M-SPACES WITH QUASI-INTERIOR POINTS

by W. A. FELDMAN and J. F. PORTER

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In this paper we characterize those locally convex lattices which can be represented as dense sublatices containing 1 in a space $C(X)$ and whose topologies can be recognized as topologies of uniform convergence on selections of compact subsets of X. Here $C(X)$ is the lattice of continuous real-valued functions on a completely regular space X. The class of such locally convex lattices includes the classical order unit spaces investigated by Kakutani [**3**], arbitrary products of order unit spaces, for example $\prod L^\infty$, and the order partition spaces studied in [**1**].

We remark that Jameson [**2**] obtained a representation theorem for arbitrary M-spaces as sublattices of $C(X)$ with topologies of uniform convergence on certain compact subsets. In his general setting the sublattices need not be dense nor separate the points of X. Also, Kuller [**4**] obtained an algebraic representation for a complete locally convex lattice with topology τ satisfying what he called condition (U) and observed a topological correspondence analogous to that obtained by Michael [**5**] for locally m-convex algebras: namely τ is finer than the topology of uniform convergence on certain compact sets. We show in Theorem 1 that condition (U) for a locally convex lattice (not necessarily complete) is equivalent to being representable in our sense, so that τ is in fact equal to the topology of uniform convergence on a collection of compact sets. We also show in Theorem 1 that this representability is equivalent to having a quasi-interior point and what we call a unit condition.

Given such a representation, we prove (Theorem 2) that the carrier set X and the compact subsets are unique, but that the possible topologies on X form a closed interval between a weakest σ and a strongest ω.

In §2 we show (Theorem 3) that the completion of a space having a quasi-interior point and unit condition can be identified with $C(X, \omega)$ in the topology of uniform convergence on these compact sets.

In §3 we contrast this topology with the topology of uniform convergence on all compact subsets.

One might wish to compare this representation with the representation of Banach lattices having quasi-interior points as continuous extended real-valued functions considered by Lotz and Schaefer [**6**].

1. Stone-Weierstrass Embeddability. A positive element e in a locally convex lattice V is a *quasi-interior point* if the order ideal generated by e is dense in V. We will say that V satisfies the *unit condition* if for each lattice semi-norm p in a generating (not

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necessarily directed) collection, the quotient norm space $V/\ker(p)$ has a unit (i.e., a largest element in its unit ball). Clearly, p is M-convex.

For brevity in what follows, we introduce some notations. Given a topological space (Y, τ), we will use the symbol $\mathcal{K}$ to denote a collection of compact subsets of (Y, τ) whose union is Y. By $C_{\mathcal{K}}(Y, \tau)$ we will mean the space of all real-valued continuous functions on (Y, τ) with the topology of uniform convergence on the sets in $\mathcal{K}$. Without loss of generality, we will assume that $\mathcal{K}$ contains all closed subsets and finite unions of its members. We will say that a topological vector lattice V is *Stone-Weierstrass embedded* in $C_{\mathcal{K}}(Y, \tau)$ if there is a lattice isomorphism and homeomorphism of V onto a dense sublattice of $C_{\mathcal{K}}(Y, \tau)$ containing 1.

We recall (see **[4]**) that a locally convex lattice V satisfies *condition* (U) if there is a positive element e in V such that for each lattice seminorm p in a generating collection, $p(e)>0$ and if $p(v)\le p(e)$ then $\bar{v}\le\bar{e}$ in $V/\ker(p)$.

THEOREM 1. *For a locally convex lattice V, the following are equivalent.*

(1) *V can be Stone-Weierstrass embedded in a $C_{\mathcal{K}}(X, \sigma)$ for a completely regular topology σ.*

(2) *V has a quasi-interior point and satisfies the unit condition.*

(3) *V satisfies condition (U).*

Proof. That (1) implies (3) is clear.

To see that (2) implies (1), let $\mathcal{P}$ be a collection of lattice seminorms generating the topology of V such that for each p in $\mathcal{P}$, the quotient V_p of V by the kernel of p has a unit. By Kakutani's order unit theorem, we can embed $(V_p, \bar{p})$ in $C(K)$ for some compact K. Let I_p denote the order ideal in V_p generated by the image $\bar{e}$ of e. Since the ideal generated by e is dense in V, I_p is dense in V_p, so that the image of $\bar{e}$ in $C(K)$ is never zero. it follows that $\bar{e}$ is an order unit for V_p and $V_p=I_p$. We can thus embed V_p in $C(K_p)$, where K_p is the set of lattice homomorphisms y on V_p having $y(\bar{e})=1$ with the weak topology from V_p. We conclude that for all v in V, $p(v)=\bar{p}(\bar{v})$ is equivalent to $\sup\{|y(v)|: y\in K_p\}$. Let Z be the set of lattice homomorphisms x on V having $x(e)=1$, with the weak topology σ from V. Where j_p is the adjoint of the quotient map from V into V_p, mapping K_p into Z, we let X be the union of subspaces $j_p(K_p)$ over the seminorms p in $\mathcal{P}$. Clearly the map from V into $C(X, \sigma)$ given by $v(x)=x(v)$ is a lattice homomorphism. To see that it is one-to-one, we note that $v\neq 0$ implies $p(v)\neq 0$ for some p in $\mathcal{P}$, so that $y(\bar{v})\neq 0$ for some y in K_p, implying that $v(j_p(y))\neq 0$. Finally, letting $\mathcal{K}$ denote the collection of compact sets $j_p(K_p)$, we see that V is homeomorphic to its image in $C_{\mathcal{K}}(X, \sigma)$, since $p(v)$ is equivalent to $\sup\{|v(x)| : x\in j_p(K_p)\}$.

To show that (3) implies (2), we first observe that the unit ball of $(V_p, \bar{p})$ has a largest element $\bar{e}/p(e)$, since if $p(\bar{v})\le 1$ then $p(p(e)v)\le p(e)$ so that, by condition (U), $v\le \bar{e}/p(e)$. It now follows that for each v in V there is an N such that $\bar{v}\le N\bar{e}$, so that $\bar{v}-\bar{v}\wedge N\bar{e}=0$, implying $p(v-v\wedge Ne)=0$. Thus the conditions of (2) are satisfied.

LEMMA. *Let topological vector lattice V be Stone-Weierstrass embedded in $C_{\mathcal{K}}(Y, \tau)$ with d the pre-image of 1. Then Y is the set of all continuous lattice homomorphisms z on V such that $z(d)=1$.*

Proof. Let z be a continuous lattice homomorphism on V with $z(d)=1$. There is a compact set K in $\mathcal{K}$ such that $|z(v)|<\lambda p(v)=\lambda \sup\{|v(x)|: x\in K\}$ for some $\lambda>0$. The induced continuous lattice homomorphism $\bar{z}$ on the quotient of V by the kernel of p, given by $\bar{z}(\bar{v})=z(v)$, can be continuously extended to $C(K)$ in the supremum norm topology, since the quotient is dense in $C(K)$. Since $\bar{z}(\bar{d})=1$, $\bar{z}$ is a point-evaluation functional; hence z is in Y. Clearly, the points in Y satisfy the condition.

THEOREM 2. *Let a topological vector lattice V be Stone-Weierstrass embedded in $C_{\mathcal{K}}(X,\sigma)$ where σ is the weak topology induced on X by V. There is a finest topology ω on X containing σ such that each K in $\mathcal{K}$ is compact in ω. The space V is Stone-Weierstrass embedded in $C_{\mathcal{K}'}(Y,\tau)$ if and only if there is a bijection $\psi: Y\to X$ such that the mappings*

$$(X,\omega)\xrightarrow{\psi^{-1}}(Y,\tau)\xrightarrow{\psi}(X,\sigma)$$

are continuous and $\psi\mathcal{K}'=\mathcal{K}$.

Proof. Let ω be the supremum of all topologies $\tau\supseteq\sigma$ on X such that each member of $\mathcal{K}$ is compact in τ. For each such τ, and K in $\mathcal{K}$, τ restricted to K equals σ restricted to K. Each basic open set in ω restricted to K is of the form $K\cap\left(\bigcap_{i=1}^{n} O_i\right)=\bigcap_{i=1}^{n}(K\cap O_i)$ for O_i in some τ_i and hence is open in σ restricted to K. This proves the first assertion of the theorem.

Suppose V is Stone-Weierstrass embedded in $C_{\mathcal{K}}(X,\sigma)$ and in $C_{\mathcal{K}'}(Y,\tau)$ with embedding maps j and j', respectively, and let e and e' be the corresponding pre-images of 1 in V. We define mappings $\psi: Y\to X$ and $\phi: X\to Y$ by setting $\psi(y)=y/y(e)$ and $\phi(x)=x/x(e')$, where x and y are viewed as lattice homomorphisms on V. These mappings are well-defined by the lemma and the fact that e and e' are quasi-interior points (so that $y(e)$ and $x(e')$ are never zero). It is easy to verify ψ and ϕ are inverses of each other. As a mapping from (Y,τ) to $(X,\psi\tau)$, ψ is a homeomorphism, so that its adjoint $\psi^*: C(X,\psi\tau)\to C(Y,\tau)$ is an isomorphism. We define $q: C(Y,\tau)\to C(Y,\tau)$ by setting $q(f)=f/je$. One can verify that the following diagram is commutative:

$$\begin{array}{ccccc}
C_{\mathcal{K}'}(Y,\tau) & \xrightarrow{q} & C_{\mathcal{K}'}(Y,\tau) & \xrightarrow{(\psi^*)^{-1}=\phi^*} & C_{\psi\mathcal{K}'}(X,\psi\tau)\\
\uparrow{\scriptstyle j'} & & & & \downarrow{\scriptstyle i}\\
v & \xrightarrow{j} & C_{\mathcal{K}}(X,\sigma) & \xrightarrow{\iota} & F(X)
\end{array}$$

Here $F(X)$ is the set of all function on X, and i and ι are inclusions. It follows that for each v in V, jv is continuous on $(X,\psi\tau)$. Thus $\psi\tau\supseteq\sigma$. Hence $\psi:(Y,\tau)\to(X,\sigma)$ is continuous. Since the mappings of the diagram are homeomorphisms (into), for K in $\mathcal{K}$ there is a K' in $\mathcal{K}'$ such that $p_K\circ j$ is dominated by a multiple of $p_{\psi K'}\circ(\phi^* qj')=p_{\psi K'}\circ j$. Since V separates X and contains the constant function 1, we conclude $K\subseteq\psi K'$. Similarly, each $\psi K'$ in $\psi\mathcal{K}'$ is contained in some K in $\mathcal{K}$. Thus $\psi\mathcal{K}'=\mathcal{K}$. Since each K in $\mathcal{K}$ is compact in $\psi\tau$, it follows that $\omega\supseteq\psi\tau$. Thus $\psi^{-1}:(X,\omega)\to(Y,\tau)$ is continuous. This proves the sufficiency of the second assertion; the necessity can easily be readily verified.

For convenience, we will call a locally convex lattice having a quasi-interior point and satisfying the unit condition an *M^*-space.* We see in Theorem 2 that the carrier set X and the collection $\mathcal{K}$ are essentially unique for an M^*-space. However, the topologies for X can range from σ to ω; we will call this collection of topologies the *carrier interval* $[\sigma, \omega]$. For an example in which this carrier interval is non-trivial, consider the usual topology τ on the interval $[0, 1]$ and the M^*-space $C_{\mathcal{K}}([0, 1], \tau)$ where $\mathcal{K}$ consists of all finite subsets of $[0, 1]$. Then σ is the usual topology, ω is the discrete topology, and $C([0, 1], \omega) \neq C([0, 1], \sigma)$.

2. The Completion of an M^*-space. Let M^*-space V be Stone-Weierstrass embedded in $C_{\mathcal{K}}(X, \tau)$, with carrier interval $[\sigma, \omega]$. We recall that τ is a *k-topology* on X if the open sets of τ are precisely those sets O such that $O \cap K$ is relatively open for each compact subset K of X. In analogy, we will say that τ is a *$\mathcal{K}$-topology* on X if the open sets of τ are precisely those sets O such that $O \cap K$ is relatively open for each (compact) set K in $\mathcal{K}$. Clearly, each $\mathcal{K}$-topology is a k-topology.

The next proposition is used in the proof of Theorem 3.

PROPOSITION 1. *The topology ω is a $\mathcal{K}$-topology, the only $\mathcal{K}$-topology in the carrier interval $[\sigma, \omega]$.*

Proof. Let ω^* be the $\mathcal{K}$-topology on X whose open sets are precisely those sets U such that $U \cap K$ is relatively open in ω for each K in $\mathcal{K}$. Clearly ω^* is finer than ω and, since each member of $\mathcal{K}$ is compact in ω^*, we have $\omega^* = \omega$. For τ in $[\sigma, \omega]$ and K in $\mathcal{K}$, as noted in the proof of Theorem 2 a set is relatively open in (K, τ) if and only if it is relatively open in (K, ω). Thus if τ is a $\mathcal{K}$-topology, it must be ω.

THEOREM 3. *The completion of an M^*-space is lattice isomorphic and homeomorphic to $C_{\mathcal{K}}(X, \omega)$.*

Proof. In view of Theorem 2, it is sufficient to prove that $C_{\mathcal{K}}(X, \omega)$ is complete. Let $\{f_\alpha\}$ be a Cauchy net in $C_{\mathcal{K}}(X, \omega)$. The restriction of $\{f_\alpha\}$ to a set K in $\mathcal{K}$ is Cauchy in the supremum norm space $C(K)$ and thus has a limit f_K in $C(K)$. The function f defined on X by $f(x) = f_K(x)$ for x in K is continuous on (X, ω) since ω is a $\mathcal{K}$-topology. It is clear that $\{f_\alpha\}$ converges to f in $C_{\mathcal{K}}(X, \omega)$.

Proposition 1 and Theorem 3 imply the well-known result that $C(K)$ with the topology of compact convergence is complete if X is a k-space.

We observe that ω need not be completely regular. Let the real line $\mathbf{R}$ have the topology δ generated by the usual open sets togeather with the set $\mathbf{R}\backslash\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$. Since δ is finer than the usual topology, $C(\mathbf{R}, \delta)$ separates the points of $\mathbf{R}$. Thus $\mathbf{R}$ is the carrier set of $C_{\mathcal{K}}(\mathbf{R}, \delta)$ where $\mathcal{K}$ is all the compact sets of $(\mathbf{R}, \delta)$, and δ is in the carrier interval $[\sigma, \omega]$. Clearly, δ is first countable, so that $(\mathbf{R}, \delta)$ is a k-space, implying here that δ is a $\mathcal{K}$-topology. It follows from Proposition 1 that δ is the unique $\mathcal{K}$-topology ω. But $\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ cannot be separated from 0, so that ω is not completely regular.

Although ω need not be completely regular it can be replaced by an associated completely regular topology $\omega^{\#}$ (the weak topology induced on X by $C(X, \omega)$) so that $C_{\mathscr{K}}(X, \omega) = C_{\mathscr{K}}(X, \omega^{\#})$.

The next proposition can now be easily proved.

PROPOSITION 2. *Let V be an M^*-space with carrier interval $[\sigma, \omega]$. Then the following are equivalent*:

(1) $\sigma = \omega^{\#}$;

(2) $C(X, \sigma) = C(X, \omega)$;

(3) $C_{\mathscr{K}}(X, \sigma)$ *is complete.*

Moreover, completeness of V implies these conditions.

3. The topology of compact convergence. The question arises: Are the compact sets of ω precisely the collection $\mathscr{K}$? The following example shows that this is not always so. Let τ be the usual topology of the interval $[0, 1]$ and let the compact sets of $\mathscr{K}$ be the convergent sequences of $[0, 1]$, augmented by their limits. For the M^*-space $C_{\mathscr{K}}([0, 1], \tau)$, $\sigma = \tau$. Also, $\sigma = \omega$ since these topologies agree on the sets in $\mathscr{K}$ and σ is first countable. But there are compact sets in $([0, 1], \omega)$ which are not in $\mathscr{K}$.

We remark that in the above example, $C([0, 1], \tau)$ is complete in both the topology of uniform convergence on the sets in $\mathscr{K}$ and the finer topology of uniform convergence on all compact sets, by Theorem 3.

Clearly, if all M-convex seminorms on an M^*-space are continuous, then the compact sets of ω are precisely those of $\mathscr{K}$. More generally, it would be of interest to know conditions for an M^*-space to be represented with the topology of uniform convergence on all compact subsets of (X, ω). The orthogonality condition studied in [1] provides one such condition in a restricted setting.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, AR. 72701