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ISOMETRIC MAPPINGS OF NON-COMMUTATIVE L_p SPACES

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If the L_p spaces of two measure spaces are "the same", to what extent can we identify the underlying measure spaces? This question has been partially answered by Schneider [**7**] (extending results of Forelli [**2**]). He proves that a linear isometry between the L_p spaces of two finite measure spaces is in fact an (isometric) homomorphism between the corresponding L_∞ spaces, if it preserves the identity.

Kadison [**4**] and later Russo [**10**], have considered what might be called non-commutative analogues of the above problem. Their point of view is different from ours, however, since their "measure spaces" are already in bijective correspondence by assumption, and their goal is to determine how much of the algebraic structure is transferred by this bijection.

In this paper, we attempt to extend Schneider's result to the non-commutative case, thus strengthening Theorem 2 of Russo [**10**]. Specifically, we consider two finite Von Neumann algebras $\mathscr{A}_1$, $\mathscr{A}_2$ with faithful traces m_1, m_2, and a *-linear map T from a *-subalgebra $\mathscr{U}$ of $\mathscr{A}_1$ to L_p $(\mathscr{A}_2, m_2)$ for some $p > 2$, which preserves the identity and the L_p-norm (see Segal [**8**] for the relevant definitions). We prove that T must be a Jordan homomorphism, and must preserve the operator norm (and thus, by the Riesz-Thorin-Kunze theorem [**5**], all L_q-norms for $q > 2$). In the absence of commutativity, we cannot conclude that T is an associative homomorphism without some extra assumptions. In fact, if $\mathscr{A}_2$ is a factor, then we can show that T must be either an (associative) homomorphism or an antihomomorphism.

The results of this paper are similar to well known results of Kadison [**4**]. However, our hypotheses are weaker, in that he considers the mapping T to be an isometric bijection between $\mathscr{A}_1$ and $\mathscr{A}_2$. Furthermore, his results are not applicable to our problem (but see Corollary 2.1 (ii)), because we need to prove first that T is a Jordan homomorphism (using a method entirely different from Kadison's) in order to be able to conclude that it preserves the operator norm. A similar relation exists between our results and results of B. Russo [**10**]. We note that our Theorem 2 is stronger, since, starting from weaker assumptions (namely, that T maps a *-subalgebra $\mathscr{U}$ of $\mathscr{A}_1$ *into* L_p $(\mathscr{A}_2, m_2)$ rather than $\mathscr{A}_1$ onto itself, and that T is *-linear, rather than positivity preserving) we are able to get stronger conclusions (namely, positivity preservation, and

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preservation of all L_p-norms, for p in $[2, \infty]$). Finally, we note that M. Broise [**9**] has obtained partial results in the semi-finite case.

Throughout this paper, we let $\mathscr{A}_1, \mathscr{A}_2$ be two finite Von Neumann algebras. Thus there exist faithful, central, normal states m_i on $\mathscr{A}_i$ $(i = 1, 2)$. Then if $\mathscr{A}_i$ acts on the Hilbert space $\mathscr{H}_i$, $(\mathscr{H}_i, \mathscr{A}_i, m_i)$ $(i = 1, 2)$ are finite regular gage spaces in the sense of Segal [**8**].

We need a technical result constituting an extension to the present, non-commutative case, of results of Schneider [**7**] and Forelli [**2**];

THEOREM 1. *Let $0 < p < \infty$, $f_i \in L_p(\mathscr{H}_i, \mathscr{A}_i, m_i)$ $(i = 1, 2)$ f_i normal. Suppose that there is a positive constant A, such that, whenever $z \in \mathbf{C}$ is such that $|z| < A$, we have*

$$||1 + zf_1||_{L_p(m_1)} = ||1 + zf_2||_{L_p(m_2)}.$$

Then

(a) $||f_1||_{L_2(m_1)} = ||f_2||_{L_2(m_2)}$

(b) *If $p > 2$, then* $||f_1||_{L_4(m_1)} = ||f_2||_{L_4(m_2)}$.

Proof. Let $\mathscr{B}_i \subseteq \mathscr{A}_i$ be the Von Neumann algebra generated by the spectral projections of f_i (that is by the projections $e_\lambda{}^i$ such that $f_i = \int_{\mathbf{C}} \lambda de_\lambda{}^i$. Since f_i may be identified with a closed densely defined operator acting on $\mathscr{H}_i$ (this is because the gages are finite; see [**6**, Theorems 4 and 5]), it follows that $e_\lambda{}^i \in \mathscr{A}_i$).

Then $(\mathscr{H}_i, \mathscr{B}_i, m_i|_{\mathscr{B}i})$ is a commutative finite regular gage space. It is therefore [**8**, pp. 402–3] algebraically equivalent to the gage space built on a finite measure space $(\mathscr{X}_i, \sigma_i)$. Since f_i is measurable with respect to $\mathscr{B}_i$ [**8**, Definition 2.1] it follows by [**8**, Theorem 2] that f_i corresponds, under the above equivalence, to a measurable function φ_i on $(\mathscr{X}_i, \sigma_i)$.

We now apply the commutative theorem of Forelli-Schneider to the functions φ_i on the measure spaces $(\mathscr{X}_i, \sigma_i)$. Note that, if $z \in \mathbf{C}$ is such that $|z| < A$,

$$\begin{aligned} ||1 + z\varphi_1||_{L_p(\sigma_1)} &= \left[\int |1 + z\varphi_1(x)|^p d\sigma_1(x)\right]^{1/p} \\ &= [m_1(|1 + zf_1|^p)]^{1/p} \quad \text{by the above equivalence} \\ &= [m_2(|1 + zf_2|^p)]^{1/p} = \left[\int |1 + z\varphi_2(x)|^p d\sigma_2(x)\right]^{1/p} \\ &= ||1 + z\varphi_2||_{L_p(\sigma_2)} < \infty \quad \text{since } f_i \in L_p(\mathscr{H}_i, \mathscr{A}_i, m_i). \end{aligned}$$

Thus the hypotheses of [**2**, Proposition 1] and [**7**, Theorem A] are satisfied, and so we conclude

(a) $||\varphi_1||_{L_2(\sigma_1)} = ||\varphi_2||_{L_2(\sigma_2)}$

and

(b) If $p > 2$, then $||\varphi_1||_{L_4(\sigma_1)} = ||\varphi_2||_{L_4(\sigma_2)}$.

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The desired conclusion now follows from the fact that if $0 < q < \infty$,

$$\|f_i\|^q_{L_q(m_i)} = m_i(|f_i|^q) = \int |\varphi_i(x)|^q d\sigma_i(x) = \|\varphi_i\|^q_{L_q(\sigma_i)}.$$

THEOREM 2. *Let $\mathscr{U} \subseteq \mathscr{A}_1$ be a unital *-subalgebra. For some p in $(2, \infty)$, let*

$$T: \mathscr{U} \to L_p(\mathscr{A}_2, m_2)$$

*be a *-linear map such that $T(1) = 1$. Suppose that*

$$\|Tf\|_{L_p(m_2)} = \|f\|_{L_p(m_1)} \quad \text{for every normal } f \in \mathscr{U}.$$

Then T is a Jordan homomorphism, that is,

$$T(fg + gf) = TfTg + TgTf, \quad f, g \in \mathscr{U}.$$

Remark 1. Young [**12**] has shown, based on the coincidence of the L_p topology and the strong topology on the unit ball of $\mathscr{A}_1$ (Dixmier [**3**]) that T admits an extension T_e to the weak closure $\mathscr{U}^-$ of $\mathscr{U}$, which is also an L_p-isometry. By Corollary 2.1 (see below) $T_e(\mathscr{U}^-) \subseteq \mathscr{A}_2$. By Dixmier's result, T_e will be ultraweakly continuous at 0, hence everywhere in $\mathscr{A}_1$. This provides a quicker, if indirect, proof of Lemma 3.1 of Størmer [**11**].

Remark 2. Russo [**10**] provides an example showing that our assumptions are too weak for the case $p = 2$. In this case, the stronger assumptions of his Theorem 2 are essential.

Proof. (i) Let $f \in \mathscr{U}$ be self-adjoint, and $z \in \mathbf{C}$. Since $T(1 + zf) = 1 + zTf$, we have (since Tf is also self-adjoint)

$$\|1 + zf\|_{L_p(m_1)} = \|1 + zTf\|_{L_p(m_2)}.$$

Thus Theorem 1 (b) shows

$$\|1 + zf\|_{L_4(m_1)} = \|1 + zTf\|_{L_4(m_2)} < \infty, \quad \text{since } f \in \mathscr{A}_1 \subseteq L_4(m_1).$$

Now

$$|1 + zf|^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k f^j f^k$$

and so

$$\|1 + zf\|_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k m_1(f^j f^k).$$

Similarly

$$\|1 + zTf\|_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k m_2((Tf)^j (Tf)^k).$$

Therefore

(1) $\quad m_1(f^j f^k) = m_2((Tf)^j (Tf)^k), \quad j, k = 0, 1, 2.$

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(ii) Putting $j = 1$, $k = 2$, in (1) yields

$$m_2((Tf)^3) = m_1(f^3).$$

Replacing f with $f + ag$, a real, f, g self-adjoint, expanding and comparing terms in a^2, we find

$$m_2(Tf(Tg)^2 + TgTfTg + (Tg)^2Tf) = m_1(fg^2 + gfg + g^2f)$$

or, in view of the centrality of the traces

$$(2) \quad m_2(Tf(Tg)^2) = m_1(fg^2).$$

On the other hand, putting $j = k = 1$ in (1) yields

$$m_2((Tf)^2) = m_1(f^2)$$

which, upon "linearization" and use of centrality as above, yields

$$m_2(TfTg) = m_1(fg).$$

Replacing g by g^2 above, and comparing the result with (2) we find

$$m_2(Tf(Tg)^2) = m_2(TfT(g^2))$$

and, replacing f by g^2, we get

$$(3) \quad m_2(T(g^2)(Tg)^2) = m_2((T(g^2))^2).$$

Finally, if we put $j = k = 2$ *in* (1), we find

$$m_2((Tg)^4) = m_1(g^4)$$

while (2) with $f = g^2$ becomes

$$m_2(T(g^2)(Tg)^2) = m_1(g^4)$$

hence

$$(4) \quad m_2((Tg)^4) = m_2(T(g^2)(Tg)^2).$$

Therefore

$$||(Tg)^2 - T(g^2)||_2^2 = m_2((Tg)^4 - (Tg)^2T(g^2) - T(g^2)(Tg)^2 + (T(g^2))^2) = 0$$

by (3) and (4), and so $(Tg)^2 = T(g^2)$ for every self-adjoint g in $\mathscr{U}$.

(iii) Now let $f \in \mathscr{U}$ be arbitrary, and write $f = f_1 + if_2$ with $f_1, f_2 \in \mathscr{U}$ self-adjoint. Since $f_1 + f_2$ is self-adjoint, part (ii) yields

$$T((f_1 + f_2)^2) = (T(f_1 + f_2))^2 = (Tf_1 + Tf_2)^2.$$

That is,

$$\begin{aligned} T(f_1^2 + f_2^2 + f_1f_2 + f_2f_1) &= T(f_1^2) + T(f_2^2) + T(f_1f_2 + f_2f_1) \\ &= (Tf_1)^2 + (Tf_2)^2 + (Tf_1Tf_2 + Tf_2Tf_1). \end{aligned}$$

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Thus

$$T(f_1f_2 + f_2f_1) = Tf_1Tf_2 + Tf_2Tf_1.$$

Therefore,

$$\begin{aligned} T(f^2) = T((f_1 + if_2)^2) &= T(f_1^2 - f_2^2 + i(f_1f_2 + f_2f_1)) \\ &= (Tf_1)^2 - (Tf_2)^2 + i(Tf_1Tf_2 + Tf_2Tf_1) \\ &= (Tf_1 + iTf_2)^2 = (Tf)^2. \end{aligned}$$

Finally, if $f, g \in \mathscr{U}$ are arbitrary, we have

$$\begin{aligned} T((f+g)^2) &= T(f^2 + g^2 + fg + gf) = (Tf)^2 + (Tg)^2 + T(fg + gf) \\ &= (T(f+g))^2 = (Tf)^2 + (Tg)^2 + TfTg + TgTf. \end{aligned}$$

Therefore,

$$TfTg + TgTf = T(fg + gf).$$

COROLLARY 2.1. (i) *If $f \in \mathscr{U}$ is self-adjoint*, $||Tf||_\infty = ||f||_\infty$.
(ii) *For every $f \in \mathscr{U}$*, $||Tf||_\infty = ||f||_\infty$. *Hence* $T(\mathscr{U}) \subseteq \mathscr{A}_2$.
(iii) *T is positivity preserving.*

Proof. (i) Let $l \in \mathbf{N}$. We have

$$\begin{aligned} ||Tf||^{2l}_{L_{2l}(m_2)} &= m_2(|Tf|^{2l}) = m_2((Tf^*)^l(Tf)^l) \\ &= m_2((T(f^l))^*(T(f^l))) \quad \text{by Theorem 2} \\ &= ||T(f^l)||^2_{L_2(m_2)} = ||f^l||^2_{L_2(m_1)} \quad \text{by Theorem 1 (a)} \\ &= m_1(f^{*l}f^l) = m_1(|f|^{2l}) = ||f||^{2l}_{L_{2l}(m_1)}. \end{aligned}$$

Thus

$$||Tf||_{L_{2l}(m_2)} = ||f||_{L_{2l}(m_1)}.$$

The result follows by letting l tend to infinity.

(ii) If $f \in \mathscr{U}$ is arbitrary, write $f = f_1 + if_2$ with f_k self-adjoint. Since $f_1 = \frac{1}{2}(f + f^*)$, $f_2 = (1/2i)(f - f^*)$, it follows that $||f_k||_\infty \leqq ||f||_\infty$. Therefore

$$||Tf||_\infty \leqq ||Tf_1||_\infty + ||Tf_2||_\infty = ||f_1||_\infty + ||f_2||_\infty \leqq 2||f||_\infty.$$

This shows that $T(\mathscr{U}) \subseteq \mathscr{A}_2$.

Now the proof of Kadison [**4**, Theorem 5] is applicable, and shows that T is actually isometric. (Although he assumes T to be a bijection, the argument proving that T is isometric does not depend on this, but only on the fact that T is a Jordan homomorphism and that it is isometric on self-adjoint elements, which follows from part (i) of the corollary.)

(iii) In view of part (ii), there is no loss of generality in assuming $\mathscr{U}$ to be uniformly closed.

If $f \in \mathscr{U}$ is positive, there is a unique $g \in \mathscr{U}$ such that $g^2 = f$ and $g \geqq 0$. Now $T(f) = T(g^2) = (Tg)^2$ is positive since Tg is self-adjoint. This completes the proof.

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THEOREM 3 *There exists an orthogonal central projection $p \in \mathscr{A}_2$ such that the map*

$$T_1 : f \rightarrow T(f)p$$
$$(\text{respectively } T_2 : f \rightarrow T(f)(1 - p))$$

*is a *-homomorphism (respectively a *-anti-homomorphism) and $T = T_1 + T_2$ as linear maps.*

Proof. We have shown that the image of $\mathscr{U}$ under T consists of bounded operators. Therefore, the extension T_e of T to $\mathscr{U}^-$ (cf. the first remark following Theorem 2) satisfies the hypotheses of Theorem 3.3 of Størmer [**11**].

COROLLARY 3.1. *Suppose that, in addition to the assumptions of Theorem* 2, *$\mathscr{A}_2$ is a factor. Then T must be either an (associative) homomorphism or an antihomomorphism.*

Proof. As T has now been proved to be a Jordan homomorphism from $\mathscr{U}$ into $\mathscr{A}_2$, this is an immediate consequence of Theorem 3.

Remark. It is not possible, without some extra assumption, to exclude one or the other possibility. For example, the identity mapping is an homomorphism of any factor onto itself, and it clearly preserves the L_p-norm. As an example of an antihomomorphism consider the mapping T defined as follows: $\mathscr{A}_1$ is a factor on a Hilbert space $\mathscr{H}$; v is an antilinear antiunitary operator from $\mathscr{H}$ to some Hilbert space $\mathscr{K}$; we put $T(f) = v^{-1}f^*v$, $(f \in \mathscr{A}_1)$. Then $T(\mathscr{A}_1)$ is a factor and T preserves the L_p-norm for every $p > 1$. (This example is due to Dixmier [**1**]).

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