THE JOINT LAPLACE TRANSFORMS FOR DIFFUSION OCCUPATION TIMES

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Abstract

In this paper we adopt the perturbation approach of Landriault, Renaud and Zhou (2011) to find expressions for the joint Laplace transforms of occupation times for time-homogeneous diffusion processes. The expressions are in terms of solutions to the associated differential equations. These Laplace transforms are applied to study ruin-related problems for several classes of diffusion risk processes.

Keywords: Exit problem; occupation time; ruin probability; time-homogeneous diffusion

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1. Introduction

The occupation time is the amount of time a stochastic process stays within a certain range. It is an interesting topic for stochastic processes. Many explicit results on Laplace transforms for occupation times have been obtained for some well-known examples of diffusion processes; see, e.g. [6] for a collection of such results. Some results related to occupation times for general diffusion processes can be found in [18] and [19]. More recently, Laplace transforms have also been obtained for occupation times of processes with jumps; see [7], [9], [14], [15], [17], [23], and [24] for results along this line and their applications in risk theory and finance. To the best of the authors' knowledge, joint Laplace transforms have not been studied systematically for general diffusion processes.

The standard way of finding the Laplace transform for occupation times is to solve the associated (integro-) differential equation via the Feynman–Kac representation. The excursion theory has also been applied to obtain occupation time related Laplace transforms; see [18] and [19]. An alternative perturbation approach was recently proposed in [15] for the spectrally negative Lévy processes. With this approach, the Laplace transforms for occupation times can be obtained by an approximation argument based on the solutions to the exit problems, which is an excursion theory argument at its heart. Since the exit problems for time-homogeneous diffusion processes can also be solved explicitly, in this paper we adopt the strategy of [15] to study the joint Laplace transforms for diffusion occupation times. These Laplace transforms are expressed in terms of solutions to the differential equation associated to the diffusion generator.

The results in this paper find applications in the study of ruin problems for the so-called diffusion risk process with random observation times and the so-called omega model, where,

for both models, the ruin probability can be expressed in terms of its occupation time. Our results can also be applied to study the Brownian surplus process with two-valued drift.

The rest of the paper is arranged as follows. In Section 2 we introduce the time-homogeneous diffusion processes and their exit problems. In Section 3, the desired Laplace transforms of diffusion occupation times are found. In Section 4, possible applications of the results obtained in Section 3 to several diffusion risk processes are discussed. In Section 5, explicit expressions are presented for the examples of Brownian motion with drift and Brownian motion with two-valued drift.

2. Time-homogeneous diffusion processes

We now introduce the one-dimensional diffusion process X considered in this paper. For $-\infty \leq l_1 < l_2 \leq \infty$, write I for the interval with endpoints l_1 and l_2. For simplicity, we consider only the case in which both l_1 and l_2 are inaccessible, i.e. neither of them can be reached in finite time. In the rest of paper, we mostly choose $l_1 = -\infty$ and $l_2 = \infty$. The I-valued regular time-homogeneous diffusion process $X = \{X_t, \ t \geq 0\}$, defined on a filtered probability space $\{\Omega, \mathbb{P}, \{\mathcal{F}_t, \ t \geq 0\}\}$, is specified by the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \tag{2.1}$$

where $X_0 = x_0$ is the initial value and $\{W_t, \ t \geq 0\}$ is a standard Brownian motion. Throughout the paper, we assume that (2.1) allows a unique strong solution, which is guaranteed if there exists a constant $K > 0$ such that, for all $x, y \in I$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \qquad \mu^2(x) + \sigma^2(x) \leq K^2(1 + x^2). \tag{2.2}$$

Let $T_x = \inf\{t \geq 0 \colon X_t = x\}$ be the first hitting time of X at level x with the convention that $\inf \varnothing = \infty$. The Laplace transforms associated with the two-sided exit problem for the diffusion process X were first solved in [8]. Suppose that $g_{-,r}(\cdot)$ and $g_{+,r}(\cdot)$ are two independent, positive, and convex solutions of the equation

$$\tfrac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = rg(x), \qquad r > 0, \tag{2.3}$$

with $g_{-,r}(\cdot)$ strictly decreasing and $g_{+,r}(\cdot)$ strictly increasing. For many particular diffusions of interest, the differential equation (2.3) yields explicit expressions for $g_{-,r}(\cdot)$ and $g_{+,r}(\cdot)$; see [6]. Define a pair of Laplace exponents

$$\psi_r^{\pm}(\cdot) = \pm \frac{g'_{\pm,r}(\cdot)}{g_{\pm,r}(\cdot)}, \qquad r > 0. \tag{2.4}$$

In particular, denoting by

$$G(x) = \exp\left[-\int^x \frac{2\mu(y)}{\sigma^2(y)}\,\mathrm{d}y\right],$$

and letting $r \to 0+$, we choose

$$g_{-,0}(x) \begin{cases} \equiv 1 & \text{if } \int_x^\infty G(y)\,\mathrm{d}y = \infty, \\[2mm] = \int_x^\infty G(y)\,\mathrm{d}y & \text{otherwise,} \end{cases} \tag{2.5}$$

and

$$g_{+,0}(x) \begin{cases} \equiv 1 & \text{if } \int_{-\infty}^{x} G(y)\,dy = \infty, \\ = \int_{-\infty}^{x} G(y)\,dy & \text{otherwise.} \end{cases} \tag{2.6}$$

Hence,

$$\psi_0^-(x) = \frac{G(x)}{\int_x^{\infty} G(y)\,dy} \quad \text{and} \quad \psi_0^+(x) = \frac{G(x)}{\int_{-\infty}^{x} G(y)\,dy}. \tag{2.7}$$

Furthermore, for $r > 0$, define a function

$$f_r(y, z) = g_{-,r}(y)g_{+,r}(z) - g_{-,r}(z)g_{+,r}(y). \tag{2.8}$$

Note that the function $f_r(y, z)$ is strictly decreasing in y and strictly increasing in z, and $f_r(y, z) = 0$ if and only if $y = z$. For ease of notation, denote by

$$f_{1,r}(y, z) = \frac{\partial}{\partial y} f_r(y, z), \qquad f_{2,r}(y, z) = \frac{\partial}{\partial z} f_r(y, z), \quad \text{and} \quad f_{12,r}(y, z) = \frac{\partial^2}{\partial y \partial z} f_r(y, z).$$

Throughout the paper, for ease of notation, we write $\mathbb{E}_{x_0}[\cdot] = \mathbb{E}[\cdot \mid X_0 = x_0]$ for the conditional expectation, $\mathbb{P}_{x_0}\{\cdot\}$ for the corresponding probability, and $\mathbb{E}_{x_0}[\cdot\,; C] = \mathbb{E}_{x_0}[\cdot\,\mathbf{1}_C]$ with $\mathbf{1}_C$ denoting the indicator function of a set $C \subset \Omega$. Furthermore, we drop the subscript indicating the initial value if $x_0 = 0$.

For $a < x < b$, we have

$$\mathbb{E}_x[e^{-rT_a}; T_a < T_b] = \frac{f_r(x, b)}{f_r(a, b)} \quad \text{and} \quad \mathbb{E}_x[e^{-rT_b}; T_b < T_a] = \frac{f_r(a, x)}{f_r(a, b)}. \tag{2.9}$$

In particular, letting $r \to 0+$ in (2.9) yields the two-sided exit probabilities

$$\mathbb{P}_x\{T_b < T_a\} = \frac{\int_a^x G(y)\,dy}{\int_a^b G(y)\,dy} \quad \text{and} \quad \mathbb{P}_x\{T_a < T_b\} = \frac{\int_x^b G(y)\,dy}{\int_a^b G(y)\,dy};$$

see, e.g. [6], [10], and [13]. Moreover, for $r > 0$, we have

$$\lim_{x \to -\infty} g_{-,r}(x) = \lim_{x \to \infty} g_{+,r}(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g_{-,r}(x) = \lim_{x \to -\infty} g_{+,r}(x) = 0. \tag{2.10}$$

In addition,

$$\lim_{r \to \infty} \psi_r^{\pm}(\cdot) = \infty; \tag{2.11}$$

see Equation (12) of [19]. Therefore, letting $b \to \infty$ in the first relation of (2.9) and $a \to -\infty$ in the second relation of (2.9), for $a < x < b$, we have

$$\mathbb{E}_x e^{-rT_a} = \frac{g_{-,r}(x)}{g_{-,r}(a)} \quad \text{and} \quad \mathbb{E}_x e^{-rT_b} = \frac{g_{+,r}(x)}{g_{+,r}(b)}. \tag{2.12}$$

In the following, we collect some identities that will be used later. They can be easily verified by (2.4) and (2.8).

Lemma 2.1. *For any $x \neq y$ and $r > 0$, we have*

$$f_{2,r}(x, x) f_{1,r}(y, y) - f_{2,r}(x, y) f_{1,r}(x, y) = -f_{12,r}(x, y) f_r(x, y), \tag{2.13}$$

$$-f_{1,r}(x, y)\psi_r^-(y) - f_{2,r}(x, y)\psi_r^-(x) = f_{12,r}(x, y) + f_r(x, y)\psi_r^-(x)\psi_r^-(y), \tag{2.14}$$

and

$$f_{1,r}(x, y)\psi_r^+(y) + f_{2,r}(x, y)\psi_r^+(x) = f_{12,r}(x, y) + f_r(x, y)\psi_r^+(x)\psi_r^+(y). \tag{2.15}$$

3. The occupation times for diffusion processes

Throughout the paper, denote by e_δ an independent, exponential random variable with rate $\delta > 0$ and assume that $\lambda, \lambda_1, \lambda_2 > 0$. We first solve for $\mathbb{E} \exp[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\, ds]$ using the perturbation approach from [15].

Theorem 3.1. *It holds that*

$$\mathbb{E} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\, ds\right] = \frac{\delta \psi_{\delta+\lambda}^+(0)/(\delta + \lambda) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}.$$

Proof. Define an approximation L_ε^δ for the occupation time $\int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\, ds$ intuitively as follows. Up to time e_δ, L_ε^δ counts both the time X spends below level 0 and the time it takes to move from level 0 to level ε, but not from ε to 0. To make it rigorous, let θ be the shift operator such that $X_t \circ \theta_s = X_{s+t}$. Since X starts from 0, we put $T_0^1 = 0$, $T_\varepsilon^1 = T_\varepsilon$, and

$$T_0^{n+1} = T_\varepsilon^n + T_0 \circ \theta_{T_\varepsilon^n}, \qquad T_\varepsilon^{n+1} = T_0^{n+1} + T_\varepsilon \circ \theta_{T_0^{n+1}}, \quad \text{for } n \geq 1.$$

Then

$$L_\varepsilon^\delta = \sum_{n=1}^\infty (T_\varepsilon^n \wedge e_\delta - T_0^n \wedge e_\delta).$$

Hence, by the memoryless property we obtain

$$\begin{aligned}
\mathbb{E} e^{-\lambda L_\varepsilon^\delta} &= \mathbb{E}[e^{-\lambda L_\varepsilon^\delta}; T_\varepsilon < e_\delta] + \mathbb{E}[e^{-\lambda e_\delta}; e_\delta < T_\varepsilon] \\
&= \mathbb{E}[e^{-\lambda T_\varepsilon}; T_\varepsilon < e_\delta] \mathbb{E}_\varepsilon e^{-\lambda L_\varepsilon^\delta} + \mathbb{E} e^{-\lambda e_\delta} - \mathbb{E}[e^{-\lambda e_\delta}; T_\varepsilon < e_\delta] \\
&= \mathbb{E} e^{-(\delta+\lambda)T_\varepsilon} \mathbb{E}_\varepsilon e^{-\lambda L_\varepsilon^\delta} + \frac{\delta}{\delta + \lambda}(1 - \mathbb{E} e^{-(\delta+\lambda)T_\varepsilon}) \\
&= \frac{g_{+,\delta+\lambda}(0)}{g_{+,\delta+\lambda}(\varepsilon)} \mathbb{E}_\varepsilon e^{-\lambda L_\varepsilon^\delta} + \frac{\delta}{\delta + \lambda}\left(1 - \frac{g_{+,\delta+\lambda}(0)}{g_{+,\delta+\lambda}(\varepsilon)}\right),
\end{aligned} \tag{3.1}$$

where the last step is due to (2.12). Similarly, we have

$$\begin{aligned}
\mathbb{E}_\varepsilon e^{-\lambda L_\varepsilon^\delta} &= \mathbb{E}_\varepsilon[e^{-\lambda L_\varepsilon^\delta}; T_0 < e_\delta] + \mathbb{E}_\varepsilon[e^{-\lambda L_\varepsilon^\delta}; e_\delta < T_0] \\
&= \mathbb{P}_\varepsilon\{T_0 < e_\delta\} \mathbb{E} e^{-\lambda L_\varepsilon^\delta} + \mathbb{P}_\varepsilon\{e_\delta < T_0\} \\
&= \mathbb{E}_\varepsilon e^{-\delta T_0} \mathbb{E} e^{-\lambda L_\varepsilon^\delta} + 1 - \mathbb{E}_\varepsilon e^{-\delta T_0} \\
&= \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)} \mathbb{E} e^{-\lambda L_\varepsilon^\delta} + 1 - \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)}.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), solving for $\mathbb{E} e^{-\lambda L_\varepsilon^\delta}$, and taking the limit, we obtain

$$\begin{aligned}
\mathbb{E} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\, ds\right] &= \lim_{\varepsilon \to 0+} \mathbb{E} e^{-\lambda L_\varepsilon^\delta} \\
&= \lim_{\varepsilon \to 0+}\left\{\left[\frac{\delta}{\delta+\lambda}\left(1 - \frac{g_{+,\delta+\lambda}(0)}{g_{+,\delta+\lambda}(\varepsilon)}\right) + \frac{g_{+,\delta+\lambda}(0)}{g_{+,\delta+\lambda}(\varepsilon)}\left(1 - \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)}\right)\right]\right. \\
&\qquad \left. \times \left[1 - \frac{g_{+,\delta+\lambda}(0)}{g_{+,\delta+\lambda}(\varepsilon)} \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)}\right]^{-1}\right\} \\
&= \frac{\delta \psi_{\delta+\lambda}^+(0)/(\delta + \lambda) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}.
\end{aligned}$$

This completes the proof of Theorem 3.1.

Taking the limit as $\delta \to 0+$ in the equality in Theorem 3.1, we obtain the following result.

Corollary 3.1. *It holds that*

$$\mathbb{E}\exp\left[-\lambda\int_0^\infty \mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right] = \frac{\psi_0^-(0)}{\psi_\lambda^+(0)+\psi_0^-(0)},$$

where $\psi_0^-(0)$ is specified in (2.7).

Corollary 3.2. *In general, we have, for $x > 0$,*

$$\mathbb{E}_x\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right] = \frac{g_{-,\delta}(x)}{g_{-,\delta}(0)}\frac{\delta\psi_{\delta+\lambda}^+(0)/(\delta+\lambda)+\psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0)+\psi_\delta^-(0)}+1-\frac{g_{-,\delta}(x)}{g_{-,\delta}(0)}, \quad (3.3)$$

and, for $x < 0$,

$$\mathbb{E}_x\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right] = \frac{g_{+,\delta+\lambda}(x)}{g_{+,\delta+\lambda}(0)}\frac{\delta\psi_{\delta+\lambda}^+(0)/(\delta+\lambda)+\psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0)+\psi_\delta^-(0)}$$
$$+\frac{\delta}{\delta+\lambda}\left(1-\frac{g_{+,\delta+\lambda}(x)}{g_{+,\delta+\lambda}(0)}\right).$$

Proof. For $x > 0$, by comparing T_0 with e_δ and using the memoryless property, we obtain

$$\mathbb{E}_x\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right] = \mathbb{P}_x\{T_0<e_\delta\}\mathbb{E}\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]+\mathbb{P}_x\{e_\delta<T_0\}.$$

Then (3.3) follows from (2.12) and Theorem 3.1. For $x < 0$,

$$\mathbb{E}_x\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]$$
$$= \mathbb{E}_x[\mathrm{e}^{-\lambda T_0};T_0<e_\delta]\mathbb{E}\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]+\mathbb{E}_x[\mathrm{e}^{-\lambda e_\delta};e_\delta<T_0]$$
$$= \mathbb{E}_x\mathrm{e}^{-(\delta+\lambda)T_0}\mathbb{E}\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]+\frac{\delta}{\delta+\lambda}(1-\mathbb{E}_x\mathrm{e}^{-(\delta+\lambda)T_0}).$$

Then (3.3) also follows from (2.12) and Theorem 3.1.

Using the Laplace transforms on exit times in Section 2, all the results below could be easily extended to a general initial surplus $x \in \mathbb{R}$. However, to keep the paper short, we skip this minor generalization.

The strategy employed in the proof of Theorem 3.1 can be further exploited to find the joint Laplace transforms on the occupation times in multiple regions.

Theorem 3.2. *For any $b > 0$, we have*

$$\mathbb{E}\exp\left[-\lambda_1\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s-\lambda_2\int_0^{e_\delta}\mathbf{1}_{\{X_s>b\}}\,\mathrm{d}s\right]$$
$$= \left[\frac{\delta}{\delta+\lambda_1}\left(\frac{f_{2,\delta}(0,b)}{f_\delta(0,b)}+\psi_{\delta+\lambda_2}^-(b)\right)\psi_{\delta+\lambda_1}^+(0)\right.$$
$$\left.-\left(\frac{\lambda_2}{\delta+\lambda_2}\frac{f_{2,\delta}(0,0)}{f_\delta(0,b)}+\frac{f_{1,\delta}(0,b)}{f_\delta(0,b)}\right)\psi_{\delta+\lambda_2}^-(b)-\frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}\right]$$
$$\times\left[\left(\frac{f_{2,\delta}(0,b)}{f_\delta(0,b)}+\psi_{\delta+\lambda_2}^-(b)\right)\psi_{\delta+\lambda_1}^+(0)-\frac{f_{1,\delta}(0,b)}{f_\delta(0,b)}\psi_{\delta+\lambda_2}^-(b)-\frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}\right]^{-1}$$

and

$$\mathbb{E}_b \exp\left[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, ds - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s > b\}} \, ds\right]$$

$$= \left[\left(\frac{\lambda_1}{\delta + \lambda_1} \frac{f_{1,\delta}(b, b)}{f_\delta(0, b)} + \frac{f_{2,\delta}(0, b)}{f_\delta(0, b)}\right) \psi_{\delta+\lambda_1}^+(0)\right.$$

$$\left. + \frac{\delta}{\delta + \lambda_2}\left(-\frac{f_{1,\delta}(0, b)}{f_\delta(0, b)} + \psi_{\delta+\lambda_1}^+(0)\right) \psi_{\delta+\lambda_2}^-(b) - \frac{f_{12,\delta}(0, b)}{f_\delta(0, b)}\right]$$

$$\times \left[\left(\frac{f_{2,\delta}(0, b)}{f_\delta(0, b)} + \psi_{\delta+\lambda_2}^-(b)\right) \psi_{\delta+\lambda_1}^+(0) - \frac{f_{1,\delta}(0, b)}{f_\delta(0, b)} \psi_{\delta+\lambda_2}^-(b) - \frac{f_{12,\delta}(0, b)}{f_\delta(0, b)}\right]^{-1}.$$

Proof. Similar to the proof of Theorem 3.1, for arbitrarily small $\varepsilon > 0$, we approximate the occupation time $\int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, ds$ by $L_{\varepsilon, 0+}^\delta$, the sum of durations up to time e_δ for all the nonoverlapping excursions of the process X that start from 0 and end at ε. Similarly, we approximate $\int_0^{e_\delta} \mathbf{1}_{\{X_s > b\}} \, ds$ by $L_{\varepsilon, b-}^\delta$, the sum of durations up to time e_δ for all the excursions of X that start from b and end at $b - \varepsilon$.

For ease of notation, we define $I_0(x) = \mathbb{E}_x \exp[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, ds - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s > b\}} \, ds]$ and its approximation $I_\varepsilon(x) = \mathbb{E}_x \exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]$ for some $x \in \mathbb{R}$. By the strong Markov property, we have

$$I_\varepsilon(0) = \mathbb{E}[\exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]; T_\varepsilon < e_\delta] + \mathbb{E}[\exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]; e_\delta < T_\varepsilon]$$

$$= \mathbb{E}[e^{-\lambda_1 T_\varepsilon}; T_\varepsilon < e_\delta] I_\varepsilon(\varepsilon) + \mathbb{E}[e^{-\lambda_1 e_\delta}; e_\delta < T_\varepsilon]$$

$$= \mathbb{E}e^{-(\delta+\lambda_1) T_\varepsilon} I_\varepsilon(\varepsilon) + \frac{\delta}{\delta + \lambda_1}(1 - \mathbb{E}e^{-(\delta+\lambda_1) T_\varepsilon})$$

$$= \frac{g_{+,\delta+\lambda_1}(0)}{g_{+,\delta+\lambda_1}(\varepsilon)} I_\varepsilon(\varepsilon) + \frac{\delta}{\delta + \lambda_1}\left(1 - \frac{g_{+,\delta+\lambda_1}(0)}{g_{+,\delta+\lambda_1}(\varepsilon)}\right), \tag{3.4}$$

where the last step is due to (2.12). Furthermore,

$$I_\varepsilon(\varepsilon) = \mathbb{E}_\varepsilon[\exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]; T_0 < T_b \wedge e_\delta]$$

$$+ \mathbb{E}_\varepsilon[\exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]; T_b < T_0 \wedge e_\delta]$$

$$+ \mathbb{E}_\varepsilon[\exp[-\lambda_1 L_{\varepsilon, 0+}^\delta - \lambda_2 L_{\varepsilon, b-}^\delta]; e_\delta < T_0 \wedge T_b]$$

$$= \mathbb{P}_\varepsilon\{T_0 < T_b \wedge e_\delta\} I_\varepsilon(0) + \mathbb{P}_\varepsilon\{T_b < T_0 \wedge e_\delta\} I_\varepsilon(b) + \mathbb{P}_\varepsilon\{e_\delta < T_0 \wedge T_b\}$$

$$= \mathbb{E}_\varepsilon[e^{-\delta T_0}; T_0 < T_b] I_\varepsilon(0) + \mathbb{E}_\varepsilon[e^{-\delta T_b}; T_b < T_0] I_\varepsilon(b) + 1 - \mathbb{E}_\varepsilon e^{-\delta(T_0 \wedge T_b)}$$

$$= \frac{f_\delta(\varepsilon, b)}{f_\delta(0, b)} I_\varepsilon(0) + \frac{f_\delta(0, \varepsilon)}{f_\delta(0, b)} I_\varepsilon(b) + 1 - \frac{f_\delta(0, \varepsilon) + f_\delta(\varepsilon, b)}{f_\delta(0, b)}, \tag{3.5}$$

where the last step is due to (2.9). Substituting (3.5) into (3.4), solving for $I_\varepsilon(0)$, and taking the limit, we obtain

$$I_0(0) = \lim_{\varepsilon \to 0+} I_\varepsilon(0)$$

$$= \lim_{\varepsilon \to 0+} \left\{\left[\frac{\delta}{\delta + \lambda_1}\left(1 - \frac{g_{+,\delta+\lambda_1}(0)}{g_{+,\delta+\lambda_1}(\varepsilon)}\right)\right.\right.$$

$$\left. + \frac{g_{+,\delta+\lambda_1}(0)}{g_{+,\delta+\lambda_1}(\varepsilon)}\left(\frac{f_\delta(0, \varepsilon)}{f_\delta(0, b)} I_\varepsilon(b) + 1 - \frac{f_\delta(0, \varepsilon) + f_\delta(\varepsilon, b)}{f_\delta(0, b)}\right)\right]$$

$$\left. \times \left[1 - \frac{g_{+,\delta+\lambda_1}(0)}{g_{+,\delta+\lambda_1}(\varepsilon)} \frac{f_\delta(\varepsilon, b)}{f_\delta(0, b)}\right]^{-1}\right\}$$

$$= \left[\frac{\delta}{\delta + \lambda_1} \psi_{\delta + \lambda_1}^+(0) + \frac{f_{2,\delta}(0,0)}{f_\delta(0,b)} I_0(b) - \frac{f_{1,\delta}(0,b) + f_{2,\delta}(0,0)}{f_\delta(0,b)} \right]$$

$$\times \left[\psi_{\delta + \lambda_1}^+(0) - \frac{f_{1,\delta}(0,b)}{f_\delta(0,b)} \right]^{-1}. \tag{3.6}$$

Here, (3.6) is the first equation of the target terms $I_0(0)$ and $I_0(b)$ we obtained.

Similarly,

$$\begin{aligned}
I_\varepsilon(b) &= \mathbb{E}_b[\exp[-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta]; T_{b-\varepsilon} < e_\delta] \\
&\quad + \mathbb{E}_b[\exp[-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta]; e_\delta < T_{b-\varepsilon}] \\
&= \mathbb{E}_b[e^{-\lambda_2 T_{b-\varepsilon}}; T_{b-\varepsilon} < e_\delta] I_\varepsilon(b-\varepsilon) + \mathbb{E}_b[e^{-\lambda_2 e_\delta}; e_\delta < T_{b-\varepsilon}] \\
&= \mathbb{E}_b e^{-(\delta+\lambda_2) T_{b-\varepsilon}} I_\varepsilon(b-\varepsilon) + \frac{\delta}{\delta+\lambda_2}(1 - \mathbb{E}_b e^{-(\delta+\lambda_2) T_{b-\varepsilon}}) \\
&= \frac{g_{-,\delta+\lambda_2}(b)}{g_{-,\delta+\lambda_2}(b-\varepsilon)} I_\varepsilon(b-\varepsilon) + \frac{\delta}{\delta+\lambda_2}\left(1 - \frac{g_{-,\delta+\lambda_2}(b)}{g_{-,\delta+\lambda_2}(b-\varepsilon)}\right). \tag{3.7}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_\varepsilon(b-\varepsilon) &= \mathbb{E}_{b-\varepsilon}[\exp[-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta]; T_b < T_0 \wedge e_\delta] \\
&\quad + \mathbb{E}_{b-\varepsilon}[\exp[-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta]; T_0 < T_b \wedge e_\delta] \\
&\quad + \mathbb{E}_{b-\varepsilon}[\exp[-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta]; e_\delta < T_0 \wedge T_b] \\
&= \mathbb{P}_{b-\varepsilon}\{T_b < T_0 \wedge e_\delta\} I_\varepsilon(b) + \mathbb{P}_{b-\varepsilon}\{T_0 < T_b \wedge e_\delta\} I_\varepsilon(0) + \mathbb{P}_{b-\varepsilon}\{e_\delta < T_0 \wedge T_b\} \\
&= \mathbb{E}_{b-\varepsilon}[e^{-\delta T_b}; T_b < T_0] I_\varepsilon(b) + \mathbb{E}_{b-\varepsilon}[e^{-\delta T_0}; T_0 < T_b] I_\varepsilon(0) + 1 - \mathbb{E}_{b-\varepsilon} e^{-\delta(T_0 \wedge T_b)} \\
&= \frac{f_\delta(0, b-\varepsilon)}{f_\delta(0,b)} I_\varepsilon(b) + \frac{f_\delta(b-\varepsilon, b)}{f_\delta(0,b)} I_\varepsilon(0) + 1 - \frac{f_\delta(0, b-\varepsilon) + f_\delta(b-\varepsilon, b)}{f_\delta(0,b)}. \tag{3.8}
\end{aligned}$$

Substituting (3.8) into (3.7), solving for $I_\varepsilon(b)$, and taking the limit, we obtain

$$\begin{aligned}
I_0(b) &= \lim_{\varepsilon \to 0+} I_\varepsilon(b) \\
&= \lim_{\varepsilon \to 0+} \Bigg\{ \left[\frac{\delta}{\delta+\lambda_2}\left(1 - \frac{g_{-,\delta+\lambda_2}(b)}{g_{-,\delta+\lambda_2}(b-\varepsilon)}\right) \right. \\
&\quad + \left. \frac{g_{-,\delta+\lambda_2}(b)}{g_{-,\delta+\lambda_2}(b-\varepsilon)}\left(\frac{f_\delta(b-\varepsilon, b)}{f_\delta(0,b)} I_\varepsilon(0) + 1 - \frac{f_\delta(0, b-\varepsilon) + f_\delta(b-\varepsilon, b)}{f_\delta(0,b)}\right) \right] \\
&\quad \times \left[1 - \frac{g_{-,\delta+\lambda_2}(b)}{g_{-,\delta+\lambda_2}(b-\varepsilon)}\frac{f_\delta(0, b-\varepsilon)}{f_\delta(0,b)} \right]^{-1} \Bigg\} \\
&= \left[\frac{\delta}{\delta+\lambda_2} \psi_{\delta+\lambda_2}^-(b) - \frac{f_{1,\delta}(b,b)}{f_\delta(0,b)} I_0(0) + \frac{f_{2,\delta}(0,b) + f_{1,\delta}(b,b)}{f_\delta(0,b)} \right] \\
&\quad \times \left[\psi_{\delta+\lambda_2}^-(b) + \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)} \right]^{-1}. \tag{3.9}
\end{aligned}$$

Here, (3.9) is the second equation of the target terms $I_0(0)$ and $I_0(b)$ we obtained. Therefore, solving the linear system composed of (3.6) and (3.9), and simplifying the result using identity (2.13) with $x = 0$, $y = b$, and $r = \delta$, we complete the proof of Theorem 3.2.

Remark 3.1. When $\lambda_2 \to 0+$, the first joint Laplace transform in Theorem 3.2 is reduced to Theorem 3.1 using identity (2.14) with $x = 0$, $y = b$, and $r = \delta$. Furthermore, when $\delta \to 0+$, by (2.5), (2.6), and (2.8), both of the Laplace transforms in Theorem 3.2 are equal to 0.

Let $b \to 0+$ in Theorem 3.2, and by the facts that $f_{12,\delta}(0,0) = 0$ and $f_{1,\delta}(0,0) = -f_{2,\delta}(0,0)$, we obtain the following compact expression.

Corollary 3.3. *It holds that*

$$\mathbb{E} \exp\left[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{X_s<0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s>0\}} \, \mathrm{d}s\right]$$
$$= \frac{\delta\psi_{\delta+\lambda_1}^+(0)/(\delta+\lambda_1) + \delta\psi_{\delta+\lambda_2}^-(0)/(\delta+\lambda_2)}{\psi_{\delta+\lambda_1}^+(0) + \psi_{\delta+\lambda_2}^-(0)}.$$

By (2.11), letting $\lambda_2 \to \infty$ and $\lambda_1 \to \infty$ in the first and second equations of Theorem 3.2, respectively, we obtain the following result.

Corollary 3.4. *For any $a < 0$, we have*

$$\mathbb{E}\left[\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s<0\}} \, \mathrm{d}s\right]; e_\delta < T_b\right] = \frac{\delta f_\delta(0,b)\psi_{\delta+\lambda}^+(0)/(\delta+\lambda) - f_{2,\delta}(0,0) - f_{1,\delta}(0,b)}{f_\delta(0,b)\psi_{\delta+\lambda}^+(0) - f_{1,\delta}(0,b)}$$

and

$$\mathbb{E}_b\left[\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s>b\}} \, \mathrm{d}s\right]; e_\delta < T_0\right]$$
$$= \frac{\delta f_\delta(0,b)\psi_{\delta+\lambda}^-(b)/(\delta+\lambda) + f_{1,\delta}(b,b) + f_{2,\delta}(0,0)}{f_\delta(0,b)\psi_{\delta+\lambda}^-(b) + f_{2,\delta}(0,b)}.$$

Next we consider the joint Laplace transforms on the occupation times in a bounded region and an unbounded region.

Theorem 3.3. *For any $a < 0$, we have*

$$\mathbb{E} \exp\left[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{a \le X_s<0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s<a\}} \, \mathrm{d}s\right]$$
$$= \left[\frac{\delta(\delta+\lambda_2)f_{2,\delta+\lambda_1}(a,0) + \delta(\lambda_1-\lambda_2)f_{2,\delta+\lambda_1}(0,0)}{(\delta+\lambda_1)(\delta+\lambda_2)f_{\delta+\lambda_1}(a,0)}\psi_{\delta+\lambda_2}^+(a) - \frac{f_{1,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\psi_\delta^-(0)\right.$$
$$\left. + \psi_{\delta+\lambda_2}^+(a)\psi_\delta^-(0) - \frac{\delta f_{12,\delta+\lambda_1}(a,0)}{(\delta+\lambda_1)f_{\delta+\lambda_1}(a,0)}\right]$$
$$\times \left[\frac{f_{2,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\psi_{\delta+\lambda_2}^+(a) - \frac{f_{1,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\psi_\delta^-(0) + \psi_{\delta+\lambda_2}^+(a)\psi_\delta^-(0)\right.$$
$$\left. - \frac{f_{12,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\right]^{-1}$$

and

$$\mathbb{E}_a \exp\left[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{a \le X_s<0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s<a\}} \, \mathrm{d}s\right]$$
$$= \left[\frac{\delta f_{2,\delta+\lambda_1}(a,0)}{(\delta+\lambda_2)f_{\delta+\lambda_1}(a,0)}\psi_{\delta+\lambda_2}^+(a) - \frac{\delta f_{1,\delta+\lambda_1}(a,0) - \lambda_1 f_{2,\delta+\lambda_1}(a,a)}{(\delta+\lambda_1)f_{\delta+\lambda_1}(a,0)}\psi_\delta^-(0)\right.$$
$$\left. + \frac{\delta}{\delta+\lambda_2}\psi_{\delta+\lambda_2}^+(a)\psi_\delta^-(0) - \frac{\delta f_{12,\delta+\lambda_1}(a,0)}{(\delta+\lambda_1)f_{\delta+\lambda_1}(a,0)}\right]$$
$$\times \left[\frac{f_{2,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\psi_{\delta+\lambda_2}^+(a) - \frac{f_{1,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\psi_\delta^-(0) + \psi_{\delta+\lambda_2}^+(a)\psi_\delta^-(0)\right.$$
$$\left. - \frac{f_{12,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\right]^{-1}.$$

Proof. For arbitrarily small $\varepsilon > 0$, we approximate the occupation time $\int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, ds$ by $L_{\varepsilon,a,0}^\delta$, the sum of durations up to time e_δ for all the excursions of X that either start from 0 and end at ε, avoiding a, or start from 0 and end at a, avoiding ε. Similarly, we approximate $\int_0^{e_\delta} \mathbf{1}_{\{X_s < a\}} \, ds$ by $L_{\varepsilon,a+}^\delta$, the sum of durations up to time e_δ for all the excursions of X that start from a and end at $a + \varepsilon$.

We define $J_0(x) = \mathbb{E}_x \exp[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, ds - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s < a\}} \, ds]$ and its approximation $J_\varepsilon(x) = \mathbb{E}_x \exp[-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta]$ for some $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
J_\varepsilon(0) &= \mathbb{E}[\exp[-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta]; T_\varepsilon < T_a \wedge e_\delta] \\
&\quad + \mathbb{E}[\exp[-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta]; T_a < T_\varepsilon \wedge e_\delta] \\
&\quad + \mathbb{E}[\exp[-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta]; e_\delta < T_\varepsilon \wedge T_a] \\
&= \mathbb{E}[e^{-\lambda_1 T_\varepsilon}; T_\varepsilon < T_a \wedge e_\delta] J_\varepsilon(\varepsilon) + \mathbb{E}[e^{-\lambda_1 T_a}; T_a < T_\varepsilon \wedge e_\delta] J_\varepsilon(a) \\
&\quad + \mathbb{E}[e^{-\lambda_1 e_\delta}; e_\delta < T_\varepsilon \wedge T_a] \\
&= \mathbb{E}[e^{-(\delta+\lambda_1) T_\varepsilon}; T_\varepsilon < T_a] J_\varepsilon(\varepsilon) + \mathbb{E}[e^{-(\delta+\lambda_1) T_a}; T_a < T_\varepsilon] J_\varepsilon(a) \\
&\quad + \frac{\delta(1 - \mathbb{E}e^{-(\delta+\lambda_1)(T_\varepsilon \wedge T_a)})}{\delta + \lambda_1} \\
&= \frac{f_{\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,\varepsilon)} J_\varepsilon(\varepsilon) + \frac{f_{\delta+\lambda_1}(0,\varepsilon)}{f_{\delta+\lambda_1}(a,\varepsilon)} J_\varepsilon(a) + \frac{\delta}{\delta + \lambda_1} \left(1 - \frac{f_{\delta+\lambda_1}(a,0) + f_{\delta+\lambda_1}(0,\varepsilon)}{f_{\delta+\lambda_1}(a,\varepsilon)} \right).
\end{aligned}
$$
(3.10)

Furthermore,

$$
\begin{aligned}
J_\varepsilon(\varepsilon) &= \mathbb{P}_\varepsilon\{T_0 < e_\delta\} J_\varepsilon(0) + \mathbb{P}_\varepsilon\{e_\delta < T_0\} \\
&= \mathbb{E}_\varepsilon e^{-\delta T_0} J_\varepsilon(0) + 1 - \mathbb{E}_\varepsilon e^{-\delta T_0} \\
&= \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)} J_\varepsilon(0) + 1 - \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)},
\end{aligned}
$$
(3.11)

where the last step is due to (2.12). Substituting (3.11) into (3.10), solving for $J_\varepsilon(0)$, and taking the limit, we obtain

$$
\begin{aligned}
J_0(0) &= \lim_{\varepsilon \to 0+} J_\varepsilon(0) \\
&= \lim_{\varepsilon \to 0+} \left\{ \left[\frac{f_{\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,\varepsilon)} \left(1 - \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)} \right) + \frac{f_{\delta+\lambda_1}(0,\varepsilon)}{f_{\delta+\lambda_1}(a,\varepsilon)} J_\varepsilon(a) \right. \right. \\
&\qquad\qquad \left. + \frac{\delta}{\delta + \lambda_1} \left(1 - \frac{f_{\delta+\lambda_1}(a,0) + f_{\delta+\lambda_1}(0,\varepsilon)}{f_{\delta+\lambda_1}(a,\varepsilon)} \right) \right] \\
&\qquad\qquad \left. \times \left[1 - \frac{f_{\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,\varepsilon)} \frac{g_{-,\delta}(\varepsilon)}{g_{-,\delta}(0)} \right]^{-1} \right\} \\
&= \left[\psi_\delta^-(0) + \frac{f_{2,\delta+\lambda_1}(0,0)}{f_{\delta+\lambda_1}(a,0)} J_0(a) + \frac{\delta}{\delta + \lambda_1} \frac{f_{2,\delta+\lambda_1}(a,0) - f_{2,\delta+\lambda_1}(0,0)}{f_{\delta+\lambda_1}(a,0)} \right] \\
&\quad \times \left[\psi_\delta^-(0) + \frac{f_{2,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)} \right]^{-1}.
\end{aligned}
$$
(3.12)

Here, (3.12) is the first equation of the target terms $J_0(0)$ and $J_0(a)$ we obtained.

Similarly,

$$
\begin{aligned}
J_\varepsilon(a) &= \mathbb{E}_a[\exp[-\lambda_1 L^\delta_{\varepsilon,a,0} - \lambda_2 L^\delta_{\varepsilon,a+}]; T_{a+\varepsilon} < e_\delta] \\
&\quad + \mathbb{E}_a[\exp[-\lambda_1 L^\delta_{\varepsilon,a,0} - \lambda_2 L^\delta_{\varepsilon,a+}]; e_\delta < T_{a+\varepsilon}] \\
&= \mathbb{E}_a[e^{-\lambda_2 T_{a+\varepsilon}}; T_{a+\varepsilon} < e_\delta]J_\varepsilon(a+\varepsilon) + \mathbb{E}_a[e^{-\lambda_2 e_\delta}; e_\delta < T_{a+\varepsilon}] \\
&= \mathbb{E}_a e^{-(\delta+\lambda_2)T_{a+\varepsilon}} J_\varepsilon(a+\varepsilon) + \frac{\delta}{\delta+\lambda_2}(1 - \mathbb{E}_a e^{-(\delta+\lambda_2)T_{a+\varepsilon}}) \\
&= \frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)} J_\varepsilon(a+\varepsilon) + \frac{\delta}{\delta+\lambda_2}\left(1 - \frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\right). \qquad (3.13)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
J_\varepsilon(a+\varepsilon) &= \mathbb{E}_{a+\varepsilon}[\exp[-\lambda_1 L^\delta_{\varepsilon,a,0} - \lambda_2 L^\delta_{\varepsilon,a+}]; T_a < T_0 \wedge e_\delta] \\
&\quad + \mathbb{E}_{a+\varepsilon}[\exp[-\lambda_1 L^\delta_{\varepsilon,a,0} - \lambda_2 L^\delta_{\varepsilon,a+}]; T_0 < T_a \wedge e_\delta] \\
&\quad + \mathbb{E}_{a+\varepsilon}[\exp[-\lambda_1 L^\delta_{\varepsilon,a,0} - \lambda_2 L^\delta_{\varepsilon,a+}]; e_\delta < T_a \wedge T_0] \\
&= \mathbb{E}_{a+\varepsilon}[e^{-\lambda_1 T_a}; T_a < T_0 \wedge e_\delta]J_\varepsilon(a) + \mathbb{E}_{a+\varepsilon}[e^{-\lambda_1 T_0}; T_0 < T_a \wedge e_\delta]J_\varepsilon(0) \\
&\quad + \mathbb{E}_{a+\varepsilon}[e^{-\lambda_1 e_\delta}; e_\delta < T_a \wedge T_0] \\
&= \mathbb{E}_{a+\varepsilon}[e^{-(\delta+\lambda_1)T_a}; T_a < T_0]J_\varepsilon(a) + \mathbb{E}_{a+\varepsilon}[e^{-(\delta+\lambda_1)T_0}; T_0 < T_a]J_\varepsilon(0) \\
&\quad + \frac{\delta}{\delta+\lambda_1}(1 - \mathbb{E}_{a+\varepsilon}[e^{-(\delta+\lambda_1)(T_a \wedge T_0)}]) \\
&= \frac{f_{\delta+\lambda_1}(a+\varepsilon,0)}{f_{\delta+\lambda_1}(a,0)} J_\varepsilon(a) + \frac{f_{\delta+\lambda_1}(a,a+\varepsilon)}{f_{\delta+\lambda_1}(a,0)} J_\varepsilon(0) \\
&\quad + \frac{\delta}{\delta+\lambda_1}\left(1 - \frac{f_{\delta+\lambda_1}(a,a+\varepsilon) + f_{\delta+\lambda_1}(a+\varepsilon,0)}{f_{\delta+\lambda_1}(a,0)}\right). \qquad (3.14)
\end{aligned}
$$

Substituting (3.14) into (3.13), solving for $J_\varepsilon(a)$, and taking the limit, we obtain

$$
\begin{aligned}
J_0(a) &= \lim_{\varepsilon\to 0+} J_\varepsilon(a) \\
&= \lim_{\varepsilon\to 0+}\left\{\left[\frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\frac{\delta}{\delta+\lambda_1}\left(1 - \frac{f_{\delta+\lambda_1}(a,a+\varepsilon)+f_{\delta+\lambda_1}(a+\varepsilon,0)}{f_{\delta+\lambda_1}(a,0)}\right)\right.\right. \\
&\qquad\qquad \left.+ \frac{\delta}{\delta+\lambda_2}\left(1 - \frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\right)\right] \\
&\qquad\qquad \left.\times\left[1 - \frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\frac{f_{\delta+\lambda_1}(a+\varepsilon,0)}{f_{\delta+\lambda_1}(a,0)}\right]^{-1}\right\} \\
&\quad + \lim_{\varepsilon\to 0+}\left\{J_\varepsilon(0)\left[\frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\frac{f_{\delta+\lambda_1}(a,a+\varepsilon)}{f_{\delta+\lambda_1}(a,0)}\right]\right. \\
&\qquad\qquad \left.\times\left[1 - \frac{g_{+,\delta+\lambda_2}(a)}{g_{+,\delta+\lambda_2}(a+\varepsilon)}\frac{f_{\delta+\lambda_1}(a+\varepsilon,0)}{f_{\delta+\lambda_1}(a,0)}\right]^{-1}\right\} \\
&= \left[\frac{\delta}{\delta+\lambda_1}\frac{-f_{1,\delta+\lambda_1}(a,0)-f_{2,\delta+\lambda_1}(a,a)}{f_{\delta+\lambda_1}(a,0)} + \frac{\delta}{\delta+\lambda_2}\psi^+_{\delta+\lambda_2}(a) + \frac{f_{2,\delta+\lambda_1}(a,a)}{f_{\delta+\lambda_1}(a,0)}J_0(0)\right] \\
&\quad \times\left[\psi^+_{\delta+\lambda_2}(a) - \frac{f_{1,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}\right]^{-1}. \qquad (3.15)
\end{aligned}
$$

Here, (3.15) is the second equation of the target terms $J_0(0)$ and $J_0(a)$ we obtained. Finally, solving the linear system composed of (3.12) and (3.15), and simplifying the result using identity (2.13) with $x = a$, $y = 0$, and $r = \delta + \lambda_1$, we complete the proof of Theorem 3.3.

Remark 3.2. When $\lambda_1 = \lambda_2 = \lambda$, the first joint Laplace transform in Theorem 3.3 is reduced to Theorem 3.1 using identity (2.15) with $x = a$, $y = 0$, and $r = \delta + \lambda$. Furthermore, when $\lambda_1 \to 0+$, $\lambda_2 = \lambda$, and $a = 0$, the second joint Laplace transform in Theorem 3.3 is reduced to Theorem 3.1 using identity (2.14) with $x = a$, $y = 0$, and $r = \delta$.

Letting $\delta \to 0+$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.5. *For any $a < 0$, we have*

$$\mathbb{E} \exp\left[-\lambda_1 \int_0^\infty \mathbf{1}_{\{a \leq X_s < 0\}} \, ds - \lambda_2 \int_0^\infty \mathbf{1}_{\{X_s < a\}} \, ds \right]$$

$$= \frac{-f_{1,\lambda_1}(a,0)\psi_0^-(0) + f_{\lambda_1}(a,0)\psi_{\lambda_2}^+(a)\psi_0^-(0)}{f_{2,\lambda_1}(a,0)\psi_{\lambda_2}^+(a) - f_{1,\lambda_1}(a,0)\psi_0^-(0) + f_{\lambda_1}(a,0)\psi_{\lambda_2}^+(a)\psi_0^-(0) - f_{12,\lambda_1}(a,0)}$$

and

$$\mathbb{E}_a \exp\left[-\lambda_1 \int_0^\infty \mathbf{1}_{\{a \leq X_s < 0\}} \, ds - \lambda_2 \int_0^\infty \mathbf{1}_{\{X_s < a\}} \, ds \right]$$

$$= \frac{f_{2,\delta+\lambda_1}(a,a)\psi_0^-(0)}{f_{2,\lambda_1}(a,0)\psi_{\lambda_2}^+(a) - f_{1,\lambda_1}(a,0)\psi_0^-(0) + f_{\lambda_1}(a,0)\psi_{\lambda_2}^+(a)\psi_0^-(0) - f_{12,\lambda_1}(a,0)}.$$

Letting $\lambda_2 \to 0+$ in Theorem 3.3, and using the identity $f_{2,\delta+\lambda}(a,a) = -f_{1,\delta+\lambda}(a,a)$, we obtain the following result.

Corollary 3.6. *For any $a < 0$, we have*

$$\mathbb{E} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, ds \right]$$

$$= \left[\frac{\delta f_{2,\delta+\lambda}(a,0) + \lambda f_{2,\delta+\lambda}(0,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a)\psi_\delta^-(0) \right.$$

$$\left. - \frac{\delta f_{12,\delta+\lambda}(a,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \right]$$

$$\times \left[\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a)\psi_\delta^-(0) - \frac{f_{12,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \right]^{-1}$$

and

$$\mathbb{E}_a \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, ds \right]$$

$$= \left[\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{\delta f_{1,\delta+\lambda}(a,0) + \lambda f_{1,\delta+\lambda}(a,a)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a)\psi_\delta^-(0) \right.$$

$$\left. - \frac{\delta f_{12,\delta+\lambda}(a,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \right]$$

$$\times \left[\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a)\psi_\delta^-(0) - \frac{f_{12,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \right]^{-1}.$$

Letting $\delta \to 0+$ in Corollary 3.6, we obtain the following simplified expression.

Corollary 3.7. *For any $a < 0$, we have*

$$\mathbb{E} \exp\left[-\lambda \int_0^\infty \mathbf{1}_{\{a \le X_s < 0\}} \, ds\right]$$

$$= \frac{f_{2,\lambda}(0,0)\psi_0^+(a) - f_{1,\lambda}(a,0)\psi_0^-(0) + f_\lambda(a,0)\psi_0^+(a)\psi_0^-(0)}{f_{2,\lambda}(a,0)\psi_0^+(a) - f_{1,\lambda}(a,0)\psi_0^-(0) + f_\lambda(a,0)\psi_0^+(a)\psi_0^-(0) - f_{12,\lambda}(a,0)}$$

and

$$\mathbb{E}_a \exp\left[-\lambda \int_0^\infty \mathbf{1}_{\{a \le X_s < 0\}} \, ds\right]$$

$$= \frac{f_{2,\lambda}(a,0)\psi_0^+(a) - f_{1,\lambda}(a,a)\psi_0^-(0) + f_\lambda(a,0)\psi_0^+(a)\psi_0^-(0)}{f_{2,\lambda}(a,0)\psi_0^+(a) - f_{1,\lambda}(a,0)\psi_0^-(0) + f_\lambda(a,0)\psi_0^+(a)\psi_0^-(0) - f_{12,\lambda}(a,0)}.$$

Remark 3.3. As $a \to -\infty$, by (2.10), the first Laplace transforms in Corollaries 3.6 and 3.7 are reduced to Theorem 3.1 and Corollary 3.1, respectively.

Letting $\lambda_2 \to \infty$ in the first equation of Theorem 3.3, by (2.11), we obtain the following result.

Corollary 3.8. *For any $a < 0$, we have*

$$\mathbb{E}\left[\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \le X_s < 0\}} \, ds\right]; e_\delta < T_a\right]$$

$$= \frac{\delta(f_{2,\delta+\lambda}(a,0) - f_{2,\delta+\lambda}(0,0))/(\delta+\lambda) + f_{\delta+\lambda}(a,0)\psi_\delta^-(0)}{f_{2,\delta+\lambda}(a,0) + f_{\delta+\lambda}(a,0)\psi_\delta^-(0)}.$$

4. Some applications to diffusion risk processes

The Laplace transforms in Section 3 are motivated by the study of two classes of risk models that we are going to introduce below.

4.1. The risk process with random observations

Let the surplus process X in a risk model be the time-homogeneous diffusion process defined in (2.1) with initial value $x_0 > 0$, and let N be an independent Poisson process with constant rate $\lambda > 0$ of consecutive arrival times $0 < \tau_1 < \tau_2 < \cdots$. Suppose that the values of the process X are only observed at the times (τ_i). The ruin time for this model is defined as

$$\tau_\lambda = \inf_{i \ge 1}\{\tau_i : X_{\tau_i} < 0\},$$

with the convention that $\inf \varnothing = \infty$.

There is a natural connection between the survival probability for this risk process and the occupation time for the underlying process X. Actually, for any $t > 0$,

$$\mathbb{P}_{x_0}\{\tau_\lambda > t\} = \mathbb{E}_{x_0}\mathbb{P}\{\{\tau_i\} \cap \{s \le t : X_s < 0\} = \varnothing \mid X\}$$

$$= \mathbb{E}_{x_0}\mathbb{E}\left[\exp\left[-\lambda \int_0^t \mathbf{1}_{\{X_s < 0\}} \, ds\right] \;\middle|\; X\right]$$

$$= \mathbb{E}_{x_0}\exp\left[-\lambda \int_0^t \mathbf{1}_{\{X_s < 0\}} \, ds\right].$$

Therefore, the Laplace transform for the ruin time τ_λ is

$$\mathbb{E}_{x_0} e^{-\delta \tau_\lambda} = \mathbb{P}_{x_0}\{\tau_\lambda < e_\delta\} = 1 - \mathbb{E}_{x_0} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, \mathrm{d}s\right].$$

Furthermore, by Corollary 3.2,

$$\mathbb{E}_{x_0} e^{-\delta \tau_\lambda} = \frac{\lambda}{\delta + \lambda} \frac{g_{-,\delta}(x_0)}{g_{-,\delta}(0)} \frac{\psi^+_{\delta+\lambda}(0)}{\psi^+_{\delta+\lambda}(0) + \psi^-_\delta(0)}. \tag{4.1}$$

Note that, when $\lambda \to \infty$ and $\delta \to 0+$, (4.1) is reduced to the usual ruin probability by (2.5) and (2.11).

Moreover, for $b > 0$, write

$$\tau_{\lambda,b} = \inf_{i \geq 1}\{\tau_i : X_{\tau_i} < 0 \text{ or } X_{\tau_i} > b\}$$

for the 'two-sided exit time' of this model. Then, by conditioning, its Laplace transform is given by

$$\mathbb{E}_{x_0} e^{-\delta \tau_{\lambda,b}} = \mathbb{P}_{x_0}\{\tau_{\lambda,b} < e_\delta\} = 1 - \mathbb{E}_{x_0} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, \mathrm{d}s - \lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s > b\}} \, \mathrm{d}s\right],$$

which can be found by Theorem 3.2.

Fixing a level $a \leq 0$ and a positive integer m, for the abovementioned diffusion risk model with exponential interobservation times, we can further propose another generalized ruin as the event that either there is at least one observation of X below level a or there are at least m observations of X between level a and level 0. Note that such a ruin has the spirit of a new ruin model subject to Chapters 7 and 11 of the bankruptcy code which was first introduced in [16]. Write $\tau_\lambda^{m,a}$ for the first Poisson arrival time τ_i when such a ruin occurs. By definition, $\tau_\lambda^{1,-\infty} = \tau_\lambda$.

Conditioning on X and using the properties for Poisson processes, for any $t > 0$, we obtain

$$\mathbb{P}_{x_0}\{\tau_\lambda^{m,a} > t\}$$

$$= \sum_{i=0}^{m-1} \mathbb{E}_{x_0}\left[\frac{(\lambda \int_0^t \mathbf{1}_{\{a \leq X_s < 0\}} \, \mathrm{d}s)^i}{i!} \exp\left[-\lambda \int_0^t \mathbf{1}_{\{a \leq X_s < 0\}} \, \mathrm{d}s\right] \exp\left[-\lambda \int_0^t \mathbf{1}_{\{X_s < a\}} \, \mathrm{d}s\right]\right]$$

$$= \sum_{i=0}^{m-1} \mathbb{E}_{x_0}\left[\frac{(\lambda \int_0^t \mathbf{1}_{\{a \leq X_s < 0\}} \, \mathrm{d}s)^i}{i!} \exp\left[-\lambda \int_0^t \mathbf{1}_{\{X_s < 0\}} \, \mathrm{d}s\right]\right].$$

Therefore, the Laplace transform for the ruin time $\tau_\lambda^{m,a}$ is

$$\mathbb{E}_{x_0} e^{-\delta \tau_\lambda^{m,a}} = \mathbb{P}_{x_0}\{\tau_\lambda^{m,a} < e_\delta\}$$

$$= 1 - \sum_{i=0}^{m-1} \mathbb{E}_{x_0}\left[\frac{(\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, \mathrm{d}s)^i}{i!} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, \mathrm{d}s\right]\right].$$

In order to find the above Laplace transform for the generalized ruin time $\tau_\lambda^{m,a}$, we only need to recover the joint Laplace transform $\mathbb{E}_{x_0} \exp[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{a \leq X_s < 0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s < a\}} \, \mathrm{d}s]$, which can be found by Theorem 3.3.

Remark 4.1. Note that this generalized risk model with random observations was first introduced in [2] and [3] for compound Poisson risk processes with independent Erlang(n) interobservation times. Our model with exponential interarrival times is a special case with $n = 1$. Unfortunately, it is not apparent that the ruin probability for this general risk model with Erlang(n) interarrival times in [2] and [3] can be associated to the occupation times in a nice way due to the lack-of-memoryless property.

4.2. The omega risk process

Let X be the time-homogeneous diffusion process defined in (2.1) with initial value $x_0 > 0$. The so-called omega model was first introduced in [4]. We refer the reader to [1] and [12] for more recent results on the omega model. The event of bankruptcy is introduced for such a model. In contrast to traditional ruin, the bankruptcy is not immediate even if the surplus is negative. To specify the bankruptcy probability, a bankruptcy rate function $\omega(x) \geq 0$, $x \leq 0$, is introduced, which is typically a decreasing function. Then, whenever the surplus is at level $x \leq 0$, $\omega(x)\,\mathrm{d}t$ is the probability of bankruptcy within $\mathrm{d}t$ time units. To be more precise, we can define an auxiliary process N on the same probability space as X so that, given X, process N follows the law of a Poisson process with time-dependent intensity $\omega(X_s)\,\mathbf{1}_{\{X_s < 0\}}$, $s > 0$, i.e. N is a Cox process. We then define the bankruptcy time, denoted by τ_ω, as the first arrival time for N. One can also simply define

$$\tau_\omega = \inf\left\{t \geq 0 \colon \int_0^t \omega(X_s)\,\mathrm{d}s > e_1\right\}$$

for an independent rate-one exponential random variable e_1; see [17]. Then the risk model considered in [15] corresponds to the omega risk process with a constant rate function ω.

The probability of bankruptcy for the omega risk process can be determined as follows: for any $t > 0$,

$$\mathbb{P}_{x_0}\{\tau_\omega > t\} = \mathbb{P}_{x_0}\{N_t = 0\} = \mathbb{E}_{x_0}\exp\left[-\int_0^t \omega(X_s)\,\mathbf{1}_{\{X_s < 0\}}\,\mathrm{d}s\right].$$

Therefore, the Laplace transform for the bankruptcy time τ_ω is

$$\mathbb{E}_{x_0}\mathrm{e}^{-\delta\tau_\omega} = \mathbb{P}_{x_0}\{\tau_\omega < e_\delta\} = 1 - \mathbb{E}_{x_0}\exp\left[-\int_0^{e_\delta} \omega(X_s)\,\mathbf{1}_{\{X_s < 0\}}\,\mathrm{d}s\right]. \qquad (4.2)$$

For a constant bankruptcy rate $\omega(x) \equiv \omega$, the bankruptcy probability of the omega risk process coincides with the ruin probability of the risk process with random observations, which has been pointed out in [12]. Therefore, by (4.1) and (4.2),

$$\mathbb{E}_{x_0}\mathrm{e}^{-\delta\tau_\omega} = 1 - \mathbb{E}_{x_0}\exp\left[-\omega\int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\,\mathrm{d}s\right] = \frac{\omega}{\delta + \omega}\frac{g_{-,\delta}(x_0)}{g_{-,\delta}(0)}\frac{\psi_{\delta+\omega}^+(0)}{\psi_{\delta+\omega}^+(0) + \psi_\delta^-(0)}.$$

For a piecewise-constant bankruptcy rate

$$\omega(x) = \lambda_1\,\mathbf{1}_{\{a \leq x < 0\}} + \lambda_2\,\mathbf{1}_{\{-\infty < x < a\}} \quad \text{for some } a < 0;$$

the corresponding Laplace transform (4.2) can be found by Theorem 3.3.

The omega risk process with rate function $\omega(x) = \lambda\,\mathbf{1}_{\{a \leq x < 0\}} + \infty\,\mathbf{1}_{\{-\infty < x < a\}}$ is of particular interest. For this process, bankruptcy occurs at rate λ when the surplus is between a and 0, and bankruptcy occurs immediately once the surplus reaches level a. It can also be treated as a Chapters 7 and 11 type ruin of [16] for the current model. For such an omega risk process

with diffusion surplus process, the Laplace transform for the bankruptcy time is given by

$$\lim_{\lambda_2 \to \infty} \mathbb{E}_{x_0} \exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \le X_s < 0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s < a\}} \, \mathrm{d}s \right]$$

$$= \mathbb{E}_{x_0}\left[\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{a \le X_s < 0\}} \, \mathrm{d}s \right]; e_\delta < T_a \right],$$

which can be found by Corollary 3.8.

5. Examples

We apply the results in Section 3 to two examples in order to obtain more explicit expressions and to compare them with the known results.

5.1. Brownian motion with drift

Let $X_t = \mu t + W_t$ be a Brownian motion with drift. The corresponding differential equation (2.3) is reduced to

$$\tfrac{1}{2} g''(x) + \mu g'(x) = r g(x), \qquad r > 0.$$

Defining

$$\beta_r^\pm = -\mu \pm \sqrt{\mu^2 + 2r},$$

we have

$$g_{\pm,r}(x) = \mathrm{e}^{\beta_r^\pm x} \quad \text{and} \quad \psi_r^\pm(\cdot) \equiv \pm\beta_r^\pm.$$

Moreover,

$$f_r(y, z) = \mathrm{e}^{\beta_r^- y + \beta_r^+ z} - \mathrm{e}^{\beta_r^- z + \beta_r^+ y},$$

$$f_{1,r}(y, z) = \beta_r^- \mathrm{e}^{\beta_r^- y + \beta_r^+ z} - \beta_r^+ \mathrm{e}^{\beta_r^- z + \beta_r^+ y},$$

$$f_{2,r}(y, z) = \beta_r^+ \mathrm{e}^{\beta_r^- y + \beta_r^+ z} - \beta_r^- \mathrm{e}^{\beta_r^- z + \beta_r^+ y},$$

and

$$f_{12,r}(y, z) = -2r\mathrm{e}^{\beta_r^- y + \beta_r^+ z} + 2r\mathrm{e}^{\beta_r^- z + \beta_r^+ y}.$$

By Theorem 3.2, we obtain

$$\mathbb{E} \exp\left[-\lambda_1 \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}} \, \mathrm{d}s - \lambda_2 \int_0^{e_\delta} \mathbf{1}_{\{X_s > b\}} \, \mathrm{d}s \right]$$

$$= \frac{C_1 \mathrm{e}^{\beta_\delta^+ b} + C_2 \mathrm{e}^{\beta_\delta^- b} + \lambda_2(\beta_\delta^+ - \beta_\delta^-)\beta_{\delta+\lambda_2}^- / (\delta + \lambda_2)}{D_1 \mathrm{e}^{\beta_\delta^+ b} + D_2 \mathrm{e}^{\beta_\delta^- b}},$$

where

$$C_1 = \frac{\delta}{\delta + \lambda_1} \beta_\delta^+ \beta_{\delta+\lambda_1}^+ + \beta_\delta^- \beta_{\delta+\lambda_2}^- - \frac{\delta}{\delta + \lambda_1} \beta_{\delta+\lambda_1}^+ \beta_{\delta+\lambda_2}^- + 2\delta,$$

$$C_2 = -\frac{\delta}{\delta + \lambda_1} \beta_\delta^- \beta_{\delta+\lambda_1}^+ - \beta_\delta^+ \beta_{\delta+\lambda_2}^- + \frac{\delta}{\delta + \lambda_1} \beta_{\delta+\lambda_1}^+ \beta_{\delta+\lambda_2}^- - 2\delta,$$

$$D_1 = \beta_\delta^+ \beta_{\delta+\lambda_1}^+ + \beta_\delta^- \beta_{\delta+\lambda_2}^- - \beta_{\delta+\lambda_1}^+ \beta_{\delta+\lambda_2}^- + 2\delta,$$

and

$$D_2 = -\beta_\delta^- \beta_{\delta+\lambda_1}^+ - \beta_\delta^+ \beta_{\delta+\lambda_2}^- + \beta_{\delta+\lambda_1}^+ \beta_{\delta+\lambda_2}^- - 2\delta.$$

By Theorem 3.3, we obtain

$$\mathbb{E}\exp\left[-\lambda_1\int_0^{e_\delta}\mathbf{1}_{\{a\le X_s<0\}}\,\mathrm{d}s-\lambda_2\int_0^{e_\delta}\mathbf{1}_{\{X_s<a\}}\,\mathrm{d}s\right]$$

$$=\frac{C_3 e^{\beta_{\delta+\lambda_1}^+ a}+C_4 e^{\beta_{\delta+\lambda_1}^- a}+\delta(\lambda_1-\lambda_2)(\beta_{\delta+\lambda_1}^+-\beta_{\delta+\lambda_1}^-)\beta_{\delta+\lambda_2}^+/(\delta+\lambda_1)(\delta+\lambda_2)}{D_3 e^{\beta_{\delta+\lambda_1}^+ a}+D_4 e^{\beta_{\delta+\lambda_1}^- a}},$$

where

$$C_3=-\beta_\delta^-\beta_{\delta+\lambda_1}^++\beta_\delta^-\beta_{\delta+\lambda_2}^+-\frac{\delta}{\delta+\lambda_1}\beta_{\delta+\lambda_1}^-\beta_{\delta+\lambda_2}^+-2\delta,$$

$$C_4=\beta_\delta^-\beta_{\delta+\lambda_1}^--\beta_\delta^-\beta_{\delta+\lambda_2}^++\frac{\delta}{\delta+\lambda_1}\beta_{\delta+\lambda_1}^+\beta_{\delta+\lambda_2}^++2\delta,$$

$$D_3=-\beta_\delta^-\beta_{\delta+\lambda_1}^++\beta_\delta^-\beta_{\delta+\lambda_2}^+-\beta_{\delta+\lambda_1}^-\beta_{\delta+\lambda_2}^+-2(\delta+\lambda_1),$$

and

$$D_4=\beta_\delta^-\beta_{\delta+\lambda_1}^--\beta_\delta^-\beta_{\delta+\lambda_2}^++\beta_{\delta+\lambda_1}^+\beta_{\delta+\lambda_2}^++2(\delta+\lambda_1).$$

In the following we present several results on the occupation times of X with relatively short expressions.

By Corollary 3.1, we have

$$\mathbb{E}\exp\left[-\lambda\int_0^\infty\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]=\frac{2\mu}{\mu+\sqrt{\mu^2+2\lambda}}\quad\text{if }\mu\ge0,$$

which agrees with Formula 1.4.3 of [6, p. 255].

By Corollary 3.2, we have

$$\mathbb{E}_x\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right]=\begin{cases}1-\dfrac{\lambda}{\delta+\lambda}\dfrac{\beta_{\delta+\lambda}^+}{\beta_{\delta+\lambda}^+-\beta_\delta^-}e^{\beta_\delta^- x}&\text{if }x\ge0,\\[2mm]\dfrac{\delta}{\delta+\lambda}+\dfrac{\lambda}{\delta+\lambda}\dfrac{-\beta_\delta^-}{\beta_{\delta+\lambda}^+-\beta_\delta^-}e^{\beta_{\delta+\lambda}^+ x}&\text{if }x\le0,\end{cases}$$

which agrees with Formula 1.4.1 of [6, pp. 254, 255].

By Corollary 3.3,

$$\mathbb{E}\exp\left[-\lambda_1\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s-\lambda_2\int_0^{e_\delta}\mathbf{1}_{\{X_s>0\}}\,\mathrm{d}s\right]=\frac{2\delta}{-\beta_{\delta+\lambda_1}^-\beta_{\delta+\lambda_2}^+},$$

which agrees with Formula 1.6.1 of [6, p. 258].

By Corollary 3.4,

$$\mathbb{E}\left[\exp\left[-\lambda\int_0^{e_\delta}\mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s\right];e_\delta<T_b\right]$$

$$=\frac{(\delta\beta_{\delta+\lambda}^+/(\delta+\lambda)-\beta_\delta^-)e^{\beta_\delta^+ b}+(\beta_\delta^+-\delta\beta_{\delta+\lambda}^+/(\delta+\lambda))e^{\beta_\delta^- b}-\beta_\delta^++\beta_\delta^-}{(\beta_{\delta+\lambda}^+-\beta_\delta^-)e^{\beta_\delta^+ b}+(\beta_\delta^+-\beta_{\delta+\lambda}^+)e^{\beta_\delta^- b}}.$$

By Corollary 3.7,

$$\mathbb{E}\exp\left[-\lambda\int_0^\infty \mathbf{1}_{\{a\leq X_s<0\}}\,\mathrm{d}s\right] = \frac{\mu\beta_\lambda^+ \mathrm{e}^{\beta_\lambda^+ a} - \mu\beta_\lambda^- \mathrm{e}^{\beta_\lambda^- a}}{(\mu\beta_\lambda^+ - \lambda)\mathrm{e}^{\beta_\lambda^+ a} - (\mu\beta_\lambda^- - \lambda)\mathrm{e}^{\beta_\lambda^- a}} \quad \text{if } \mu \geq 0,$$

which agrees with Formula 1.7.3 of [6, p. 262].

By Corollary 3.8,

$$\mathbb{E}\left[\exp\left[-\lambda\int_0^{e_\delta} \mathbf{1}_{\{a\leq X_s<0\}}\,\mathrm{d}s\right]; e_\delta < T_a\right]$$
$$= \frac{(\beta_\delta^- - \delta\beta_{\delta+\lambda}^-/(\delta+\lambda))\mathrm{e}^{\beta_{\delta+\lambda}^+ a} + (\delta\beta_{\delta+\lambda}^+/(\delta+\lambda) - \beta_\delta^-)\mathrm{e}^{\beta_{\delta+\lambda}^- a} - \delta(\beta_{\delta+\lambda}^+ - \beta_{\delta+\lambda}^-)/(\delta+\lambda)}{(\beta_\delta^- - \beta_{\delta+\lambda}^-)\mathrm{e}^{\beta_{\delta+\lambda}^+ a} + (\beta_{\delta+\lambda}^+ - \beta_\delta^-)\mathrm{e}^{\beta_{\delta+\lambda}^- a}},$$

which agrees with Formula 2.7.1(1) of [6, p. 301].

5.2. Brownian motion with two-valued drift

Let X be a Brownian motion with two-valued drift, specified by the stochastic differential equation

$$\mathrm{d}X_t = (\mu_L\,\mathbf{1}_{(-\infty,0)}(X_t) - \mu_R\,\mathbf{1}_{(0,\infty)}(X_t))\,\mathrm{d}t + \mathrm{d}W_t, \tag{5.1}$$

where $\mu_L, \mu_R \in \mathbb{R}$ and W_t is a standard one-dimensional Brownian motion. The Brownian motion with two-valued drift, referred to as refracted Brownian motion, is also of interest to risk theory; see, for example, [5] and [11]. Although the Lipschitz assumption (2.2) for the drift function $\mu(\cdot) = \mu_L\,\mathbf{1}_{(-\infty,0)}(\cdot) - \mu_R\,\mathbf{1}_{(0,\infty)}(\cdot)$ fails, (5.1) still has a unique strong solution; see [20] and [22]. We refer the reader to [21] for recent work on Brownian motion with two-valued drift.

In this two-valued drift model, for $r > 0$, suppose that $g_{-,r}(\cdot)$ and $g_{+,r}(\cdot)$ are two independent, positive, and convex solutions of the equation

$$\tfrac{1}{2}g''(x) + (\mu_L\,\mathbf{1}_{(-\infty,0)}(x) - \mu_R\,\mathbf{1}_{(0,\infty)}(x))g'(x) - rg(x) = 0,$$

with $g_{-,r}(\cdot)$ strictly decreasing and $g_{+,r}(\cdot)$ strictly increasing. We choose

$$g_{-,r}(x) = \exp[(\mu_R - \sqrt{\mu_R^2 + 2r}\,)x]\,\mathbf{1}_{\{x>0\}}$$
$$+ [c_-\exp[(-\mu_L + \sqrt{\mu_L^2 + 2r}\,)x]$$
$$+ (1 - c_-)\exp[(-\mu_L - \sqrt{\mu_L^2 + 2r}\,)x]]\,\mathbf{1}_{\{x<0\}} \tag{5.2}$$

and

$$g_{+,r}(x) = [c_+\exp[(\mu_R + \sqrt{\mu_R^2 + 2r}\,)x] + (1 - c_+)\exp[(\mu_R - \sqrt{\mu_R^2 + 2r}\,)x]]\,\mathbf{1}_{\{x>0\}}$$
$$+ \exp[(-\mu_L + \sqrt{\mu_L^2 + 2r}\,)x]\,\mathbf{1}_{\{x<0\}}, \tag{5.3}$$

where

$$c_- = \frac{\mu_R - \sqrt{\mu_R^2 + 2r} + \mu_L + \sqrt{\mu_L^2 + 2r}}{2\sqrt{\mu_L^2 + 2r}}$$

$$\text{and} \quad c_+ = \frac{-\mu_L + \sqrt{\mu_L^2 + 2r} - \mu_R + \sqrt{\mu_R^2 + 2r}}{2\sqrt{\mu_R^2 + 2r}},$$

so that the functions $g_{\pm,r}(\cdot)$ are differentiable at 0. Hence, it is easy to see that

$$\psi_r^-(0) = -\frac{g'_{-,r}(0)}{g_{-,r}(0)} = -\mu_R + \sqrt{\mu_R^2 + 2r} \quad \text{and} \quad \psi_r^+(0) = \frac{g'_{+,r}(0)}{g_{+,r}(0)} = -\mu_L + \sqrt{\mu_L^2 + 2r}.$$

By Theorem 3.1, we obtain

$$\mathbb{E}\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s < 0\}}\, \mathrm{d}s\right] = \frac{\delta\psi_{\delta+\lambda}^+(0)/(\delta+\lambda) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}$$

$$= \frac{\delta(-\mu_L + \sqrt{\mu_L^2 + 2(\delta+\lambda)})/(\delta+\lambda) - \mu_R + \sqrt{\mu_R^2 + 2\delta}}{-\mu_L + \sqrt{\mu_L^2 + 2(\delta+\lambda)} - \mu_R + \sqrt{\mu_R^2 + 2\delta}}$$

and

$$\mathbb{E}\exp\left[-\lambda \int_0^{e_\delta} \mathbf{1}_{\{X_s > 0\}}\, \mathrm{d}s\right] = \frac{\delta\psi_{\delta+\lambda}^-(0)/(\delta+\lambda) + \psi_\delta^+(0)}{\psi_{\delta+\lambda}^-(0) + \psi_\delta^+(0)}$$

$$= \frac{\delta(-\mu_R + \sqrt{\mu_R^2 + 2(\delta+\lambda)})/(\delta+\lambda) - \mu_L + \sqrt{\mu_L^2 + 2\delta}}{-\mu_R + \sqrt{\mu_R^2 + 2(\delta+\lambda)} - \mu_L + \sqrt{\mu_L^2 + 2\delta}}.$$

We have thus recovered Equation (15) of [21]. Furthermore, we can obtain the probability density functions for the occupation times $\int_0^t \mathbf{1}_{\{X_s < 0\}}$ and $\int_0^t \mathbf{1}_{\{X_s > 0\}}$ by inverting the corresponding Laplace transform as in [21].

In addition, by (5.2) and (5.3), we can easily derive all the joint Laplace transforms in Theorems 3.2 and 3.3 for Brownian motion with two-valued drift.

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