl(\exp\biggl(\frac{-|x-y|^2}{4s_j}\biggr) - \exp\biggl(-\frac{1}{4}\biggl(s_j|x+y|^2 + \frac{|x-y|^2}{s_j}\biggr)\biggr)\biggr) f(y)\,\mathrm{d}y \\
&\qquad - \frac{1}{\sqrt{s_{j+1}}}\int_{-\infty}^{\infty}\biggl(\exp\biggl(-\frac{|x-y|^2}{4s_{j+1}}\biggr) \\
&\qquad\qquad - \exp\biggl(-\frac{1}{4}\biggl(s_{j+1}|x+y|^2 + \frac{|x-y|^2}{s_{j+1}}\biggr)\biggr)\biggr) f(y)\,\mathrm{d}y\biggr) \\
&\quad + ((1-s_{j+1}^2)^{1/2} - (1-s_j^2)^{1/2}) \\
&\quad\times \frac{1}{\sqrt{s_j}}\int_{-\infty}^{\infty}\exp\biggl(-\frac{1}{4}\biggl(s_j|x+y|^2 + \frac{|x-y|^2}{s_j}\biggr)\biggr) f(y)\,\mathrm{d}y \\
&\quad + (1-(1-s_{j+1}^2)^{1/2}) \\
&\quad\times \int_{-\infty}^{\infty}\biggl(\frac{1}{\sqrt{s_j}}\exp\biggl(-\frac{1}{4}\biggl(s_j|x+y|^2 + \frac{|x-y|^2}{s_j}\biggr)\biggr) \\
&\qquad\qquad - \frac{1}{\sqrt{s_{j+1}}}\exp\biggl(-\frac{1}{4}\biggl(s_{j+1}|x+y|^2 + \frac{|x-y|^2}{s_{j+1}}\biggr)\biggr)\biggr) f(y)\,\mathrm{d}y\biggr] \\
&= B_{1,j}(x) + B_{2,j}(x) + B_{3,j}(x), \quad x\in\mathbb{R},\ j\in\mathbb{N}.
\end{aligned}$$

In order to estimate $B_{1,j}(x)$, $j \in \mathbb{N}$, we consider the function

$$\varphi(s;x,y) = \frac{1}{\sqrt{s}}\biggl(\mathrm{e}^{-|x-y|^2/(4s)} - \exp\biggl(-\frac{1}{4}\biggl(s|x+y|^2 + \frac{|x-y|^2}{s}\biggr)\biggr)\biggr), \quad s>0,\ x,y\in[-1,1].$$

We can write

$$B_{1,j}(x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{t_{j+1}}^{t_j}\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s;x,y)\,\mathrm{d}s\, f(y)\,\mathrm{d}y, \quad x\in[-1,1].$$

Since $1-\exp(-\frac{1}{4}s|x+y|^2) \leqslant C|x+y|^2 s \leqslant Cs$, $s\in(0,1)$, $x,y\in[-1,1]$, a straightforward manipulation leads to

$$\begin{aligned}
\biggl|\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s;x,y)\biggr| &\leqslant C\biggl(s^{-3/2}\exp\biggl(-\frac{|x-y|^2}{8s}\biggr)\biggl(1-\exp\biggl(-\frac{1}{4}s|x+y|^2\biggr)\biggr) \\
&\qquad + |x+y|^2 s^{-1/2}\exp\biggl(-\frac{1}{4}\biggl(s|x+y|^2 + \frac{|x-y|^2}{s}\biggr)\biggr)\biggr) \\
&\leqslant C\biggl(s^{-3/2}\exp\biggl(-\frac{|x-y|^2}{8s}\biggr)s + |x+y|^2 s^{-1/2}\exp\biggl(-\frac{1}{4}\frac{|x-y|^2}{s}\biggr)\biggr) \\
&\leqslant Cs^{-1/2}\exp\biggl(-\frac{|x-y|^2}{8s}\biggr).
\end{aligned}$$

Hence, we get

$$\sum_{j=1}^{\infty}|B_{1,j}(x)| \leqslant C\int_{-1}^{1}|f(y)|\int_{0}^{1}s^{-1/2}\exp\left(-\frac{|x-y|^2}{8s}\right)\mathrm{d}s\,\mathrm{d}y \\ \leqslant C\|f\|_\infty, \quad x\in[-1,1].$$

For $B_{2,j}(x)$, $j\in\mathbb{N}$ and $x\in[-1,1]$, we have

$$|B_{2,j}(x)| \leqslant \frac{C(s_j^2-s_{j+1}^2)}{s_j^{1/2}[(1-s_{j+1}^2)^{1/2}+(1-s_j^2)^{1/2}]}\int_{-\infty}^{\infty}\exp\left(-\frac{|x-y|^2}{4s_j}\right)\mathrm{d}y\|f\|_\infty \\ \leqslant C(s_j-s_{j+1})\|f\|_\infty.$$

Then,

$$\sum_{j=1}^{\infty}|B_{2,j}(x)| \leqslant C\|f\|_\infty\sum_{j=1}^{\infty}(s_j-s_{j+1}) \leqslant C\|f\|_\infty, \quad x\in[-1,1].$$

In order to estimate $B_{3,j}(x)$, $j\in\mathbb{N}$, we define the function

$$\phi(s;x,y)=\frac{1}{\sqrt{s}}\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{|x-y|^2}{s}\right)\right), \quad s>0,\ x,y\in[-1,1].$$

It is not hard to see that, for $j\in\mathbb{N}$, $s\in(t_{j+1},t_j)$, $x,y\in[-1,1]$.

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\phi(s;x,y)\right| \leqslant Cs^{-3/2}\exp\left(-\frac{1}{8}\left(s|x+y|^2+\frac{|x-y|^2}{s}\right)\right) \\ \leqslant Ct_{j+1}^{-3/2}\exp\left(-\frac{1}{8}\left(t_{j+1}|x+y|^2+\frac{|x-y|^2}{t_j}\right)\right),$$

Hence, for $j\in\mathbb{N}$ and $x,y\in[-1,1]$, the mean-value theorem implies

$$|B_{3,j}(x)| \leqslant C\frac{s_{j+1}^2(s_j-s_{j+1})}{1+(1-s_j^2)^{1/2}}\int_{-1}^{1}t_{j+1}^{-3/2}\exp\left(-\frac{1}{8}\left(s_{j+1}|x+y|^2+\frac{|x-y|^2}{s_j}\right)\right)\mathrm{d}y\|f\|_\infty \\ \leqslant C(s_j-s_{j+1})\|f\|_\infty.$$

Therefore, we obtain

$$\sum_{j=1}^{\infty}|B_{3,j}(x)| \leqslant C\|f\|_\infty, \quad x\in[-1,1].$$

By putting together the above estimations we conclude that

$$\left(\sum_{j=1}^{\infty}|W_{s_j}^{\Delta}(f)(x)-W_{t(s_j)}^{H}(f)(x)-(W_{s_{j+1}}^{\Delta}(f)(x)-W_{t(s_{j+1})}^{H}(f)(x))|^\rho\right)^{1/\rho} \\ \leqslant C\sum_{j=1}^{\infty}|W_{s_j}^{\Delta}(f)(x)-W_{t(s_j)}^{H}(f)(x)-(W_{s_{j+1}}^{\Delta}(f)(x)-W_{t(s_{j+1})}^{H}(f)(x))| \\ \leqslant C\|f\|_\infty, \quad x\in[-1,1].$$

□

Proof of Theorem 1.1 (ii). It is sufficient to combine Lemmas 2.3 and 2.5. Indeed, let $\{s_j\}_{j\in\mathbb{N}}$ be the sequence and let G be the function in Lemma 2.3. We can write

$$\begin{aligned}
&\biggl(\sum_{j=1}^{\infty}|W^H_{t(s_j)}(G)(x)-W^H_{t(s_{j+1})}(G)(x)|^\rho\biggr)^{1/\rho}\\
&\quad\geqslant\biggl(\sum_{j=1}^{\infty}|W^\Delta_{s_j}(G)(x)-W^\Delta_{s_{j+1}}(G)(x)|^\rho\biggr)^{1/\rho}\\
&\qquad-\biggl(\sum_{j=1}^{\infty}|W^\Delta_{s_j}(G)(x)-W^H_{t(s_j)}(G)(x)-(W^\Delta_{s_{j+1}}(G)(x)-W^H_{t(s_{j+1})}(G)(x))|^\rho\biggr)^{1/\rho}\\
&\quad\geqslant\biggl(\sum_{j=1}^{\infty}|W^\Delta_{s_j}(G)(x)-W^\Delta_{s_{j+1}}(G)(x)|^\rho\biggr)^{1/\rho}-C\|G\|_\infty,\quad x\in[0,1].
\end{aligned}$$

Hence, $\mathcal{V}_\rho(W^H_t)(G)\notin L^\infty(0,1)$. □

Proof of Theorem 1.1 (iii). We shall proceed as in the proof of [**2**, Theorem 1.3]. Let $B_r=B(x_0,r)$ be an interval contained in $[-1,1]$ with $x_0\in\mathbb{R}$ and $r>0$, and $B^*_r=B(x_0,2r)$. We write $f=f_1+f_2$, where $f_1=f\chi_{B^*_r}$. Then

$$\mathcal{V}_\rho(W^H_t)(f)\leqslant\mathcal{V}_\rho(W^H_t)(f_1)+\mathcal{V}_\rho(W^H_t)(f_2).$$

As $\mathcal{V}_\rho(W^H_t)$ is bounded in $L^2(\mathbb{R})$ we have

$$\begin{aligned}
\frac{1}{|B_r|}\int_{B_r}\mathcal{V}_\rho(W^H_t)(f_1)(x)\,\mathrm{d}x&\leqslant\biggl(\frac{1}{|B_r|}\int_{B_r}(\mathcal{V}_\rho(W^H_t)(f_1)(x))^2\,\mathrm{d}x\biggr)^{1/2}\\
&\leqslant C\biggl(\frac{1}{|B_r|}\int_{B^*_r}|f(x)|^2\,\mathrm{d}x\biggr)^{1/2}\leqslant C\|f\|_\infty.
\end{aligned}$$

On the other hand, Hölder's inequality leads to

$$|W^H_t(f_2)(x)-W^H_s(f_2)(x)|\leqslant 2^{1/\rho}\biggl(\int_{-\infty}^{\infty}|W^H_t(x,y)-W^H_s(x,y)|\,|f_2(y)|^\rho\,\mathrm{d}y\biggr)^{1/\rho}$$

for every $x\in\mathbb{R}$ and $s,t>0$. Also, we have

$$\int_0^1\biggl|\frac{\mathrm{d}}{\mathrm{d}t}W^H_t(x,y)\biggr|\,\mathrm{d}t\leqslant C\frac{1}{|x-y|},\quad x,y\in\mathbb{R}. \tag{2.3}$$

Indeed, for every $x,y\in\mathbb{R}$ and $t\in(0,1)$, we obtain

$$\begin{aligned}
\biggl|\frac{\mathrm{d}}{\mathrm{d}t}W^H_t(x,y)\biggr|&\leqslant C\biggl(\biggl(\frac{t}{1-t^2}\biggr)^{1/2}\frac{t^2+1}{t^2}+\biggl(\frac{1-t^2}{t}\biggr)^{1/2}\biggl(|x+y|^2+\frac{|x-y|^2}{t^2}\biggr)\biggr)\\
&\qquad\times\exp\biggl(-\frac14\biggl(\frac{|x-y|^2}{t}+|x+y|^2t\biggr)\biggr)\\
&\leqslant C(1-t)^{-1/2}t^{-3/2}\exp\biggl(-\frac{|x-y|^2}{8t}\biggr).
\end{aligned}$$

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Then, by using [**7**, Lemma 1.1] we obtain (2.3). Hence, for every $x \in B_r$, we have

$$\begin{aligned}
\mathcal{V}_\rho(W_t^H)(f_2)(x) &= \sup_{\{t_j\}\searrow 0} \biggl(\sum_{j=1}^{\infty} |W_{t_j}^H(f_2)(x) - W_{t_{j+1}}^H(f_2)(x)|^\rho\biggr)^{1/\rho} \\
&\leqslant 2^{1/\rho} \sup_{\{t_j\}\searrow 0} \biggl(\sum_{j=1}^{\infty} \int_{-\infty}^{\infty}\int_{t_{j+1}}^{t_j} \biggl|\frac{\mathrm{d}}{\mathrm{d}t} W_t^H(x,y)\biggr| \,\mathrm{d}t |f_2(y)|^\rho \,\mathrm{d}y\biggr)^{1/\rho} \\
&\leqslant 2^{1/\rho} \biggl(\int_{-\infty}^{\infty}\int_{0}^{1} \biggl|\frac{\mathrm{d}}{\mathrm{d}t} W_t^H(x,y)\biggr| \,\mathrm{d}t |f_2(y)|^\rho \,\mathrm{d}y\biggr)^{1/\rho} \\
&\leqslant C\biggl(\int_{-\infty}^{\infty} \frac{1}{|x-y|} |f_2(y)|^\rho \,\mathrm{d}y\biggr)^{1/\rho} \\
&\leqslant C\|f\|_\infty \biggl(\log\frac{1}{2r}\biggr)^{1/\rho}.
\end{aligned}$$

Then we conclude that

$$\frac{1}{|B_r|}\int_{B_r} \mathcal{V}_\rho(W_t^H)(f_2)(x)\,\mathrm{d}x \leqslant C\|f\|_\infty \biggl(\log\frac{1}{2r}\biggr)^{1/\rho},$$

and the proof of (iii) is completed. □

3. Growth of Hermitian singular integrals

The logarithmic behaviour at the origin of the Hilbert transform $\mathfrak{H}$ is shown by the following easy example. Let $f = \chi_{(0,1)}$ and $0 < r < 1$. Then we have

$$\begin{aligned}
\frac{1}{r}\int_{-r}^{0} \mathfrak{H}(-\chi_{(0,1)})(x)\,\mathrm{d}x &= \frac{1}{r}\int_{-r}^{0}\int_{0}^{1} \frac{1}{y-x}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{r}\int_{-r}^{0} \ln\biggl(\frac{1-x}{-x}\biggr)\mathrm{d}x \\
&= \frac{1}{r}\int_{0}^{r} \ln\frac{1+x}{x}\,\mathrm{d}x \\
&= \frac{(1+r)\log(1+r) - r\log r}{r} \\
&\sim \log\frac{e}{r} \quad \text{as } r \to 0.
\end{aligned}$$

The next proposition shows that this behaviour is shared by the Hermitian singular integral $\partial_x(H)^{-1/2}$.

Proposition 3.1. *For every $x_0 \in B(0,1)$ there exists a bounded function f such that*

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} \partial_x(H)^{-1/2} f(x)\,\mathrm{d}x \sim \log\frac{e}{r} \quad \text{as } r \to 0.$$

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Proof. Observe that

$$\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{\partial}{\partial x}\exp\left(-\frac{(x-y)^2}{4s}\right)\frac{\mathrm{d}s}{s} = -\frac{\sqrt{2}}{\pi}\frac{1}{x-y}, \quad x,y\in\mathbb{R},\ x\neq y.$$

Hence, by using the above example it is sufficient to prove that, for a certain $C>0$,

$$\left|R_H(x,y) - \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{\partial}{\partial x}\exp\left(-\frac{(x-y)^2}{4s}\right)\frac{\mathrm{d}s}{s}\right| \leqslant C, \quad x,y\in[-1,1]. \tag{3.1}$$

We write

$$R_H(x,y) = -\frac{1}{\pi}\int_0^1 \frac{\partial}{\partial x}\left(\exp\left(-\frac{1}{4}\left(\frac{(x-y)^2}{s}+s(x+y)^2\right)\right)\right)\eta(s)\,\mathrm{d}s, \quad x,y\in\mathbb{R},\ x\neq y,$$

where

$$\eta(s) = \left(\log\frac{1+s}{1-s}\right)^{-1/2}((1-s^2)s)^{-1/2}, \quad s\in(0,1).$$

Hence,

$$\begin{aligned}
&\left|R_H(x,y) + \frac{\sqrt{2}}{\pi}\frac{1}{x-y}\right| \\
&\quad\leqslant C\biggl(\int_0^{1/2}\left|\frac{\partial}{\partial x}\left(\exp\left(-\frac{1}{4s}|x-y|^2\right)\left(1-\exp\left(-\frac{s|x+y|^2}{4}\right)\right)\right)\right|\frac{\mathrm{d}s}{s} \\
&\qquad + \int_0^{1/2}\left|\frac{\partial}{\partial x}\left(\exp\left(-\frac{1}{4s}|x-y|^2\right)\right)\left(\frac{1}{\sqrt{2}s}-\eta(s)\right)\right|\mathrm{d}s \\
&\qquad + \int_{1/2}^1\left|\frac{\partial}{\partial x}\left(\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{1}{s}|x-y|^2\right)\right)\right)\right|\eta(s)\,\mathrm{d}s \\
&\qquad + \int_{1/2}^\infty\left|\frac{\partial}{\partial x}\left(\exp\left(-\frac{|x-y|^2}{4s}\right)\right)\right|\frac{\mathrm{d}s}{s}\biggr) \\
&\quad = C(I_1(x,y)+I_2(x,y)+I_3(x,y)+I_4(x,y)), \quad x,y\in\mathbb{R},\ x\neq y.
\end{aligned}$$

We analyse each term separately. By the mean-value theorem and (1.3) we have that

$$\begin{aligned}
I_1(x,y) &\leqslant C\biggl(\int_0^{1/2}\exp\left(-\frac{|x-y|^2}{4s}\right)\frac{|x-y|}{s}\left(1-\exp\left(-\frac{s|x+y|^2}{4}\right)\right)\frac{\mathrm{d}s}{s} \\
&\qquad + \int_0^{1/2}\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{1}{s}|x-y|^2\right)\right)|x+y|\,\mathrm{d}s\biggr) \\
&\leqslant C, \quad x,y\in[-1,1].
\end{aligned}$$

For the remainder summands we use

$$\eta(s)\sim\frac{1}{\sqrt{2}s} \quad \text{as } s\to 0^+, \qquad \int_{1/2}^1\eta(s)\,\mathrm{d}s<\infty$$

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and we have

$$I_2(x,y) \leqslant C\int_0^{1/2} \frac{|x-y|}{s}\exp\left(-\frac{|x-y|^2}{4s}\right)\mathrm{d}s \leqslant C, \quad x,y\in[-1,1];$$

$$\begin{aligned}I_3(x,y) &\leqslant \int_{1/2}^1 \exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{1}{s}|x-y|^2\right)\right)\left(s|x+y|+\frac{1}{s}|x-y|\right)\eta(s)\,\mathrm{d}s\\ &\leqslant \int_{1/2}^1\left(s+\frac{1}{s}\right)\eta(s)\,\mathrm{d}s \leqslant C, \quad x,y\in[-1,1];\\ I_4(x,y) &\leqslant \int_{1/2}^\infty \exp\left(-\frac{1}{4s}|x-y|^2\right)|x-y|\frac{\mathrm{d}s}{s^2}\\ &\leqslant \int_{1/2}^\infty \frac{1}{s^{3/2}}\,\mathrm{d}s \leqslant C, \quad x,y\in[-1,1].\end{aligned}$$

Thus, (3.1) is established. □

4. Finiteness of the classical square function

Let f be the function defined by

$$f(y) = \sum_{k\in\mathbb{Z}}(-1)^k\chi_{[a^{2k},a^{2k+1}]}(y), \tag{4.1}$$

where $a>0$ is a real number that we shall fix later.

The following useful properties can easily be proved.

Lemma 4.1. *Let f be the function defined in (4.1) and let $t_j=a^{2j}$, $j\in\mathbb{Z}$. Then*

(i) $f(t_jy)=(-1)^jf(y)$, $j\in\mathbb{Z}$,

(ii) $\displaystyle\int_{\mathbb{R}}\frac{t_j}{t_j^2+y^2}f(y)\,\mathrm{d}y = (-1)^j\int_0^\infty\frac{1}{1+y^2}f(y)\,\mathrm{d}y$,

(iii) $\displaystyle\left|\int_{\mathbb{R}}\frac{t_j}{t_j^2+y^2}f(y)\,\mathrm{d}y - \int_{\mathbb{R}}\frac{t_{j+1}}{t_{j+1}^2+y^2}f(y)\,\mathrm{d}y\right| = 2\int_0^\infty\frac{1}{1+y^2}f(y)\,\mathrm{d}y.$

Lemma 4.2. *There exist $a_0>1$ and a positive constant C such that*

$$\left|\int_0^\infty\frac{1}{1+y^2}f(y)\,\mathrm{d}y\right| \geqslant C$$

for all functions f defined in (4.1) with $a>a_0$.

Proof. Observe that

$$\int_1^\infty\frac{1}{1+y^2}\,\mathrm{d}y = \frac{\pi}{4}.$$

Now we choose $a>a_0$ such that

$$\int_1^{a_0}\frac{1}{1+y^2}\,\mathrm{d}y \geqslant \frac{\pi}{5}.$$

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Hence,

$$\int_a^\infty \frac{1}{1+y^2}\,\mathrm{d}y = \int_0^{1/a} \frac{1}{1+y^2}\,\mathrm{d}y \leqslant \frac{\pi}{20}.$$

Then, as $f(y)=0$ for $y \in \bigcup_{j\in\mathbb{Z}}[a^{2j+1}, a^{2j+2}]$, we have

$$\begin{aligned}\int_0^\infty \frac{1}{1+y^2} f(y)\,\mathrm{d}y &= \left(\int_0^1 + \int_1^a + \int_a^\infty\right) \frac{1}{1+y^2} f(y)\,\mathrm{d}y \\ &\geqslant \int_1^a \frac{1}{1+y^2}\,\mathrm{d}y - \left(\int_0^{1/a} \frac{1}{1+y^2}\,\mathrm{d}y + \int_{a^2}^\infty \frac{1}{1+y^2}\,\mathrm{d}y\right) \\ &\geqslant \frac{\pi}{5} - \frac{\pi}{10} = \frac{\pi}{10}.\end{aligned}$$

□

Lemma 4.3. *There exists a function $F \in L^\infty(\mathbb{R})$ with support in $[-1,1]$ and a decreasing sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,1)$ convergent to zero such that*

$$\sum_{j=1}^\infty |P^\Delta_{t_{j+1}}(F)(x) - P^\Delta_{t_j}(F)(x)|^2 \notin L^\infty([0,1]).$$

Proof. Let $a>1$ be as in Lemma 4.2. We define the function

$$F = \sum_{k=-\infty}^{-1} (-1)^k \chi_{[a^{2k}, a^{2k+1})}.$$

It is clear that $F = f\chi_{[0,1/a)} = f\chi_{[0,1)}$, where f is the function defined in (4.1). Moreover, form Lemma 4.1, for every $j \in \mathbb{Z}$ and $y \in \mathbb{R}$, we have $F(a^{2j}y) = (-1)^j f(y)\chi_{[0,a^{-2j})}(y)$. Therefore, for each $j \in \mathbb{Z}$, we can write

$$\begin{aligned}\int_{-\infty}^\infty \frac{a^{2j}}{(a^{2j})^2+y^2} F(y)\,\mathrm{d}y &= \int_{-\infty}^\infty \frac{1}{1+u^2} F(a^{2j}u)\,\mathrm{d}u \\ &= (-1)^j \int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[0,a^{-2j})}(u)\,\mathrm{d}u.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{-\infty}^\infty \frac{a^{2j}}{(a^{2j})^2+y^2} F(y)\,\mathrm{d}y - \int_{-\infty}^\infty \frac{a^{2(j+1)}}{(a^{2(j+1)})^2+y^2} F(y)\,\mathrm{d}y \\ &\quad = (-1)^j \left(\int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[0,a^{-2j}]}\,\mathrm{d}u + \int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[0,a^{-2(j+1)}]}\,\mathrm{d}u\right) \\ &\quad = (-1)^j \left(2\int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[0,a^{-2(j+1)}]}\,\mathrm{d}u + \int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[a^{-2(j+1)},a^{-2j}]}\,\mathrm{d}u\right).\end{aligned}$$

Observe that

$$\int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\chi_{[0,a^{-2(j+1)}]}\,\mathrm{d}u \to \int_{-\infty}^\infty \frac{1}{1+u^2} f(u)\,\mathrm{d}u \quad \text{as } j \to -\infty$$

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and

$$\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[a^{-2(j+1)},a^{-2j}]}\,\mathrm{d}u\right| \leqslant \left(\frac{1}{1+(a^{-2(j+1)})^2}\right)^{1/4}\int_{-\infty}^{\infty}\left(\frac{1}{1+u^2}\right)^{3/4}\mathrm{d}u \to 0 \quad \text{as } j\to-\infty.$$

Hence, there exist $C>0$ and $j_0\in\mathbb{N}$ such that

$$\left|\int_{-\infty}^{\infty}\frac{a^{2j}}{(a^{2j})^2+y^2}F(y)\,\mathrm{d}y-\int_{-\infty}^{\infty}\frac{a^{2(j+1)}}{(a^{2(j+1)})^2+y^2}F(y)\,\mathrm{d}y\right|\geqslant C,\quad j\leqslant -j_0. \tag{4.2}$$

As a consequence we can ensure that given any positive constant $M>0$ there exists a finite subset J of $\mathbb{N}$ such that

$$\sum_{-j\in J}\left|\int_{-\infty}^{\infty}\frac{a^{2j}}{(a^{2j})^2+y^2}F(y)\,\mathrm{d}y-\int_{-\infty}^{\infty}\frac{a^{2(j+1)}}{(a^{2(j+1)})^2+y^2}F(y)\,\mathrm{d}y\right|^2\geqslant M.$$

By proceeding as above we get

$$\sum_{-j\in J}\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[0,a^{-2j})}(u)\,\mathrm{d}u+\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[0,a^{-2(j+1)})}(u)\,\mathrm{d}u\right|^2\geqslant M.$$

On the other hand, the obvious changes of variables give

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\frac{a^{2j}}{(a^{2j})^2+|x-y|^2}F(y)\,\mathrm{d}y-\int_{-\infty}^{\infty}\frac{a^{2(j+1)}}{(a^{2(j+1)})^2+|x-y|^2}F(y)\,\mathrm{d}y\right|\\
&\quad=\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}F(a^{2j}u+x)\,\mathrm{d}u-\int_{-\infty}^{\infty}\frac{1}{1+u^2}F(a^{2(j+1)}u+x)\,\mathrm{d}u\right|\\
&\quad=\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u+a^{-2j}x)\chi_{[0,a^{-2j}]}(u+a^{-2j}x)\,\mathrm{d}u\right.\\
&\qquad\left.+\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u+a^{-2(j+1)}x)\chi_{[0,a^{-2(j+1)}]}(u+a^{-2(j+1)}x)\,\mathrm{d}u\right|.
\end{aligned}$$

We observe that

$$\lim_{k\to0}\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u+k)\chi_{[0,A)}(u+k)\,\mathrm{d}u=\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[0,A)}(u)\,\mathrm{d}u,$$

where the convergence is uniform in $A\in(0,\infty)$. Hence, there exists $\eta>0$ such that if $|x|<\eta a^{2j}$, $-j\in J$,

$$\begin{aligned}
&\sum_{-j\in J}\left|\int_{-\infty}^{\infty}\frac{a^{2j}}{(a^{2j})^2+|x-y|^2}F(y)\,\mathrm{d}y-\int_{-\infty}^{\infty}\frac{a^{2(j+1)}}{(a^{2(j+1)})^2+|x-y|^2}F(y)\,\mathrm{d}y\,\mathrm{d}y\right|^2\\
&\quad\geqslant\frac{1}{2}\sum_{-j\in J}\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[0,a^{-2j})}(u)\,\mathrm{d}u+\int_{-\infty}^{\infty}\frac{1}{1+u^2}f(u)\chi_{[0,a^{-2(j+1)})}(u)\,\mathrm{d}u\right|^2\\
&\quad\geqslant\tfrac{1}{2}M.
\end{aligned}$$

As $M>0$ is arbitrary, we obtain the lemma. □

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Remark 4.4. The above arguments can be slightly adapted in order to show that if f is the function defined in (4.1), then there exists a decreasing sequence $\{t_j\}_{j\in\mathbb{Z}} \subset (0,\infty)$ such that $t_j \to 0$, as $j \to -\infty$, and $t_j \to +\infty$, as $j \to +\infty$, such that

$$\sum_{j\in\mathbb{Z}} |P^{\Delta}_{t_{j+1}}(f)(x) - P^{\Delta}_{t_j}(f)(x)|^2 = \infty \quad \text{a.e. } x \in \mathbb{R}.$$

Proof of Theorem 1.2. Let $t_j = a^{2j}$, $j \in \mathbb{Z}$, where $a > 1$. Then

$$\begin{aligned}
\sum_j |P^{\Delta}_{t_j}(f)(x) - P^{\Delta}_{t_{j+1}}(f)(x)|^2 &= \sum_{j\in\mathbb{Z}} \left| \int_{t_{j+1}}^{t_j} \frac{\partial}{\partial t} P^{\Delta}_t(f)(x)\Big|_{t=u} \,\mathrm{d}u \right|^2 \\
&\leqslant \sum_{j\in\mathbb{Z}} \int_{t_{j+1}}^{t_j} \left| \frac{\partial}{\partial t} P^{\Delta}_t(f)(x)\Big|_{t=u} \right|^2 \mathrm{d}u |t_j - t_{j+1}| \\
&= (a^2-1) \sum_j \left(\int_{t_{j+1}}^{t_j} t_j \left| \frac{\partial}{\partial t} P^{\Delta}_t(f)(x)\Big|_{t=u} \right|^2 \mathrm{d}u \right) \\
&\leqslant C \sum_j \int_{t_{j+1}}^{t_j} u \left| \frac{\partial}{\partial t} P^{\Delta}_t(f)(x)\Big|_{t=u} \right|^2 \mathrm{d}u \Big) \\
&= C g_1(f)(x)^2.
\end{aligned}$$

In order to finish the proof, we just take f and F as the functions defined in (4.1) and in Lemma 4.3, by also taking into account Remark 4.4. □

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References

1. J. T. Campbell, R. L. Jones, K. Reinhold and M. Wierdl, Oscillation and variation for the Hilbert transform, *Duke Math. J.* **105**(1) (2000), 59–83.
2. R. Crescimbeni, R. A. Macías, T. Menárguez, J. L. Torrea and B. Viviani, The ρ-variation as an operator between maximal operators and singular integrals, *J. Evol. Eqns* **9**(1) (2009), 81–102.
3. R. L. Jones, R. Kaufman, J. M. Rosenblatt and M. Wierdl, Oscillation in ergodic theory, *Ergod. Theory Dynam. Syst.* **18** (1998), 889–935.
4. R. L. Jones and K. Reinhold, Oscillation and variation inequalities for convolution powers, *Ergod. Theory Dynam. Syst.* **21**(6) (2001), 1809–1829.
5. R. L. Jones, A. Seeger and J. Wright, Strong variational and jump inequalities in harmonic analysis, *Trans. Am. Math. Soc.* **360**(12) (2008), 6711–6742.
6. R. Oberlin, A. Seeger, T. Tao, C. Thiele and J. Wright, A variation norm Carleson theorem, preprint (arXiv:0910.1555v2; 2010).
7. K. Stempak and J. L. Torrea, Poisson integrals and Riesz transforms for Hermite function expansions with weights, *J. Funct. Analysis* **202**(2) (2003), 443–472.

8. S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, Volume 42 (Princeton University Press, 1993).
9. S. Wang, Some properties of Littlewood–Paley's g-function, *Sci. Sinica* A **28**(3) (1985), 252–262.
10. S. Wang, Some properties of the Littlewood–Paley g-function, in *Classical real analysis*, Contemporary Mathematics, Volume 42, pp. 191–202 (American Mathematical Society, Providence, RI, 1985).

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