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"TOPOLOGICALLY INDEXED FUNCTION SPACES AND ADJOINT FUNCTORS"

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ABSTRACT. Let *Top* denote the category of topological spaces and continuous maps. In this paper we discuss families of function spaces indexed by the elements of a topological space T, and their relationship to the characterization of right adjoints $Top/S \to Top/T$, where S is also a topological space. After reducing the problem to the case where S is a one-point space, we describe a class of right adjoints $Top \to Top/T$, and then show that every right adjoint $Top \to Top/T$ is isomorphic to one of this form. We conclude by giving necessary and sufficient conditions for a left adjoint $Top/T \to Top$ to be isomorphic to one of the form $-\times_T Y$, where Y is a space over T, and $\times_T$ denotes the fiber product with the product topology.

Introduction. A great deal has been written about "suitable" topologies on function spaces. Although most of it appeared during the past two decades, many of the problems can be traced back to a 1945 paper of R. H. Fox, entitled "On topologies for function spaces" [4]. In this paper Fox writes

> Given topological spaces X, T and Y and a function h from $X \times T$ to Y which is continuous in x for each fixed t, there is associated with h a function h^* from T to $F = Y^X$, the space whose elements are continuous functions from X to Y. The function h^* is defined as follows: $h^*(t) = h_t$, where $h_t(x) = h(x, t)$ for every x in X... It would be desirable to so topologize F that the functions h^* which are continuous are precisely those which correspond to continuous functions h. It has been known for a long time that this is possible if X satisfies certain conditions, chief among which is the condition of local compactness... several years ago, in a letter, Hurewicz proposed to me the problem of defining such a topology for F when X is not locally compact. At that time I showed by an example that this is not in general possible. Recently I discovered that, by restricting the range of T in a very reasonable way, one of the standard topologies for F has the desired property even for spaces X which are not locally compact...

The existence of a bijection between continuous h^* and continuous h is equivalent to the statement that X is *cartesian* in *Top* (i.e. the functor $-\times X: Top \to Top$ has a right adjoint). The characterization of cartesian spaces was dealt with by R. Brown [2], P. Wilker [13] and finally completed by B. J.

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Day and G. M. Kelly [3] in 1970. Day and Kelly essentially showed that a space Y is cartesian if and only if the collection $O(Y)$ of open subsets of Y is a continuous lattice in the sense of Scott [10].

The question of the necessity of local compactness of a cartesian space Y was answered affirmitively for separable metrizable spaces by Fox [4], and Hausdorff spaces by Day and Kelly [3]. For a non-Hausdorff space Y there are two notions of local compactness, the existence of a compact neighbourhood for each element of Y, and the existence of arbitrarily small compact neighborhoods for each element of Y. In [5], K. H. Hoffmann and J. D. Lawson showed that a sober space Y is cartesian if and only if Y is locally compact (in the latter sense). They also give an example of a cartesian space which is not locally compact.

Fox's idea of "restricting the range of T" led to J. Kelley's introduction of k-spaces [8], and later Steenrod's compactly generated spaces [12]. Each was an example of a *cartesian closed* subcategory of *Top* (i.e. one in which every object is cartesian). On the other hand, in "Quasi-topologies" [11], Spanier generalized the notion of topological space to obtain a cartesian closed category containing *Top*.

A third function space problem dealt with topologizing the sets $X\times Y$ and $Top(Y, X)$, for a fixed space Y, and all spaces X, so that $-\times Y$ is left adjoint to $Top(Y, -)$ as endofunctors of *Top* (where $Top(Y, X)$ denotes the set of continuous maps from Y to X). In particular, continuous maps with domain $X\times Y$ must be continuous in x for each y (c.f. the above quote of Fox). This problem was considered by R. Brown [1], [2], P. Wilker [13], and J. Isbell [6]. In "Function spaces and adjoints", Isbell showed that every adjoint pair of endofunctors of *Top* is of this form.

In [9], Day and Kelly's characterization is generalized to the category *Top*/T of spaces over a fixed space T. After hearing about these results, F. W. Lawvere mentioned Isbell's paper and suggested that the methods of [9] be used to characterize adjoint pairs of endofunctors of *Top*/T, as Isbell did for *Top*.

The author is grateful to Lawvere for this suggestion, and to Barry Mitchell for his helpful comments and encouragement.

1. **The reduction.** Throughout this section all categories have finite limits. If $\mathbf{A}$ is a category, then $|\mathbf{A}|$ denotes the class of objects of $\mathbf{A}$, and $\mathbf{A}(X, Y)$ denotes the set of morphisms from X to Y in $\mathbf{A}$. If $F:\mathbf{A}\to\mathbf{B}$ is a functor, and G is a right adjoint for F, we write $F\dashv G$.

If $T\in|\mathbf{A}|$, then $\mathbf{A}/T$ denotes the category whose objects are $\mathbf{A}$-morphisms $X\to T$, and morphisms are commutative triangles. An object X over T will sometimes be denoted by X without explicit reference to the projection $X\to T$. Note that if 1 denotes the terminal object of $\mathbf{A}$, then $\mathbf{A}/1\cong\mathbf{A}$.

If $p: S \to T$ is a morphism of $\mathbf{A}$, then composition with p induces a functor $\Sigma_p : \mathbf{A}/S \to \mathbf{A}/T$, which is left adjoint to the functor $p^* : \mathbf{A}/T \to \mathbf{A}/S$ defined by pulling back along p. If $T=1$ and p is the unique map $S \to 1$, Σ_p and p^* are considered as functors $\mathbf{A}/S \to \mathbf{A}$ and $\mathbf{A} \to \mathbf{A}/S$, respectively and are denoted by Σ_S and S^*, respectively. Note that Σ_S is the forgetful functor.

Given a functor $F: \mathbf{B} \to \mathbf{A}/S$, by composing with Σ_S, we obtain a functor $\tilde{F}: \mathbf{B} \to \mathbf{A}$ together with a morphism $\tilde{F}1 \to S$ (i.e. $F1$ considered as a morphism of $\mathbf{A}$). A natural transformation $\eta: F \to G$ induces a natural transformation $\tilde{\eta}: \tilde{F} \to \tilde{G}$ such that the diagram

$$\begin{array}{ccc} \tilde{F}1 & \xrightarrow{\tilde{\eta}} & \tilde{G}1 \\ & {\scriptstyle F1}\searrow \quad \swarrow{\scriptstyle G1} & \\ & S & \end{array}$$

commutes. Moreover, it is not difficult to show that a functor $F: \mathbf{B} \to \mathbf{A}/S$ has a right adjoint if and only if $\Sigma_S \circ F$ has a right adjoint [9, proposition 1.1].

Therefore, the category of left adjoints $\mathbf{B} \to \mathbf{A}/S$ is isomorphic to the category of left adjoints $F: \mathbf{B} \to \mathbf{A}$ equipped with a morphism $F1 \to S$, with the obvious morphism. In particular, taking $\mathbf{B} = \mathbf{A}/T$, we see that to characterize left adjoints $\mathbf{A}/T \to \mathbf{A}/S$ it suffices to determine all left adjoints $\mathbf{A}/T \to \mathbf{A}$.

2. **A class of adjoint pairs.** First, we consider a general construction. Let $G: \mathbf{A} \to \mathit{Sets}/T$ be a functor, where $\mathbf{A}$ is any category. In addition, suppose Z_0 is a weak terminal object of $\mathbf{A}$ (i.e. $\mathbf{A}(Z, Z_0)$ is non-empty, for all $Z \in |A|$), and a topology is given for GZ_0 such that the projection to T is continuous. If Z is any object of $\mathbf{A}$, let $\bar{G}Z$ denote the set GZ with the topology induced by the collection of all maps $G\alpha$ such that $\alpha: Z \to Z_0$ is a morphism of $\mathbf{A}$. Using the fact that Z_0 is a weak terminal object, we see that the projection $\bar{G}Z \to T$ is continuous. Thus, $\bar{G}: \mathbf{A} \to \mathit{Top}/T$ is a functor. Note that the identity $GZ \to \bar{G}Z$ need not be continuous.

Now, suppose that we are given a family $\{Y_t\}$ of spaces indexed by the elements of a space T. For the moment, we impose no topology on the set $Y = \coprod_{t \in T} Y_t$. If Z is a space, let $\mathit{Top}(\{Y_t\}, Z) = \coprod_{t \in T} \mathit{Top}(Y_t, Z)$, i.e. the collection of pairs (σ, t), where $\sigma: Y_t \to Z$ is a continuous map. Then it is easy to see that $\mathit{Top}(\{Y_t\}, -)$ defines a functor $\mathit{Top} \to \mathit{Sets}/T$.

Let $\mathbb{2}$ denote the *Sierpinski space* $\{0, 1\}$ with $\{1\}$ open but not $\{0\}$, and let $O(\{Y_t\}) = \coprod_{t \in T} O(Y_t)$, where $O(Y_t)$ is the collection of open subsets of Y_t. If Y is any space, there is a bijection between open subsets of Y and continuous maps $Y \to \mathbb{2}$. Thus, $\mathit{Top}(\{Y_t\}, \mathbb{2})$ can be identified with $O(\{Y_t\})$, and in accordance with the above principle with $Z_0 = \mathbb{2}$, a topology $\mathcal{K}$ on $O(\{Y_t\})$, such that the projection to T is continuous, induces a functor $\mathit{Top} \to \mathit{Top}/T$ which we shall denote by $\mathit{Top}_{\mathcal{K}}(\{Y_t\}, -)$. Explicitly, the topology on $\mathit{Top}_{\mathcal{K}}(\{Y_t\}, Z)$ has as a subbase consisting of the collection of subsets of the form

$$\langle K, W\rangle = \{(\sigma, t) \mid (\sigma^{-1}W, t) \in K\}$$

where $K \in \mathcal{K}$ and W is open in Z.

If A is a set, let $O(\{Y_t\})^A$ denote the product over T of A copies of $O(\{Y_t\})$ with the product topology. Elements of $O(\{Y_t\})$ will be denoted by $\langle (U_\alpha, t)\rangle_{\alpha\in A}$. Then $\mathcal{K}$ is a *topological topology* if the map $\bigcup_A : O(\{Y_t\})^A \to O(\{Y_t\})$ is continuous for every set A, and the map $\bigcap_A : O(\{Y_t\})^A \to O(\{Y_t\})$ is continuous for every finite set A. If A is empty, then $\bigcup_A, \bigcap_A : T \to O(\{Y_t\})$ are given by $t \mapsto (\phi, t)$ and $t \mapsto (Y_t, t)$, respectively. Hence, $\bigcup_\phi$ and $\bigcap_\phi$ are continuous if and only if for every K open in $O(\{Y_t\})$

$$\{t \mid (\phi, t) \in K\} \quad \text{and} \quad \{t \mid (Y_t, t) \in K\}$$

are open in T, respectively. In fact, this is precisely what we need to show that $O(\{Y_t\}) \to Top_{\mathcal{K}}(\{Y_t\}, \mathbb{2})$ is continuous.

If $p : X \to T$ is a space over T, let $p \times \{Y_t\} = \{(x, y) \mid y \in Y_{px}\}$. We shall say that a subset W of $p \times \{Y_t\}$ is *open* if $W[x] = \{y \mid (x, y) \in W\}$ is open in Y_{px}, and $x \mapsto W[x]$ defines a continuous map $X \to O(\{Y_t\})$. If W_α, $\alpha \in A$ is a family of open subsets of $p \times \{Y_t\}$, then $\bigcup W_\alpha$ and $\bigcap W_\alpha$ (if A is finite) correspond to the composites $X \to O(\{Y_t\})^A \xrightarrow{\bigcup_A} O(\{Y_t\})$ and $X \to O(\{Y_t\})^A \xrightarrow{\bigcap_A} O(\{Y_t\})$, respectively. Thus, if $\mathcal{K}$ is a topological topology, then the open subsets of $p \times \{Y_t\}$ define a topology, and we shall denote the resulting space by $p \times_{\mathcal{K}} \{Y_t\}$. Conversely, if the open subsets of $p \times \{Y_t\}$ form a topology for every space X over T, then taking $X = O(\{Y_t\})^A$ and the identity $X \to O(\{Y_t\})^A$, we see that $\mathcal{K}$ is a topological topology. In this case, it is easy to see that a continuous map $X \to X'$ over T induces a continuous map $p \times_{\mathcal{K}} \{Y_t\} \to p' \times_{\mathcal{K}} \{Y_t\}$, and it follows that $- \times_{\mathcal{K}} \{Y_t\}$ defines a functor $Top/T \to Top$.

Proposition 2.1. *If* $\{Y_t\}$ *is a family of spaces indexed by the elements of* T, *and* $\mathcal{K}$ *is a topological topology on* $O(\{Y_t\})$, *then* $- \times_{\mathcal{K}} \{Y_t\} \dashv Top_{\mathcal{K}}(\{Y_t\}, -)$.

Proof. Suppose that X is a space over T with projection p, and Z is a space. Let $\eta : X \to Top_{\mathcal{K}}(\{Y_t\}, p \times_{\mathcal{K}} \{Y_t\})$ and $\varepsilon : Top_{\mathcal{K}}(\{Y_t\}, Z) \times_{\mathcal{K}} \{Y_t\} \to Z$ be given by $\eta(x) = ((x, -), px)$ and $\varepsilon((\sigma, t), y) = \sigma y$.

If W is open in $p \times_{\mathcal{K}} \{Y_t\}$, let $w : X \to O(\{Y_t\})$ and $\chi_W : X \times_{\mathcal{K}} \{Y_t\} \to \mathbb{2}$ be the corresponding continuous maps. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Top_{\mathcal{K}}(\{Y_t\}, p \times_{\mathcal{K}} \{Y_t\}) \\ & \searrow^{w} & \downarrow{\scriptstyle Top_{\mathcal{K}}(\{Y_t\}, \chi_W)} \\ & & O(\{Y_t\}) \cong Top_{\mathcal{K}}(\{Y_t\}, \mathbb{2}) \end{array}$$

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If $K \in \mathcal{K}$, then $\eta^{-1}(\langle K, W\rangle) = \eta^{-1}\ Top_{\mathcal{K}}(\{Y_t\}, \chi_W)^{-1}(K) = w^{-1}(K)$, and hence η is continuous since w is.

Now, suppose that W is open in Z, and $\chi_W : Z \to \mathbb{2}$ is the corresponding continuous map. Then $\varepsilon^{-1}(W) = \varepsilon^{-1}\chi_W^{-1}(1)$ is open in $Top_{\mathcal{K}}(\{Y_t\}, Z) \times_{\mathcal{K}} \{Y_t\}$ since the induced map $Top_{\mathcal{K}}(\{Y_t\}, Z) \to O(\{Y_t\}) \cong Top_{\mathcal{K}}(\{Y_t\}, \mathbb{2})$ is $Top_{\mathcal{K}}(\{Y_t\}, \chi_W)$. To complete the proof one checks that the adjunction identities hold relative to η and ε.

3. **The characterization.** If $p : X \to T$ is a continuous map and $t \in T$, then the *fiber of* X *over* t is the set $X_t = p^{-1}t$ with the subspace topology. If $t : 1 \to T$ denotes the constant t valued map, then there is a bijection between elements of X_t, and morphisms $t \to p$ of Top/T.

Let $G : Top \to Top/T$ be a functor with a left adjoint F. Then there is a natural bijection

$$\theta : Top(Fp, Z) \to Top/T(p, GZ)$$

which is natural in $p : X \to T$ and Z. Taking $p = t$, we see that $(GZ)_t$ can be identified with $Top(Ft, Z)$. Applying naturality to one point embeddings $x : 1 \to X$, considered as morphisms $t \to p$, we see that $\theta(f)(x)$ is identified with $Ft \xrightarrow{Fx} Fp \xrightarrow{f} X$, where $x \in X_t$.

LEMMA 3.1. *Every space can be embedded as a subspace of a product of copies of the Sierpinski space* $\mathbb{2}$ *and the indiscrete space* 2.

Proof. It is well known that every T_0 space can be embedded in a product of $\mathbb{2}$'s. For a non-T_0 space, copies of the indiscrete space 2 are added to "separate" points.

LEMMA 3.2 (Isbell [6]). *If* F *is a left adjoint endofunctor of Top, then* F *followed by the forgetful functor Top* $\to$ *Sets can be expressed in the form* $- \times F1$.

Proof. If X is a space, then since F preserves epimorphisms, we have a continuous surjection $F(\coprod_{x \in X} 1) \to FX$. Then since F commutes with comproducts we have a natural surjection

$$X \times F1 \to FX \tag{1}$$

It is easy to see that (1) cannot identify distinct points with the same first coordinate, since every element of X is a retract as a space.

Suppose D is a directed set, and $X = D \coprod 1$ with the following topology. A subset U of X is open if $1 \notin U$, or if $1 \in U$ and U contains a subset of the form $\{x \in D \mid x \geq d\}$, for some $d \in D$. For any distinct pair of elements, X can be written as a coproduct separating the given element. Then, since F preserves coproducts, it follows that if X is of this form, (1) cannot identify two elements with different first coordinates.

Finally, using the fact that directed sets (i.e. nets) are enough to determine topologies we see that every space can be expressed as a colimit of "directed sets", and since both sides of (1) are colimit preserving it follows that (1) is a bijection for any space X.

Note that Isbell's proof of the above lemma uses Hausdorff ultraspaces rather than directed sets.

Let T-*Top* denote the category whose objects are pairs $(\{Y_t\}, \mathscr{K})$, where $\{Y_t\}$ is a family of spaces indexed by the elements of T, and $\mathscr{K}$ is a topological topology on $O(\{Y_t\})$. A morphism $f : (\{Y_t\}, \mathscr{K}) \to (\{Y'_t\}, \mathscr{K}')$ is a family of continuous maps $f_t : Y'_t \to Y_t$ such that the induced map $O(\{Y_t\}) \to O(\{Y'_t\})$ is continuous.

THEOREM 3.3. *T-Top is equivalent to the category of right adjoints $Top \to Top/T$ and natural transformations.*

Proof. Let Γ denote the following functor. If $(\{Y_t\}, \mathscr{K})$ is an object of T-*Top*, by Lemma 2.1, $\Gamma(\{Y_t\}, \mathscr{K}) = Top_{\mathscr{K}}(\{Y_t\}, -)$ is a right adjoint $Top \to Top/T$. A morphism $f : (\{Y_t\}, \mathscr{K}) \to (\{Y'_t\}, \mathscr{K}')$ induces a natural transformation $\Gamma(f) : Top_{\mathscr{K}}(\{Y_t\}, -) \to Top_{\mathscr{K}}(\{Y'_t\}, -)$ given by $\Gamma(f)_Z((\sigma, t)) = (\sigma \circ f_t, t)$, where $\sigma : Y_t \to Z$ is a continuous map.

First, we claim that Γ is full and faithful. A natural transformation $\eta : Top_{\mathscr{K}}(\{Y_t\}, -) \to Top_{\mathscr{K}'}(\{Y'_t\}, -)$ induces a natural transformation $\bar{\eta} : - \times_{\mathscr{K}} \{Y'_t\} \to - \times_{\mathscr{K}} \{Y_t\}$, and hence a family of continuous maps $\bar{\eta}_t : Y'_t \to Y_t$, since $t \times_{\mathscr{K}'} \{Y'_t\} = Y'_t$ and $t \times_{\mathscr{K}} \{Y_t\} = Y_t$. It is easy to see that the corresponding map $O(\{Y_t\}) \to O(\{Y'_t\})$ is precisely $\eta_{\mathbb{2}}$. Therefore Γ is full since $\{\bar{\eta}_t\} : (\{Y_t\}, \mathscr{K}) \to (\{Y'_t\}, \mathscr{K}')$ is a morphism, and $\Gamma(\{\bar{\eta}_t\}) = \eta$. To see that Γ is faithful, we note that if $f : (\{Y_t\}, \mathscr{K}) \to (\{Y'_t\}, \mathscr{K}')$, then $\Gamma(f)_{Y_t}((1_{Y_t}, t)) = (f_t, t)$.

To see that Γ is an equivalence of categories it remains to show that every right adjoint $G : Top \to Top/T$ is isomorphic to one of the form $Top_{\mathscr{K}}(\{Y_t\}, -)$, where $\mathscr{K}$ is a topological topology.

Suppose $F \dashv G$. As before, we can identify $(GZ)_t$ with $Top(Ft, Z)$. Letting $Y_t = Ft$ and identifying $O(Y_t)$ with $(G\mathbb{2})_t$ we obtain a family $\{Y_t\}$ of spaces together with a topology $\mathscr{K}$ on $O(\{Y_t\})$ such that the projection to T is continuous. If $f : Z \to \mathbb{2}$ is a continuous map, consider the diagram

$$\begin{array}{ccc} GZ & \xrightarrow{\phi_Z} & Top_{\mathscr{K}}(\{Y_t\}, Z) \\ {\scriptstyle Gf}\big\downarrow & & \big\downarrow{\scriptstyle Top_{\mathscr{K}}(\{Y_t\}, f)} \\ G\mathbb{2} & \xleftarrow[\phi_{\mathbb{2}}^{-1}]{} & Top_{\mathscr{K}}(\{Y_t\}, 2) \end{array} \tag{2}$$

where the rows are bijections and all maps, except possibly ϕ_Z, are continuous. But $Top_{\mathscr{K}}(\{Y_t\}, Z)$ has the topology induced by all maps

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$Top_{\mathscr{K}}(\{Y_t\}, f) : Top(\{Y_t\}, Z) \to Top(\{Y_t\}, \mathbb{2}) \to G\mathbb{2}$, and so ϕ_Z is continuous since Gf is. In particular, $\phi_{\mathbb{2}}$ is a homeomorphism.

We would like to show that ϕ_Z is a homeomorphism for every space Z. First, we show that K is a topological topology. Consider $\mathbb{2}$ as a poset with the order topology. Then $\sup : \mathbb{2}^A \to \mathbb{2}$ is continuous for every set A, and $\inf : \mathbb{2}^A \to \mathbb{2}$ is continuous if A is finite. Applying the fact that G preserves products we get continuous maps $(G\mathbb{2})^A \tilde{\to} G(\mathbb{2}^A) \xrightarrow{G(\sup)} G\mathbb{2}$ and $(G\mathbb{2})^A \tilde{\to} G(\mathbb{2}^A) \xrightarrow{G(\inf)} G(\mathbb{2})$, and by commutativity of (2), we see 'that these composites are $\bigcup_A$ and $\bigcap_A$, respectively.

Finally, since G and $Top_{\mathscr{K}}(\{Y_t\}, -)$ preserve subspaces and products (being right adjoints), by Lemma 3.1, it suffices to show that ϕ_2 is a homeomorphism, where 2 denotes the indiscrete two point space.

Recall that $Top_{\mathscr{K}}(\{Y_t\}, Z)$ has a basic family of open subsets of the form $\langle K, W\rangle = \{(\sigma, t) \mid (\sigma^{-1}W, t) \in K\}$, where $K \in \mathscr{K}$ and W is open in Z. When $Z = 2$, it is easy to see that every open subset is of the form $\coprod_{t\in U} Top(Y_t, 2)$, where U is an open subset of T.

Now, consider the functor $Top \xrightarrow{G} Top/T \xrightarrow{t^*} Top$, where $t : 1 \to T$ is the constant t valued map. By composition of adjoints, $t^* \circ G$ has a left adjoint, and so using Lemma 3.2 it is not difficult to show that $(G2)_t = (t^* \circ G)(2)$ is indiscrete, i.e. one shows that every map $X \to (G2)_t$ is continuous. Thus, every open subset of $G2$ is of the form $\coprod_{t\in U} (G2)_t$, where $U \subseteq T$. To see that U is open, it suffices to show that $G2$ admits a global section. But $Top/T(1_T, G2) \cong Top(F1_T, 2)$ which is clearly nonempty. Therefore, ϕ_2 is a homeomorphism, and the proof is complete.

Corollary 3.4. *Every left adjoint F: Top/T $\to$ Top followed by the forgetful functor Top $\to$ Sets is of the form $-\times_T Y$, for some space Y over T.*

Proof. By Theorem 3.3, $F \cong \times_{\mathscr{K}}\{Y_t\}$ where $\{Y_t\}$ is a family of spaces indexed by the elements of T, and $\mathscr{K}$ is a topological topology on $O(\{Y_t\})$. Taking $Y = \coprod_{t\in T} Y_t$ with the coproduct topology and the obvious projection to T gives the desired result.

If $(\{Y_t\}, \mathscr{K})$ is an object of T-*Top* we can also obtain a topology on the set $Y = \coprod_{t\in T} Y_t$ by identifying it with $1_T \times_{\mathscr{K}} \{Y_t\}$. First, we discuss several properties of topological topologies.

Let $\{Y_t\}$ be any family of spaces indexed by the elements of T. A subset K of $O(\{Y_t\})$ is *saturated* if whenever $(U, t) \in K$ and $U \subseteq V \in O(\{V_t\})$, then $(V, t) \in K$. K has the *finite union property* (fup) if whenever $(\bigcup_A U_\alpha, t) \in K$ and $\{U_\alpha\}_{\alpha\in A} \subseteq O(Y_t)$, then $(\bigcup_F U_\alpha, t) \in K$, for some finite $F \subseteq A$. A topology $\mathscr{K}$ on $O(\{Y_t\})$ is *saturated* or has the *finite union property* if every $K \in \mathscr{K}$ does. Note that when $T = 1$, then the collection of saturated subsets of $O(Y)$ with *fup* is known as the *Scott-topology* on the lattice $O(Y)$.

LEMMA 3.5. *Let $\mathcal{K}$ be a topology on $O(\{Y_t\})$ such that the projection to T is continuous. If $\bigcup_A : O(\{Y_t\})^A \to O(\{Y_t\})$ is continuous for all A (in particular, if $\mathcal{K}$ is topological), then $\mathcal{K}$ is saturated and has the finite union property.*

Proof. Suppose that $(U, t) \in K \in \mathcal{K}$, and A is an infinite set. Let $U_\alpha = U$ for every $\alpha \in A$. Then since $\bigcup_A$ is continuous and K is open in $O(\{Y_t\})$, we can find a basic open neighbourhood of $\langle (U_\alpha, t)\rangle_{\alpha \in A}$ which maps into K. If $U \subseteq V \in O(Y_t)$, then some element $\langle (V_\alpha, t)\rangle_{\alpha \in A}$ of this neighbourhood is such that $V_\alpha = U$ for all but one α, and $V_\alpha = V$ for that α. Thus, $(V_t, t) = \bigcup_A \langle (V_\alpha, t)\rangle_{\alpha \in A} \in K$, and so K is saturated.

To see that K has *fup*, suppose that $\{U_\alpha\}_{\alpha \in A} \subseteq O(Y_t)$ and $(\bigcup_A U_\alpha, t) \in K$. Then there is a basic open neighbourhood of $\langle (U_\alpha, t)\rangle_{\alpha \in A}$ which maps into K under $\bigcup_A$, and in this neighbourhood there is an element of the form $\langle (V_\alpha, t)\rangle_{\alpha \in A}$, where $V_\alpha = \varnothing$ for all but finitely many α, and $V_\alpha = U_\alpha$ otherwise. Therefore, $(\bigcup_F U_\alpha, t) \in K$, for some finite $F \subseteq A$, proving that K has *fup*.

Suppose that $(\{Y_t\}, \mathcal{K})$ is an object of T-*Top*, and let $Y = 1_T \times_{\mathcal{K}} \{Y_t\}$. It is easy to see that the projections $p \times_{\mathcal{K}} \{Y_t\} \to X$ and $p \times_{\mathcal{K}} \{Y_t\} \to Y$ are continuous for every $p : X \to T$ in *Top*/T. In particular, when $p = 1_T$ we have continuity of the projection $\pi : Y \to T$. Therefore, it follows that the identity map $p \times_{\mathcal{K}} \{Y_t\} \to X \times_T Y$ is continuous. Now, the space Y_t is not necessarily a subspace of Y, but the inclusion $Y_t \to Y$ is, in general, continuous. If $- \times_{\mathcal{K}} \{Y_t\}$ preserves embeddings, then clearly, Y_t is a subspace of Y. If points of T are *locally closed* (i.e. for every $t \in T$ there exist U open and F closed in T such that $\{t\} = U \cap F$), then an open subset U_t of Y_t comes from an open subset of Y, namely $U = U_t \cup \pi^{-1}(Y \backslash \{\bar{t}\})$ which is open in Y since the corresponding map $T \to O(\{Y_t\})$ is continuous by Lemma 3.5 and the fact that $\{t \mid (Y_t, t) \in K\}$ and $\{t \mid (\phi, t) \in K\}$ are open in T whenever $K \in \mathcal{K}$. Thus, we have the following corollary.

COROLLARY 3.6. *If points of T are locally closed, and F: Top/$T \to$ Top is a left adjoint, then F followed by the forgetful functor is of the form $-\times_T F(1_T)$, for some continuous map $F(1_T) \to T$. Moreover, if S is any space, then every left adjoint F: Top/$T \to$ Top/S can be expressed in the form*

$$Top/T \xrightarrow{\text{“}f^*\text{”}} Top/\Sigma_S \circ F(1_T) \xrightarrow{\Sigma_{F(1_T)}} Top/S$$

where $f : \Sigma_S \circ F(1_T) \to T$ is a continuous map and the underlying set of “f^*” is given by pulling back along f.

LEMMA 3.7 (Isbell [7]). *If Y is a space and $\mathcal{K}$ is a T_0 topological topology on $O(Y)$, then $\{\bar{U}\} = \{V \mid V \subseteq U\}$, for every $U \in O(Y)$.*

Proof. First, we note that $\{V \mid V \subseteq U\} \subseteq \{\bar{U}\}$ since $\mathcal{K}$ is saturated. On the other hand, if $V \in \{\bar{U}\}$, then $U \in \overline{\{U \cup V\}}$ since $\mathcal{K}$ is saturated, and $U \cup V \in \{\bar{U}\}$

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since $\bigcup : O(Y) \times O(Y) \to O(Y)$ is continuous. But $\mathcal{K}$ is T_0 topology, and so $U \cup V = U$. Therefore, $V \subseteq U$.

Suppose that Y and Z are spaces. Recall that the *pointwise topology* on $Top(Y, Z)$ is the topology induced by the collection of all evaluation maps $ev_y : Top(Y, Z) \to Z$, where $y \in Y$. In particular, the pointwise topology on $O(Y)$ is generated by all subsets of the form $\{U \in O(Y) \mid y \in U\}$, for some fixed $y \in Y$. It is not difficult to show that the pointwise topology is necessarily T_0.

COROLLARY 3.8. *Let F: $Top/T \to Top$ be a functor with a right adjoint G. Then $F \cong - \times_T Y$ for some cartesian space Y over T if and only if F preserves subspaces and $(G\mathbb{2})_t$ contains the pointwise topology for all $t \in T$.*

Proof. Suppose that $F \cong - \times_T Y$, for some cartesian space Y over T. Then clearly F preserves subspaces. To see that $(G\mathbb{2})_t$ contains the pointwise topology we note that the open subsets of $(G\mathbb{2})_t$ are precisely the saturated subsets of $O(Y_t)$ which have the finite union property [9].

For the converse, suppose that $(\{Y_t\}, \mathcal{K})$ corresponds to the right adjoint G, and let $Y = 1_T \times_{\mathcal{K}} \{Y_t\}$. Then as before, the obvious projection $\pi: Y \to T$ is continuous. Now, by Lemma 3.1, every space over T can be embedded as a subspace of one of the form $\pi_3 : \mathbb{2}^A \times X \times T \to T$ where A is a set, and X is an indiscrete space. Thus, since F and $- \times_T Y$ preserve subspaces, it suffices to show that the induced continuous bijection $\pi_3 \times_{\mathcal{K}} \{Y_t\} \to (\mathbb{2}^A \times X \times T) \times_T Y$ is a homeomorphism.

Suppose that W is open in $\pi_3 \times_{\mathcal{K}} \{Y_t\}$ and let $w : \mathbb{2}^A \times X \times T \to O(\{Y_t\})$ be the continuous map given by $w(\langle s_\alpha \rangle, x, t) = \{y \mid ((\langle s_\alpha \rangle, x, t), y) \in W\}$. First, we note that if $\langle s_\alpha \rangle, \langle s'_\alpha \rangle \in \mathbb{2}^A$ where $s_\alpha \leq s'_\alpha$ for all α, and $x, x' \in X$, then by Lemma 3.7 $w(\langle s_\alpha \rangle, x, t) \subseteq w(\langle s'_\alpha \rangle, x', t)$, for every $t \in T$. Fix $((\langle s_\alpha \rangle, x, t), y) \in W$, and let K be an open subset of $O(\{Y_t\})$ such that $K_t = \{(U, t) \mid y \in U\}$. Then by continuity of w there exists $\langle m_\alpha \rangle \in \mathbb{2}^A$ such that $m_\alpha \leq s_\alpha$, for all α, $m_\alpha = 0$ for all but finitely many α, and $w(\langle m_\alpha \rangle, x, t) \in K$. Now, the map $f: Y \to \pi_3 \times_{\mathcal{K}} \{Y_t\}$ defined $f(y) = ((\langle m_\alpha \rangle, x, \pi y), y)$ is continuous since it arises by applying $- \times_{\mathcal{K}} \{Y_t\}$ to the obvious continuous map $T \to \mathbb{2}^A \times X \times T$ over T. We claim that the neighbourhood $V = (\{\langle s'_\alpha \rangle \mid m_\alpha \leq s'_\alpha\} \times X \times T) \times_T f^{-1} W$ of $((\langle s_\alpha \rangle, x, t), y)$ in $(\mathbb{2}^A \times X \times T) \times_T Y$ is contained in W. To see this note that if $((\langle s'_\alpha \rangle, x', t'), y') \in V$ then $y' \in w(\langle m_\alpha \rangle, x, t') \subseteq w(\langle s'_\alpha \rangle, x', t')$, and so $((\langle s'_\alpha \rangle, x', t'), y') \in W$.

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