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Lie Algebras of Pro-Affine Algebraic Groups

Dedicated to Gerhard Hochschild on his 85th birthday

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Abstract. We extend the basic theory of Lie algebras of affine algebraic groups to the case of pro-affine algebraic groups over an algebraically closed field K of characteristic 0. However, some modifications are needed in some extensions. So we introduce the pro-discrete topology on the Lie algebra $\mathcal{L}(G)$ of the pro-affine algebraic group G over K, which is discrete in the finite-dimensional case and linearly compact in general. As an example, if L is any sub Lie algebra of $\mathcal{L}(G)$, we show that the closure of $[L, L]$ in $\mathcal{L}(G)$ is algebraic in $\mathcal{L}(G)$.

We also discuss the Hopf algebra of representative functions $H(L)$ of a residually finite dimensional Lie algebra L. As an example, we show that if L is a sub Lie algebra of $\mathcal{L}(G)$ and G is connected, then the canonical Hopf algebra morphism from $K[G]$ into $H(L)$ is injective if and only if L is algebraically dense in $\mathcal{L}(G)$.

Introduction

A pro-affine algebraic group G, over an algebraically closed field K, is an inverse limit of affine algebraic groups over K [H-M2]. This notion was introduced by Hochschild and Mostow in connection with the representation theory of groups. For example, every complex analytic group has a pro-affine algebraic group hull whose finite-dimensional rational representations are in bijective correspondence with the finite-dimensional complex analytic representations of the given analytic group [H-M3, p. 1141] (see also [Ma2], [N1]). A similar correspondence exists for residually finite dimensional Lie algebras over K in characteristic 0 by (4.2) below.

So it is of interest to extend the basic theory (found, for example, in [B], [H3], [Hu], [S]) concerning the group-Lie algebra correspondence from the category of affine algebraic groups to the category of pro-affine algebraic groups in characteristic 0. For example, if L is a sub Lie algebra of $\mathcal{L}(G)$, Theorem 2.1 shows the existence of the smallest algebraic subgroup of G whose Lie algebra contains L. In particular, $\mathcal{L}(A \cap B) = \mathcal{L}(A) \cap \mathcal{L}(B)$ whenever A and B are algebraic subgroups of G. Moreover, a morphism of connected pro-affine algebraic groups is a covering if and only if its differential is an isomorphism by (3.8).

However, some modifications are needed in extending some of the results in the affine case to the pro-affine case. So, in Section 3, we introduce the pro-discrete topology on $\mathcal{L}(G)$ which is discrete in the finite-dimensional case and linearly compact in general. For example, Theorem 3.2 shows that if M and N are connected normal algebraic subgroups of G, then $\mathcal{L}([M, N]^c) = [\mathcal{L}(M), \mathcal{L}(N)]^c$ where $[M, N]^c$ is

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the Zariski closure of $[M, N]$ in G and $[\mathcal{L}(M), \mathcal{L}(N)]^c$ is the (pro-discrete) closure of $[\mathcal{L}(M), \mathcal{L}(N)]$ in $\mathcal{L}(G)$. Theorem 3.10 shows that if A is any sub Lie algebra of $\mathcal{L}(G)$, then the (pro-discrete) closure of $[A, A]$ in $\mathcal{L}(G)$ coincides with its algebraic hull in $\mathcal{L}(G)$. In particular, if $[A, A]$ is finite-dimensional, then $[A, A]$ is algebraic in $\mathcal{L}(G)$. Theorem 3.12 (a) whose proof relies on the linear compactness of $\mathcal{L}(G)$ shows that if G is connected, then G is pro-solvable if and only if the sub Lie algebras of the *closed* derived series of $\mathcal{L}(G)$ have a zero intersection.

If $\operatorname{char}(K) = p$, some of the results in the affine case *do not extend to the pro-affine case*. For example, there exists a connected pro-affine algebraic group G such that $\mathcal{L}(G) = 0$ although G is non-trivial. Moreover, there exists a morphism f of connected pro-affine algebraic groups whose differential is surjective although f is not surjective (Example 3.6).

Most of the proofs in Sections 1–3 are obtained by reduction to the affine case. However the proof of Theorem 2.4 concerning the existence of algebraic hulls of sub Lie algebras of $\mathcal{L}(G)$, is a slight modification of the proof in the affine case as given by Hochschild in [H3]. The modification requires a version of Hilbert Nullstellensatz for commutative Hopf algebras.

In Section 4, we discuss the Hopf algebra of representative functions $H(A)$ of a Lie algebra A. Theorem 4.1 shows that if A is a sub Lie algebra of $\mathcal{L}(G)$ and G is connected, then the canonical Hopf algebra morphism from $K[G]$ into $H(A)$ is injective if and only if A is algebraically dense in $\mathcal{L}(G)$. This result is known in the case where $A = \mathcal{L}(G)$ [H2, Prop. 4.1]. Morover, in Section 4, we obtain the following results on residually finite dimensional Lie algebras over K. If L is such a Lie algebra and $G(L)$ is the pro-affine algebraic group such that $K[G(L)] = H(L)$, we have

(i) the finite-dimensional representations of L are in bijective correspondence with the finite-dimensional rational representations of $G(L)$ by (4.2),

(ii) L is algebraically dense in $\mathcal{L}\big(G(L)\big)$ and $G(L)$ is simply connected by (4.4),

(iii) if $[L, L]$ is finite-dimensional, we also have $[L, L] = \big[\mathcal{L}\big(G(L)\big), \mathcal{L}\big(G(L)\big)\big]$, and in particular,

(iv) if $[L, L]$ is finite-dimensional, then L is an ideal of $\mathcal{L}\big(G(L)\big)$ and $\mathcal{L}\big(G(L)\big)/L$ is abelian.

Property (iv) is known in the case where L is finite-dimensional (see [H1, top p. 521]).

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We shall assume that the reader is familiar with the general properties of pro-affine algebraic groups found in [H-M2, Section 2]. We adopt the following notation and conventions: K is a fixed algebraically closed field. If G is a pro-affine algebraic group over K, then $K[G]$ is its Hopf algebra of polynomial functions [HM2, p. 1127], $\mathcal{L}(G)$ is the Lie algebra of G [H2, p. 404], and G^1 is the identity component of G [H2, Thm. 2.1]. Moreover, if A is an algebraic subgroup of G, $\mathcal{L}(A)$ will be identified with its canonical image in $\mathcal{L}(G)$.

1 Preliminary Results

Let G be a pro-affine algebraic group over K. Let $\{A_i \mid i \in I\}$ be the set of all affine (finitely algebra generated) Hopf subalgebras of $K[G]$. For each i, let G_i be the affine algebraic group with $K[G_i] = A_i$. Then $K[G] = \bigcup K[G_i]$ is a directed union since every commutative Hopf algebra is the directed union of its affine Hopf subalgebras [H2, p. 400]. Hence

$$G = \varprojlim G_i \quad \text{and} \quad \mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$$

and we refer to these descriptions as the *standard limits* for G and $\mathcal{L}(G)$ respectively [Lu-Ma, p. 77]. We shall need the following result found in [H-M2, Thm. 2.1, p. 1131], [H2, p. 406].

Lemma 1.1 *Let $G = \varprojlim G_i$ and $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ be the standard limits for G and $\mathcal{L}(G)$. Then*

(a) *each projection $\pi_i : G \to G_i$ is surjective,*
(b) *If* $\operatorname{char}(K) = 0$, *each projection $\rho_i : \mathcal{L}(G) \to \mathcal{L}(G_i)$ is surjective.*

Notation Let $\pi_i : G \to G_i$ and $\rho_i : \mathcal{L}(G) \to \mathcal{L}(G_i)$ be as in (1.1). If $A \subset G$, $\pi_i(A)$ will be denoted by A_i, and if $A \subset \mathcal{L}(G)$, $\rho_i(A)$ will be denoted by A_i. With this notation, $(G)_i = G_i$ since each π_i is surjective. Moreover, if $\operatorname{char}(K) = 0$, $\big(\mathcal{L}(G)\big)_i = \mathcal{L}(G_i)$ since each ρ_i is surjective.

Lemma 1.2 *Let $G = \varprojlim G_i$ and $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ be the standard limits for G and $\mathcal{L}(G)$.*

(a) *Let A be an algebraic subgroup of G. Then $A = \varprojlim A_i$ and $\mathcal{L}(A) = \varprojlim \mathcal{L}(A_i)$. Moreover, if $A = \varprojlim A_s$ $(s \in S)$ is the standard limit for A, then $\{A_i\}$ is a cofinal subset of $\{A_s\}$.*
(b) *Let $\{A_i\}$ be a sub inverse system of $\{G_i\}$ where each A_i is an algebraic subgroup of G_i. Then $\varprojlim A_i$ is an algebraic subgroup of G. Moreover, if $\{A_i\}$ is a surjective inverse system, then each projection $\varprojlim A_i \to A_i$ is surjective.*
(c) $\mathcal{L}(G) = \mathcal{L}(G^1)$.

Proof (a) Let $\pi_i : G \to G_i$ be as in (1.1) and let $\pi : A \to A_i$ be the restriction of π_i to A. Then the transpose map $\pi^t : K[A_i] \to K[A]$ is injective since π is surjective. Thus we may identify $K[A_i]$ with its image in $K[A]$, so π^t is the identity. Let $\rho : K[G] \to K[A]$ be the restriction morphism. Then $\rho(K[G_i]) = K[A_i]$. Now we apply ρ on the directed union $K[G] = \bigcup K[G_i]$. This yields $K[A] = \bigcup K[A_i]$ as a directed union. Hence $A = \varprojlim A_i$ and $\mathcal{L}(A) = \varprojlim \mathcal{L}(A_i)$. Moreover, every finite subset of $K[A]$ lies in some $K[A_i]$. Consequently, if $A = \varprojlim A_s$ is the standard limit for A, then each $K[A_s]$ lies in some $K[A_i]$. Hence $\{A_i\}$ is a cofinal subset of $\{A_s\}$ which proves (a).

(b) Since $\{A_i\}$ is a sub inverse system of $\{G_i\}$, $\varprojlim A_i = \bigcap \pi_i^{-1}(A_i)$. But each $\pi_i^{-1}(A_i)$ is closed in G since each A_i is closed in G_i. Hence $\varprojlim A_i$ is a closed subgroup

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of G. If $\{A_i\}$ is a surjective inverse system (of algebraic groups), then each projection $\varprojlim A_i \to A_i$ is surjective [H-M1, Prop. 2.8, p. 505] which proves (b).

(c) Since $(G^1)_i = (G_i)^1$ [H2, Prop. 2.1], part (a) implies that $\mathcal{L}(G^1) = \varprojlim \mathcal{L}(G^1)_i = \varprojlim \mathcal{L}(G_i)^1$ and this last coincides with $\varprojlim \mathcal{L}(G_i) = \mathcal{L}(G)$. Hence $\mathcal{L}(G^1) = \mathcal{L}(G)$.

Theorem 1.3 *Let A be a connected algebraic subgroup of G. If A is normal in G, then $\mathcal{L}(A)$ is an ideal of $\mathcal{L}(G)$. Moreover, the converse is true if* $\operatorname{char}(K) = 0$ *and G is connected.*

Proof $A = \varprojlim A_i$ and $\mathcal{L}(A) = \varprojlim \mathcal{L}(A_i)$ by (1.2)(a). Moreover, the projection $G \to G_i$ is surjective and, if $\operatorname{char}(K) = 0$, the projections $\mathcal{L}(G) \to \mathcal{L}(G_i)$ and $\mathcal{L}(A) \to \mathcal{L}(A_i)$ are surjective by (1.1). With these facts, the proof of Theorem 1.3 can be easily reduced to the affine case.

Similarly, as in the proof of Theorem 1.3, we may reduce the following theorems to the affine case.

Theorem 1.4 *If G is abelian, then so is $\mathcal{L}(G)$. Moreover the converse is true if* $\operatorname{char}(K) = 0$ *and G is connected.*

Theorem 1.5 *Let A and B be connected algebraic subgroups of G such that $\mathcal{L}(A) \subset \mathcal{L}(B)$. If* $\operatorname{char}(K) = 0$*, then $A \subset B$.*

Let V be a G-module. Then, as in affine algebraic groups, V is called a *rational* G-module if it is a sum of finite-dimensional G-submodules which are rational in the usual sense. If V is a rational G-module, then V is an $\mathcal{L}(G)$-module in a natural way.

Theorem 1.6 *Assume* $\operatorname{char}(K) = 0$ *and G is connected. Let V be a rational G-module. Then the G-fixed part of V coincides with the $\mathcal{L}(G)$-annihilated part of V. Moreover, the G-submodules of V coincide with the $\mathcal{L}(G)$-submodules of V.*

Proof Without loss of generality, we may assume that V is of finite dimension. Let $f\colon G \longrightarrow \operatorname{Aut}(V)$ be the morphism defining the given G-action. Let $\operatorname{Fix}(G)$ be the fixed part of V under the action of G and define $\operatorname{Fix}\big(f(G)\big)$ in the same way. Let $\operatorname{Ann}\big(\mathcal{L}(G)\big)$ be the annihilated part of V under the action of $\mathcal{L}(G)$ and define $\operatorname{Ann}\Big(\mathcal{L}\big(f(G)\big)\Big)$ in the same way. Then, by the affine theory, we have $\operatorname{Fix}\big(f(G)\big) = \operatorname{Ann}\Big(\mathcal{L}\big(f(G)\big)\Big)$. But $\operatorname{Fix}(G) = \operatorname{Fix}\big(f(G)\big)$ and $\operatorname{Ann}\big(\mathcal{L}(G)\big) = \operatorname{Ann}\Big(\mathcal{L}\big(f(G)\big)\Big)$. Moreover, $\mathcal{L}(f(G) = f^o\big(\mathcal{L}(G)\big)$ by (1.1) (since $K[f(G)]$ can be viewed inside $K[G]$). Hence $\operatorname{Fix}(G) = \operatorname{Ann}\Big(f^o\big(\mathcal{L}(G)\big)\Big)$. Similarly, the G-submodules of V coincide with the $\mathcal{L}(G)$-submodules of V. This proves Theorem 1.6.

The proof of the following theorem is verbatim that of the affine case (*cf.* [H3, Thm. 2.3, p. 66].

Theorem 1.7 *If $x \in \mathcal{L}(G)$ then, as in the affine case, we have the additive Jordan components $x^{(n)}$ and $x^{(s)}$. These components are in $\mathcal{L}(G)$. Furthermore, if $f\colon G \longrightarrow H$*

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is a morphism of pro-affine algebraic group s, one has

$$f^o(x^{(n)}) = \big(f^o(x)\big)^{(n)} \quad \textit{and} \quad f^o(x^{(s)}) = \big(f^o(x)\big)^{(s)}.$$

2 Algebraic Hulls

Unless otherwise stated, we shall assume throughout this section that G is a connected pro-affine algebraic group over K and $\mathrm{char}(K) = 0$.

Let A be a sub Lie algebra of $\mathcal{L}(G)$, and let $G = \varprojlim G_i$ and $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ be the standard limits for G and $\mathcal{L}(G)$. Let $G(A_i)$ be the smallest affine algebraic subgroup of G_i whose Lie algebra contains A_i, so $\mathcal{L}\big(G(A_i)\big) = (A_i)^+$ is the algebraic hull of A_i in $\mathcal{L}(G_i)$ [B, II.7.1][H3, Thm. 2.2, p. 49]. For each $j > i$, let $\pi_{ji}\colon G_j \longrightarrow G_i$ be the (j,i) transition map in $G = \varprojlim G_i$. Then $(\pi_{ji})^o\colon \mathcal{L}(G_j) \longrightarrow \mathcal{L}(G_i)$ is the (j,i) transition map in $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$. Since $(\pi_{ji})^o(A_j) = A_i$, we must have $(\pi_{ji})^o\big((A_j)^+\big) = (A_i)^+$ and $\pi_{ji}\big(G(A_j)\big) = G(A_i)$ [B, II.7.2]. Finally, let

$$G(A) = \varprojlim G(A_i).$$

Then $G(A)$ is an algebraic subgroup of G and each projection $G(A) \longrightarrow G(A_i)$ is surjective by (1.2)(b). Hence, by (1.2)(a) $\mathcal{L}\big(G(A)\big) = \varprojlim (A_i)^+$.

We shall write A^+ for $\mathcal{L}\big(G(A)\big)$, so $A^+ = \varprojlim (A_i)^+$. In fact, A^+ is the smallest algebraic sub Lie algebra of $\mathcal{L}(G)$ containing A by the following theorem.

Theorem 2.1 *Let A be a sub Lie algebra of $\mathcal{L}(G)$. Then the smallest algebraic subgroup $G(A)$ of G whose Lie algebra contains A exists. That is, $A \subset \mathcal{L}\big(G(A)\big)$ and $G(A)$ is contained in every algebraic subgroup of G whose Lie algebra contains A. Moreover, $G(A)$ is connected.*

Proof Let $G(A)$ be the above constructed algebraic subgroup of G, so $G(A) = \varprojlim G(A_i)$ and $\mathcal{L}\big(G(A)\big) = \varprojlim (A_i)^+$. Hence $A \subset \mathcal{L}\big(G(A)\big)$ since $A \subset \varprojlim A_i \subset \varprojlim (A_i)^+$. Now suppose that $A \subset \mathcal{L}(H)$ where H is an algebraic subgroup of G. Then $A_i \subset \big(\mathcal{L}(H)\big)_i = \mathcal{L}(H_i)$ by (1.1). Hence $G(A_i) \subset H_i$ by the definition of $G(A_i)$. But $G(A) = \varprojlim G(A_i)$ by definition and $H = \varprojlim H_i$ by (1.2)(a). Hence $G(A) \subset H$. Finally, since each projection $G(A) \longrightarrow G(A_i)$ is surjective and each $G(A_i)$ is connected, it follows that $G(A)$ is connected, and Theorem 2.1 is proved.

Corollary 2.2 *Let A and B be algebraic subgroups of G. Then $\mathcal{L}(A \cap B) = \mathcal{L}(A) \cap \mathcal{L}(B)$.*

Corollary 2.3 *Suppose A and B are connected algebraic subgroups of G. If $\mathcal{L}(A) \subset \mathcal{L}(B)$, then $A \subset B$.*

Let $\tau \in \mathcal{L}(G)$. We shall view τ as a differentiation of $K[G]$. Let $\tau_{[}$ be the corresponding derivation of $K[G]$ commuting with the translation action given by $(f.x)(y) = f(yx)$. Then the dual K-space of $K[G]$ is a K-algebra with the multiplication obtained by dualizing the comultiplication in $K[G]$. In fact, $(\tau)^n = \tau \circ (\tau_{[})^{n-1}$ [H3, p. 47]. Let

$$J_\tau = \{f \in K[G] : (\tau)^n(f) = 0 \text{ for every integer } n\}$$

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where $(\tau)^0(f) = f(1_G)$ and let G_τ be the annihilator of J_τ in G. Then J_τ is a biideal of $K[G]$ [H3, bottom p. 46]. Moreover, J_τ is a Hopf ideal since $\tau\big(s(f)\big) = -\tau(f)$ where s is the antipode of $K[G]$. Hence G_τ is an algebraic subgroup of G.

Theorem 2.4 *Let $\tau \in \mathcal{L}(G)$ and let G_τ be the algebraic subgroup of G which is the annihilator of J_τ. Then G_τ is the smallest algebraic subgroup of G whose Lie algebra contains τ. Morover, G_τ is connected.*

The proof is a slight modification of the proof in the affine case as given in [H3, Thm. 2.1, p. 47]. Since our base field K is algebraically closed, we shall only need the first two and a half paragraphs of the proof. In fact, the first two paragraphs apply to any pro-affine algebraic group, whereas the last half paragraph relies on the fact to be checked that the ideal J_τ which was shown to be prime is precisely the annihilator of G_τ in $K[G]$. This is to make sure that

$$\tau \in \mathcal{L}(G_\tau).$$

In the affine case, this is true by Hilbert Nullstellensatz. In the pro-affine case, we need equivalently the fact that the K-algebra homomorphisms of $H = K[G]/J_\tau$ separate the elements of H. But this is true since H is a commutative Hopf algebra with no non-zero nilpotent elements over an algebraically closed field [H-M2, Thm. 2.1, p. 1131] or [W, Exercise 3(b), p. 119].

Remark As a corollary to Theorem 2.4 we obtain Theorem 2.1 as follows. Given a sub Lie algebra A of $\mathcal{L}(G)$, let G_A be the intersection of the family of all algebraic subgroups of G whose Lie algebras contains A. Then Theorem 2.4 implies immediately that G_A is the smallest algebraic subgroup of G whose Lie algebra contains A. Moreover, G_A is connected by (1.2)(c) since $\mathcal{L}(G_A) = \mathcal{L}\big((G_A)^1\big)$.

Corollary 2.5 *Let $f\colon G \to H$ be a morphism of pro-affine algebraic groups over K. Then $\mathcal{L}(\operatorname{Ker} f) = \operatorname{Ker} f^o$.*

The proof follows easily from Theorem 2.4 as shown in [H3, Thm. IV.2.3].

3 The Pro-Discrete Topology on $\mathcal{L}(G)$

Unless otherwise stated, we shall assume throughout this section that G is a connected pro-affine algebraic group over K and $\operatorname{char}(K) = 0$.

Let $G = \varprojlim G_i$ and $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ be the standard limits for G and $\mathcal{L}(G)$. We shall give each $\mathcal{L}(G_i)$ the discrete topology and the resulting inverse limit topology on $\mathcal{L}(G)$ will be called the *pro-discrete topology* of $\mathcal{L}(G)$. So a subset A of $\mathcal{L}(G)$ is closed if and only if $A = \varprojlim A_i$ and the (pro-discrete) closure of A in $\mathcal{L}(G)$ will be denoted by A^c which is given by $A^c = \varprojlim A_i$ [Bk, Cor. 1.4.4, p. 49].

Since $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ is an inverse limit of discrete finite-dimensional vector spaces, $\mathcal{L}(G)$ is *linearly compact* in the sense that given any family of closed cosets (of subspaces) of $\mathcal{L}(G)$ with the finite intersection property, then their intersection is non-empty [Lf, (27.1), (27.6), p. 78].

Lemma 3.1

(a) *Assume* char(K) *to be arbitrary. If H is an algebraic subgroup of $\mathcal{L}(G)$, then $\mathcal{L}(H)$ is closed in $\mathcal{L}(G)$. Moreover, the induced topology on $\mathcal{L}(H)$ coincides with the pro-discrete topology on $\mathcal{L}(H)$.*
(b) *Every finite-dimensional subspace of $\mathcal{L}(G)$ is closed.*

Proof Part (a) follows from (1.2)(a) and before giving a direct proof of (b), we note that (b) follows from the general theory of linear compactness in linearly topologized vector spaces [Lf, (27.5), (27.7), p. 78]. If A is a finite-dimensional subspace of $\mathcal{L}(G)$, then the restrictions of the transition maps $\mathcal{L}(G) \to \mathcal{L}(G_i)$ to A are injective for sufficiently large i. Consequently, $A = \varprojlim A_i$ and so A is closed in $\mathcal{L}(G)$.

Theorem 3.2 *Let M and N be connected normal algebraic subgroups of G and let $[M,N]^c$ be the Zariski closure of $[M,N]$ in G. Then $\mathcal{L}([M,N]^c) = [\mathcal{L}(M),\mathcal{L}(N)]^c$. In particular, if $[\mathcal{L}(G),\mathcal{L}(G)]$ is finite-dimensional, then*

$$\mathcal{L}([G,G]^c) = [\mathcal{L}(G),\mathcal{L}(G)].$$

Proof The proof can be easily reduced to the affine case as follows. By (1.1) and the continuity of each projection $\mathcal{L}(G) \to \mathcal{L}(G_i)$, we have $\big([\mathcal{L}(M),\mathcal{L}(N)]^c\big)_i = [\mathcal{L}(M_i),\mathcal{L}(N_i)]$. Moreover, $\big(\mathcal{L}([M,N]^c)\big)_i = \mathcal{L}([M,N]^c)_i = \mathcal{L}([M_i,N_i])$ since $[M_i,N_i]$ is closed in G_i. But $\mathcal{L}([M_i,N_i]) = [\mathcal{L}(M_i),\mathcal{L}(N_i)]$ [B, II.7.8]. Hence $\big([\mathcal{L}(M),\mathcal{L}(N)]^c\big)_i = \mathcal{L}([M,N]^c)_i$ for every i. Consequently, $[\mathcal{L}(M),\mathcal{L}(N)]^c = \mathcal{L}([M,N]^c)$ since this last is closed by (3.1)(a). In particular, if $[\mathcal{L}(G),\mathcal{L}(G)]$ is finite-dimensional, then $\mathcal{L}([G,G]^c) = [\mathcal{L}(G),\mathcal{L}(G)]$ by (3.1)(b) which proves Theorem 3.2.

Similarly, as in the proof of Theorem 3.2, we may reduce the proof of the following to the affine case [H3, Thm. 1.1, p. 107].

Theorem 3.3 *Let L be a sub Lie algebra of nilpotent elements of $\mathcal{L}(G)$. Then L is the Lie algebra of a pro-unipotent algebraic subgroup of G if and only if L is closed in $\mathcal{L}(G)$.*

Theorem 3.4 *Assume* char(K) *to be arbitrary. Let $f\colon G \longrightarrow H$ be a morphism of pro-affine algebraic groups over K. Then the differential f^o of f is continous and maps every closed subspace of $\mathcal{L}(G)$ onto a closed subspace of $\mathcal{L}(H)$.*

Proof First we assume that f is surjective. Then the transpose of f is injective, so we may identify $K[H]$ with its image in $K[G]$. Let $G = \varprojlim G_i$ be the standard limit for G, so $K[G] = \bigcup K[G_i]$ is a directed union. Then $K[H] = \bigcup (K[G_i] \cap K[H])$. But each $K[G_i] \cap K[H]$ is affine being a Hopf subalgebra of a commutative affine Hopf algebra [H3, Thm. 4.3, p. 25], [W, Exercise 10, p. 119]. Hence each $K[G_i] \cap K[H] = K[H_i]$ for some affine algebraic group H_i. Thus $K[H] = \bigcup K[H_i]$ is a directed union. Hence $H = \varprojlim H_i$ and $\mathcal{L}(H) = \varprojlim \mathcal{L}(H_i)$. If $H = \varprojlim H_s$ is the standard limit for H, then each $K[H_s]$ lies in some $K[H_i]$ since every finite subset of $K[H]$ lies in some $K[H_i]$. Hence $\{H_i\}$ is a cofinal subset of $\{H_s\}$.

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The inclusion map $K[H_i] \longrightarrow K[G_i]$ yields a surjective morphism $f_i \colon G_i \longrightarrow H_i$ and it is evident that $f = \varprojlim f_i$ and $f^o = \varprojlim f_i^o$. Now each f_i^o is continous since each of $\mathcal{L}(G_i)$ and $\mathcal{L}(H_i)$ is given the discrete topology. Hence f^o is continous with respect to the pro-discrete toplogies on $\mathcal{L}(G)$ and $\mathcal{L}(H)$ since $\{H_i\}$ is a cofinal subset of $\{H_s\}$.

Since $\mathcal{L}(G)$ is linearly compact, the continous map f^o maps every closed subspace of $\mathcal{L}(G)$ onto a linearly compact closed subspace of $\mathcal{L}(H)$ [Lf, (27.3), (27.4), p. 78]. This proves the proposition in case f is surjective.

Similarly, the proposition is valid in case f is the inclusion map of an algebraic subgroup of H by (1.2)(a) because in this case f is the inverse limit of the evident inclusion maps $G_i \longrightarrow H_i$ where $H = \varprojlim H_i$ is the standard limit for H. Finally, Theorem 3.4 follows by combining the above two cases.

Corollary 3.5 *Let $f \colon G \longrightarrow H$ be a morphism of pro-affine algebraic groups over K. If f is surjective, then so is f^o. Moreover, the converse is true if H is connected.*

Proof Suppose f is surjective. Then, as seen in the proof of Theorem 3.4, $f = \varprojlim f_i$ where each $f_i \colon G_i \longrightarrow H_i$ is surjective. Then $f^o = \varprojlim f_i^o$ and since $\operatorname{char}(K) = 0$, each f_i^o is surjective. Moreover, each projection $\mathcal{L}(G) \longrightarrow \mathcal{L}(G_i)$ and $\mathcal{L}(H) \longrightarrow \mathcal{L}(H_i)$ is surjective by (1.1). Hence $f^o\big(\mathcal{L}(G)\big)$ is algebraically dense in $\mathcal{L}(H)$. But $f^o\big(\mathcal{L}(G)\big)$ is closed in $\mathcal{L}(H)$ by (3.4). Hence $f^o\big(\mathcal{L}(G)\big) = \mathcal{L}(H)$.

Conversely, if f^o is surjective, then $\mathcal{L}\big(f(G)\big) = f^o\big(\mathcal{L}(G)\big)$ by the first part of the proof applied to $G \xrightarrow{f} f(G)$. Hence $\mathcal{L}\big(f(G)\big) = \mathcal{L}(H)$ since f^o is surjective. But H is connected. Hence $f(G) = H$ by (1.2)(c) and (2.3), so f is surjective.

Example 3.6 Assume K to be of prime characteristic. Let $G = \varprojlim G_n$, $(n \in \mathbf{N})$ where G_n is the additive group K and $G_{n+1} \longrightarrow G_n$ is given by $x \to x^p$. Then G is evidently a non-trivial connected pro-affine algebraic group although $\mathcal{L}(G) = 0$ because $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_n)$ where each transition map is the zero map (*cf.* [Ma1, Example 3.9]). Moreover, if $f \colon \{1\} \longrightarrow G$ is the trivial map, then f^o is surjective although f is not surjective.

These examples show that (2.3) and the converse part in (3.5) may fail in characteristic p *although they never fail in the affine case* for reasons of dimension.

Corollary 3.7 *Let N be a normal algebraic subgroup of G. Then $\mathcal{L}(G/N) \cong \mathcal{L}(G)/\mathcal{L}(N)$.*

Proof Apply (2.5) and (3.5) to the canonical projection $G \to G/N$.

Now we recall the following definition. A morphism $f \colon G \longrightarrow H$ of connected pro-affine algebraic groups over K is called a *covering* if f is surjective, $\operatorname{Ker} f$ is profinite, and the bijective morphism $G/\operatorname{Ker} f \longrightarrow H$ induced by f is an isomorphism of pro-affine algebraic groups [H2, p. 402]. Moreover, G is called *simply connected* if every covering of G is an isomorphism of pro-affine algebraic groups [H2, p. 411].

Proposition 3.8 (*cf.* **[H2, Prop. 3.1]**) *Let $f \colon G \longrightarrow H$ be a morphism of pro-affine algebraic groups over K. Then f is a covering if and only if f^o is an isomorphism.*

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Proof Combine (2.5) and (3.5) (with (2.3) and (1.2)(c)).

Proposition 3.9 *Let $f\colon G \to H$ be a morphism of connected pro-affine algebraic groups over K and let A be a sub Lie algebra of $\mathcal{L}(G)$. Then $f^o(A^+) = \big(f^o(A)\big)^+$ and $f\big(G(A)\big) = G\big(f(A)\big)$ in the notation of Theorem 2.1.*

Proof As in the affine case [B, II.7.2], the first part follows from Theorem 4.1 and the fact that under f^o, both the images and inverse images of algebraic Lie algebras are algebraic by virtue of (2.5), (3.5), and (2.2). Now the first part with (3.5) show that $f\big(G(A)\big)$ and $G\big(f(A)\big)$ have the same Lie algebra. But they are also connected. Hence they are equal by (2.3).

Theorem 3.10 *Let L be a sub Lie algebra of $\mathcal{L}(G)$. Then $[L,L]^c = [L,L]^+ = [L^+,L^+]^c$. In particular, if $[L,L]$ is finite-dimensional, then $[L,L] = [L,L]^+ = [L^+,L^+]$, so L is an ideal in L^+ and $[L,L]$ is algebraic in $\mathcal{L}(G)$.*

Proof Let $\pi_i\colon G \to G_i$ and $\pi_i^o\colon \mathcal{L}(G) \to \mathcal{L}(G_i)$ be as in (1.1). Then $\pi_i^o[L,L] = [L_i,L_i]$. Moreover, $\pi_i^o[L^+,L^+] = [(L_i)^+,(L_i)^+]$ by (3.9) and $[(L_i)^+,(L_i)^+] = [L_i,L_i]$ in the affine case [B, p. 109], [H3, pp. 112–3]. Similarly, $\pi_i^o[L,L]^+ = [L_i,L_i]$. Hence $[L,L]$, $[L,L]^+$, and $[L^+,L^+]$ have the same closure in $\mathcal{L}(G)$. Now we assume that $[L,L]$ is finite-dimensional. Then $[L,L]$ is closed in $\mathcal{L}(G)$ by (2.1). Hence $[L,L] = [L,L]^+ = [L^+,L^+]^c$. In particular, $[L,L]^+$ and $[L^+,L^+]$ are contained in $[L,L]$, so they are finite-dimensional and hence they are closed in $\mathcal{L}(G)$. Consequently, $[L,L] = [L^+,L^+] = [L,L]^+$, so L is an ideal in L^+ and $[L,L]$ is algebraic in $\mathcal{L}(G)$. This proves (3.10).

Lemma 3.11 *Let $f\colon \mathcal{L}(G) \to V$ be a linear map where V is a vector space (over K) such that* Ker f *is closed in $\mathcal{L}(G)$. Let $\{C_j : j \in J\}$ be a family of closed cosets of $\mathcal{L}(G)$ such that for all r, s in J, there exists $t \in J$ such that $C_t \subset C_r \cap C_s$. Then $f(\bigcap C_j) = \bigcap f(C_j)$.*

Lemma 3.11 is a consequence of the linear compactness of $\mathcal{L}(G)$ as follows (*cf.* [Lp, satz 1]). Let $y \in \bigcap f(C_j)$ so each $C_j \cap f^{-1}(y)$ is non-empty. Since Ker f is closed in $\mathcal{L}(G)$, then so is $f^{-1}(y)$. Hence each $C_j \cap f^{-1}(y)$ is a non-empty closed coset of $\mathcal{L}(G)$ (since the intersection of cosets is either empty or a coset). Now the family $\{C_j \cap f^{-1}(y)\}$ has the finite intersection property, so their intersection is non-empty by the linear compactness of $\mathcal{L}(G)$. Hence $y \in f(\bigcap C_j)$ which proves Lemma 3.11.

Let $G = \varprojlim G_i$ be the standard limit for G. Then G is called *pro-nilpotent* if each G_i is nilpotent and G is called *pro-solvable* if each G_i is solvable.

For every sub Lie algebra L of $\mathcal{L}(G)$, let $\{D^n(L)\}$ be the derived (commutator) series of L and let $\{C^n(L)\}$ be the lower central series of L.

Theorem 3.12

(a) *G is pro-nilpotent if and only if $\bigcap\Big(C^n\big(\mathcal{L}(G)\big)\Big)^c = 0$.*

(b) *G is pro-solvable if and only if $\bigcap\Big(D^n\big(\mathcal{L}(G)\big)\Big)^c = 0$.*

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Proof The proof relies on the linear compactness of $\mathcal{L}(G)$.

(a) We consider each projection $\mathcal{L}(G) \to \mathcal{L}(G_i)$ where $\mathcal{L}(G) = \varprojlim \mathcal{L}(G_i)$ is the standard limit for $\mathcal{L}(G)$. Then $\Bigl(\bigl(C^n\bigl(\mathcal{L}(G)\bigr)\bigr)^c\Bigr)_i = C^n\bigl(\mathcal{L}(G_i)\bigr)$ by (1.1) and the continuity of each projection $\mathcal{L}(G) \to \mathcal{L}(G_i)$. Consequently, by Lemma 3.11, $\bigcap\Bigl(C^n\bigl(\mathcal{L}(G)\bigr)\Bigr)^c = 0$ if and only if $\bigcap C^n\bigl(\mathcal{L}(G_i)\bigr) = 0$ for every i. But $\bigcap C^n\bigl(\mathcal{L}(G_i)\bigr) = 0$ if and only if G_i is nilpotent by the affine theory since G and hence G_i is connected. Hence $\cap\Bigl(C^n\bigl(\mathcal{L}(G)\bigr)\Bigr)^c = 0$ if and only if each G_i is nilpotent. This proves (a), and similarly, (b) is proved.

4 Representative Functions on Lie Algebras

We shall assume throughout this section that $\operatorname{char}(K) = 0$.

Let L be a Lie algebra over K and recall that L is *residually finite dimensional* if all its finite-dimensional representations separate the points of L. Let $H(L)$ be the Hopf algebra (continous) dual of the universal enveloping Hopf algebra $U(L)$ of L. Since $\operatorname{char}(K) = 0$, $H(L)$ is an integral domain [H2, p. 405]. In fact, $H(L)$ is the algebra of representative functions on L which are the matrix coordinate functions of the finite-dimensional representations of L. The proof of this is verbatim of the proof in the case where L is finite-dimensional [H1, top p. 500]. Consequently, L is residually finite dimensional if and only if the elements of $H(L)$ separate the points of L.

We define the *universal pro-affine algebraic group* $G(L)$ associated with L to be the (connected) pro-affine algebraic group over K associated with $H(L)$. Thus $K[G(L)] = H(L)$. This can be done since $H(L)$ is an integral domain Hopf algebra [H-M2, Thm. 2.1]. Thus $\mathcal{L}\bigl(G(L)\bigr)$ is the Lie algebra of all differentiations of $H(L)$. If $\mu\colon L \longrightarrow \mathcal{L}\bigl(G(L)\bigr)$ is the canonical map by evaluating at elements of L, then μ is injective if and only if L is residually finite dimensional. *So if L is residually finite dimensional, L will be identified with its canonical image in $\mathcal{L}\bigl(G(L)\bigr)$ and μ is the identity.*

Let G be a pro-affine algebraic group over K. Then $\mathcal{L}(G)$ acts naturally on $K[G]$ by its proper derivation action which commutes with the translation action $f \to f.g$ where $(f.g)(x) = f(gx)$ [H2, p. 405]. This makes $K[G]$ into a $U\bigl(\mathcal{L}(G)\bigr)$-module. Consequently, if A is any sub Lie algebra of $\mathcal{L}(G)$ we have a canonical Hopf algebra homomorphism

$$\pi_{G,A}\colon K[G] \longrightarrow H(A)$$

given by $\pi_{G,A}(f)(u) = (u.f)(1_G)$ for every u in $U(A)$.

Theorem 4.1 **(*cf.* [H2, Prop. 4.1])** *Let $\pi_{G,A}\colon K[G] \longrightarrow H(A)$ be as above where A is a sub Lie algebra of $\mathcal{L}(G)$ and assume that G is connected. Then $\pi_{G,A}$ is injective if and only if A is algebraically dense in $\mathcal{L}(G)$.*

Proof We shall write π for $\pi_{G,A}$. Suppose that A is algebraically dense in $\mathcal{L}(G)$ and consider the universal pro-affine $G(A)$ associated with A so $H(A) = K[G(A)]$. Let $\rho\colon G(A) \longrightarrow G$ be the transpose of π, and let $\mu\colon A \longrightarrow \mathcal{L}\bigl(G(A)\bigr)$ be the canonical

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map (as above). If $f \in K[G]$ and $a \in A$, we have $\rho^o\big(\mu(a)\big)(f) = \mu(a)\big(\rho^t(f)\big) = \mu(a)\big(\pi(f)\big) = \pi(f)(a) = (a.f)(1_G) = a(f)$ where a is viewed in the last equality as a differentiation on $K[G]$. Thus $\rho^o \circ \mu \colon A \longrightarrow \mathcal{L}(G)$ is the identity on A. Hence $A \subset \rho^o\Big(\mathcal{L}\big(G(A)\big)\Big) \subset \mathcal{L}(G)$. But A is algebraically dense in $\mathcal{L}(G)$ and $\rho^o\Big(\mathcal{L}\big(G(A)\big)\Big)$ is algebraic in $\mathcal{L}(G)$ by (3.5). Hence ρ^o is surjective. Consequently ρ is surjective by (3.5), so its transpose π is injective.

Conversely, suppose π is injective. Let $A^+ = \mathcal{L}(H)$ where H is some algebraic subgroup of G and consider the following diagram.

$$\begin{array}{ccc} K[G] & \xrightarrow{\ \pi\ } & H(A) \\ \Big\downarrow \rho & & \Big\downarrow \alpha \\ K[H] & \xrightarrow{\ \pi_H\ } & H\big(\mathcal{L}(H)\big) \end{array}$$

where ρ is the surjective restriction map to the algebraic subgroup H, $\pi_H = \pi_{H,\mathcal{L}(H)}$ and α is the morphism induced by the identity map $L \longrightarrow \mathcal{L}(H)$. Then, by definitions, $(\alpha \circ \pi_H \circ \rho)(f)(u) = (\pi_H \circ \rho)(f)(u) = \big(u.\rho(f)\big)(1_H) = (u.f)(1_G)$. Hence the above diagram is commutative. But π is given to be injective. Hence ρ is injective. Consequently, $\rho \colon K[G] \longrightarrow K[H]$ is an isomorphism and $G = H$. Hence A is algebraically dense in $\mathcal{L}(G)$. This completes the proof of Theorem 4.1.

Corollary 4.2 *Let L be a residually finite dimensional Lie algebra over K, so we may identify L with its canonical image in $\mathcal{L}\big(G(L)\big)$. Then $G(L)$ has the following universal property. Let H be any connected pro-affine algebraic group over K and let $i \colon L \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism whose image is algebraically dense in $\mathcal{L}(H)$. Then there exists a canonical surjective morphism $f \colon G(L) \longrightarrow H$ whose differential agrees with i on L.*

Let L be as in (4.2), so $L \subset \mathcal{L}\big(G(L)\big)$. Then $U(L)$ acts naturally on $K[G(L)]$ by the proper derivation action of $\mathcal{L}\big(G(L)\big)$ on $K[G(L)]$ which commutes with every translation action $f \to f.g$. Moreover, $H(L) = K[G(L)]$ and $U(L)$ acts naturally on $H(L)$ by the left translation action given by $(u.f)(x) = f(xu)$ for every u and x in $U(L)$.

Lemma 4.3 *The above two natural actions of $U(L)$ on $H(L) = K[G(L)]$ are identical.*

To see this, we need to recall some definitions. Let H be a Hopf algebra over K. Then the Lie algebra $\mathcal{L}(H)$ of H consists of all differentiations of H. That is, all linear maps $l \colon H \longrightarrow K$ such that $l(f.g) = l(f)\epsilon(g) + \epsilon(f)l(g)$ where ϵ is the co-unit of H. Every differentiation l of H determines a derivation $l_{[} = (\mathrm{id} \otimes l) \circ \Delta$ where Δ is the comultiplication of H. All such derivations are called *proper* and we have a Lie algebra isomorphism from $\mathcal{L}(H)$ onto the Lie algebra of all proper derivations of H [H2, p. 404], [H3, p. 36]. Now let l be an element of L. Since $L \subset \mathcal{L}\big(G(L)\big) = \mathcal{L}\big(H(L)\big)$, $l_{[}$ is a derivation of $K[G(L)] = H(L)$. With respect to $K[G(L)]$, the proper derivation action $l_{[}$ of l is the usual derivation action which commutes with every translation action $f \to f.x$ [H-M2, p. 1128]. With respect to $H(L)$, the formula $l_{[} = (\mathrm{id} \otimes l) \circ \Delta$ implies that $(l_{[}.f)(x) = \sum f_i(x) f_i'(l) = f(xl)$ for

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every x in $U(L)$ where $\Delta(f) = \sum f_i \otimes f_i'$. Hence $(u.f)(x) = f(xu)$ for every u and x in $U(L)$. This proves the Lemma.

Proposition 4.4 *Let L be a residually finite dimensional Lie algebra over K and let $G(L)$ be the universal pro-affine algebraic group associated with L. Then*

(a) $L^+ = \mathcal{L}\big(G(L)\big)$*, i.e., L is algebraically dense in $\mathcal{L}\big(G(L)\big)$, and*
(b) *$G(L)$ is simply connected.*

Proof Put $G = G(L)$. To prove (a), let $\pi\colon K[G] \longrightarrow H(L)$ be as in Theorem 4.1 where $\pi = \pi_{G,\mathcal{L}(G)}$. We claim that π is the identity map on $K[G] = H(L)$. So let $f \in K[G]$, let $u \in U(L)$, and let $i\colon K[G] \longrightarrow H(L)$ be the identity map (for clarity). Then by definitions, we have $(\pi\circ\rho)(f)(u) = (u.f)(1_G) = \varepsilon_G(u.f)$ where ε_G is the co-unit of $K[G]$. If $\varepsilon_{H(L)}$ is the co-unit of $H(L)$, we have $\varepsilon_G(u.f) = \varepsilon_{H(L)}\big(u.i(f)\big)(1) = \big(i(f)\big)(u)$ where the first equality follows from Lemma 4.3 and the other equality follows by definition. Thus, $\pi = i$. Hence π is injective. Consequently L is algebraically dense in $\mathcal{L}(G)$ by Theorem 4.1.

To prove (b), let $\beta\colon A \longrightarrow G$ be a covering and let $\beta^t\colon K[G] \longrightarrow K[A]$ be its transpose map. Then β^t is injective. To prove that β^t is surjective, let $f \in K[A]$. Since β is a covering, β^o is bijective by (3.8). Thus we may view L inside $\mathcal{L}(A)$, so β^o is the identity on L. Since L is algebraically dense in $\mathcal{L}(G)$ by part (a), it follows that L is algebraically dense in $\mathcal{L}(A)$ by (3.9). Now by the universal property of G or $G(L)$ in (4.2), there exists a surjective morphism $\sigma\colon G \to A$ whose differential agrees with β^o on L. Since β^o is the identity on L, $(\sigma\circ\beta)^o$ is the identity on L. But L is algebraically dense in $\mathcal{L}(A)$. Hence $(\sigma\circ\beta)^o$ is the identity on $\mathcal{L}(A)$. Since A is connected by assumption, it follows that $\sigma\circ\beta$ is the identity on A and β is injective. Hence, every covering $\beta\colon A \to G$ is an isomorphism. This shows that G is simply connected and Proposition 4.4 is proved.

In view of (3.9) and (3.10), Proposition 4.4 has the following consequences.

Corollary 4.5 *In the setting of Proposition 4.4, let $f\colon G(L) \to H$ be a surjective morphism onto an affine algebraic group H. Then $f^o(L)$ is algebraically dense in $\mathcal{L}(H)$.*

Corollary 4.6 (*cf.* **[H1, top p. 521]**) *In the setting of Proposition 4.4, suppose $[L, L]$ is finite-dimensional. Then $[L, L] = \big[\mathcal{L}\big(G(L)\big), \mathcal{L}\big(G(L)\big)\big] = \mathcal{L}(D)$ where D is the Zariski closure of $[G(L), G(L)]$. In particular, L is an ideal of $\mathcal{L}\big(G(L)\big)$ and $\mathcal{L}\big(G(L)\big)/L$ is abelian.*

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