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Real-dihedral harmonic Maass forms and CM-values of Hilbert modular functions

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Abstract

In this paper, we study real-dihedral harmonic Maass forms and their Fourier coefficients. The main result expresses the values of Hilbert modular forms at twisted CM 0-cycles in terms of these Fourier coefficients. This is a twisted version of the main theorem in Bruinier and Yang [*CM-values of Hilbert modular functions*, Invent. Math. **163** (2006), 229–288] and provides evidence that the individual Fourier coefficients are logarithms of algebraic numbers in the appropriate real-quadratic field. From this result and numerical calculations, we formulate an algebraicity conjecture, which is an analogue of Stark's conjecture in the setting of harmonic Maass forms. Also, we give a conjectural description of the primes appearing in CM-values of Hilbert modular functions.

1. Introduction

In the theory of modular forms, those of weight $k = 1$ are important because of their connection to Galois representations. By the theorem of Deligne–Serre [DS75], one can functorially attach to each weight-one newform f a continuous, odd, irreducible representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}),$$

where $\overline{\mathbb{Q}} \subset \mathbb{C}$ is a fixed algebraic closure of $\mathbb{Q}$. Since ρ_f is continuous, it has finite image and $\ker \rho_f$ fixes an algebraic extension $M/\mathbb{Q}$. Let $\tilde{\rho}_f$ be the composition of ρ_f and the surjection $\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$. Then the image of $\tilde{\rho}_f$ is isomorphic to either a dihedral group or one of the following groups: A_4, S_4, A_5. In the dihedral case, ρ_f is induced from a character of $\mathrm{Gal}(\overline{K}/K)$ for some quadratic field F in M. We say that f or ρ_f is imaginary-dihedral, respectively real-dihedral, if F is an imaginary, respectively a real, quadratic field. Note that f could be both imaginary-dihedral and real-dihedral.

A *harmonic Maass form* of weight $k \in \mathbb{Z}$ is a real-analytic function $\mathcal{F} : \mathcal{H} \longrightarrow \mathbb{C}$ such that it is modular and annihilated by the hyperbolic Laplacian Δ_k of weight k

$$\begin{aligned} \Delta_k &:= y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \xi_{2-k} \circ \xi_k, \\ \xi_k &:= 2iy^k\overline{\partial_{\overline{z}}}, \end{aligned} \tag{1.1}$$

where we write $z = x + iy$. Furthermore, it is only allowed to have polar-type singularities in the cusps. Since ξ_k commutes with the slash operator while changing the weight from k to $2-k$, $\xi_k(\mathcal{F})$

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is a modular form of weight $2-k$. Every harmonic Maass form $\mathcal{F}$ can be written canonically as the sum of a holomorphic part $\tilde{f}$ and a non-holomorphic part f^*. The holomorphic part $\tilde{f}$ is also known as a *mock-modular form*, which has been extensively studied by many people [BO07, BO10, DIT11] after Zwegers' ground-breaking thesis [Zwe02] (see [Zag09] for a good exposition) and has connections to many different areas of mathematics (see [Ono08] for a comprehensive overview). When $k=1$, we call $\mathcal{F}$ imaginary-dihedral, respectively real-dihedral, if $\xi_1(\mathcal{F})$ is imaginary-dihedral, respectively real-dihedral.

The imaginary-dihedral harmonic Maass forms and their Fourier coefficients have recently been studied and are well understood. Relying on the technique of Rankin–Selberg for computing heights of Heegner divisors as developed in [GZ86], we studied those with prime level in [DL15] and showed that their Fourier coefficients are logarithms of algebraic numbers in the Hilbert class field of the imaginary quadratic field K. In addition, we formulated a conjecture about the prime factorizations of the ideals generated by these algebraic numbers, which has been verified in the prime level case by Viazovska [Via12] and generalized to and proved in the square-free level case by Ehlen [Ehl13] using the techniques of theta-lifting.

In comparison, the real-dihedral case is much less well understood. The goal of this paper is to study a family of real-dihedral harmonic Maass forms and relate their Fourier coefficients to CM-values of Hilbert modular functions. Suppose $D \equiv 1 \pmod 4, p \equiv 5 \pmod 8$ are primes satisfying conditions (2.1) and (2.2). Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic field with quadratic character $\chi_D(\cdot)=(\frac{\cdot}{D})$. Denote by ϕ_p a character of conductor p and order 4, which satisfies $\phi_p(-1)=-1$ since $p \equiv 5 \pmod 8$. Then the space of cusp forms $S_1(Dp, \chi_D\phi_p)$ is one dimensional (see Proposition 2.9) and spanned by a newform

$$f_\varphi(z) := \sum_{\mathfrak{a}\subset\mathcal{O}_F} \varphi(\mathfrak{a}) q^{\mathrm{Nm}(\mathfrak{a})} = \sum_{n\geqslant 1} c_\varphi(n) q^n, \tag{1.2}$$

where $q=e^{2\pi i z}$ and φ is the ray class group character of F defined in (2.6). The representation associated to f_φ by the theorem of Deligne–Serre is

$$\rho_\varphi := \mathrm{Ind}_F^{\mathbb{Q}}(\varphi).$$

When $D=5, p=29$, the form f_φ was studied by Stark in the context of explicitly generating class fields of real-quadratic fields using special values of L-functions [Sta77a, Sta77b].

Since $S_1(Dp, \chi_D\phi_p)$ is one dimensional, there exists a harmonic Maass form $\mathcal{F}_\varphi(z)$ such that $\xi_1(\mathcal{F}_\varphi)=f_\varphi$ and its holomorphic part $\tilde{f}_\varphi$ has the following Fourier expansion at infinity:

$$\tilde{f}_\varphi(z) = c_\varphi^+(-1)q^{-1} + c_\varphi^+(0) + \sum_{\substack{n\geqslant 2\\ \chi_D(n)\neq -1}} c_\varphi^+(n) q^n. \tag{1.3}$$

Furthermore, with a mild condition on the growths of $\mathcal{F}_\varphi$ at other cusps of $\Gamma_0(Dp)$, the form $\mathcal{F}_\varphi$ is *unique* and the coefficients $c_\varphi^+(-1), c_\varphi^+(0)$ can be written explicitly as algebraic multiples of logarithms of the fundamental unit in F (see Theorem 5.6).

In the imaginary-dihedral setting, we showed that linear combinations of $c_\varphi^+(n)$ are equal to CM-values of elliptic modular functions [DL15, Theorem 2]. Thus, it is not a surprise that an analogous relation also holds in the real-dihedral setting, with elliptic modular functions replaced by Hilbert modular functions.

Let $F_2=\mathbb{Q}(\sqrt{p})$, $\mathcal{O}_{F_2}$ its ring of integers and X_{F_2} the open Hilbert modular surface whose complex points are $\mathrm{SL}_2(\mathcal{O}_{F_2})\backslash\mathcal{H}^2$. It is a connected component of the moduli space parametrizing

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isomorphisms of abelian surfaces with real multiplication. Let M_8 denote the field fixed by $\ker \tilde{\rho}_\varphi$. It contains two pairs of CM extensions K_4/F_2 and $\tilde{K}_4/\tilde{F}_2$, which are reflex fields of each other under the appropriate CM types $\Sigma = \{1, \sigma\}$ and $\tilde{\Sigma} = \sigma^3\Sigma = \{1, \sigma^{-1}\}$. Here, σ is an element of order 4 in the dihedral group $\mathrm{Gal}(M_8/\mathbb{Q}) \cong D_8$ of order 8.

Let $\mathrm{Cl}_0(K_4)$ be the kernel of the norm map $\mathrm{Nm} : \mathrm{Cl}(K_4) \longrightarrow \mathrm{Cl}(F_2)$ on class groups. Each class in $\mathrm{Cl}_0(K_4)$ gives rise to an isomorphism class of abelian surfaces on X_{F_2} with complex multiplication by (K_4, Σ), which is a 'big' CM point in the sense of [BKY12]. For $\mathcal{A} \in \mathrm{Cl}_0(K_4)$, let $Z_{\mathcal{A},\Sigma} \in X_{F_2}(\mathbb{C})$ denote the corresponding CM point. Since the 2-rank of $\mathrm{Cl}(K_4)$ is 1, it has a unique quadratic character ψ_2. Then we can define the twisted CM 0-cycle $\mathcal{CM}(K_4, \psi_2)$ by

$$\mathcal{CM}(K_4, \Sigma, \psi_2) := \sum_{\mathcal{A} \in \mathrm{Cl}_0(K_4)} \psi_2(\mathcal{A}) Z_{\mathcal{A},\Sigma}, \tag{1.4}$$

$$\mathcal{CM}(K_4, \psi_2) := \sum_{j=0}^{3} \mathcal{CM}(K_4, \sigma^j\Sigma, \psi_2). \tag{1.5}$$

It is algebraic and defined over the real quadratic field F.

For $m \in \mathbb{N}$, let T_m be the mth Hirzebruch–Zagier divisor on X_{F_2} (see (3.12) in § 3.3). A holomorphic Hilbert modular form on X_{F_2} is called normalized integral if its Fourier coefficients at the cusp infinity are integers with greatest common divisor 1. Let $\Psi(z_1, z_2)$ be a normalized integral Hilbert modular function on X_{F_2}, i.e. the ratio of two holomorphic normalized integral Hilbert modular forms. If the divisor of $\Psi(z_1, z_2)$ has the form

$$\sum_{\substack{m \geqslant 1 \\ \gcd(pD, m) = 1}} c(-m) T_m,$$

with $c(-m) \in \mathbb{Z}$, then its value at $\mathcal{CM}(K_4, \psi_2)$ is related to the coefficients $c_\varphi^+(n)$ in (1.3) as follows.

Theorem 1.1. *Let $p, D, c_\varphi^+(n)$ and Ψ be as above. Then*

$$\log |\Psi(\mathcal{CM}(K_4, \psi_2))| = -\frac{c_\varphi(p) h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \sum_{m \geqslant 1} c(-m) b_\varphi(m), \tag{1.6}$$

where $h_{\tilde{F}_2}$ and $h^+_{\tilde{F}_2}$ are the class number and narrow class number of $\tilde{F}_2 = \mathbb{Q}(\sqrt{Dp})$, respectively, and

$$b_\varphi(m) := \sum_{d \mid m} a_\varphi\left(\frac{m^2}{d^2}\right) \phi_p(d), \tag{1.7}$$

$$a_\varphi(n) := \sum_{k \in \mathbb{Z}} c_\varphi^+\left(\frac{Dn - pk^2}{4}\right) \delta_D(k), \tag{1.8}$$

$$\delta_D(k) := \begin{cases} 1, & D \nmid k, \\ 2, & D \mid k. \end{cases} \tag{1.9}$$

Remark 1.2. The terms $b_\varphi(m), a_\varphi(n)$ and the sum on the right-hand side of (1.6) all have a finite number of summands.

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In [GZ85], Gross and Zagier gave a factorization of the norms of the differences of singular moduli. The crucial input to the analytic proof is a real-analytic Eisenstein series of parallel weight one studied by Hecke [Hec24]. It is also an incoherent Eisenstein series in the sense of Kudla [Kud97]. Building on this idea, Bruinier and Yang generalized the Gross–Zagier factorization formulas [GZ85] to the setting of Hilbert modular forms [BY06]. Later, in [BY09], they combined this with the idea of Schofer [Sch09] and gave a more concise proof of the Gross–Zagier formula.

In order to prove Theorem 1.1, we will replace the diagonal restriction of the incoherent Eisenstein series in [BY06] with another real-analytic modular object of weight 2. This is constructed from $\mathcal{F}_\varphi(z)$ by applying the Shimura lift to its product with the classical theta function of weight 1/2. In [Coh77], Cohen observed that applying this construction to a cusp form of weight k yields the diagonal restriction of its Doi–Naganuma lift. This suggests that the incoherent Eisenstein series in [BY06] could come from the Doi–Naganuma lift of a modular object of weight one. We plan to address this in the future.

Another crucial observation en route to prove Theorem 1.1 is that the restriction of ρ_φ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\tilde{F}_2)$ is isomorphic to the representation induced from an unramified character $\tilde{\psi}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\tilde{K}_4)$. This special feature enables us to relate an ideal counting function with the coefficients $c_\varphi(n)$ in Proposition 4.1 and construct the automorphic Green's function from the non-holomorphic part of $\mathcal{F}_\varphi$. Furthermore, the character ψ_2 is the composition of $\tilde{\psi}$ and the type norm map (see Proposition 3.4). This explains how the left-hand side of (1.6) depends on ρ_φ.

Combining with the theory of complex multiplication [Shi98], Theorem 1.1 suggests that certain linear combinations of the $c^+_\varphi(n)$ are logarithms of algebraic numbers.

When $n \leqslant 0$, these algebraic numbers are units in the real quadratic field F. This is a particular case of Stark's conjecture for the adjoint representation $\mathrm{ad}(\rho_\varphi)$, which was proved by Stark [Sta71, II] as $\mathrm{ad}(\rho_\varphi)$ is an Artin representation with rational character. In this case, it is essentially a consequence of the analytic class number formula for F. When $n > 0$, numerical calculations suggest the following conjecture in the spirit of Stark's conjecture [Sta77a, Conjecture 2].

Conjecture 1.3. For any rational prime ℓ, there exist $u_{\mathrm{Re}}(\ell), u_{\mathrm{Im}}(\ell) \in \mathcal{O}_F[1/\ell]^\times$ such that $\mathrm{Nm}(u_{\mathrm{Re}}(\ell)) = \mathrm{Nm}(u_{\mathrm{Im}}(\ell)) = 1$ and

$$c^+_\varphi(\ell) = \log|u_{\mathrm{Re}}(\ell)| + i\log|u_{\mathrm{Im}}(\ell)|.$$

Remark 1.4. It is very intriguing to compare this with the p-adic overconvergent modular form of weight one studied in [DLR15], whose nth Fourier coefficient vanishes whenever $(\frac{n}{D}) = 1$. When $n = \ell$ is an inert prime, the ℓth Fourier coefficient is the p-adic logarithm of an ℓ-unit, which is the Gross–Stark unit studied in [DD06].

Since $c_\varphi(n) = 0$ whenever $(\frac{n}{D}) = -1$, Conjecture 1.3 is interesting for those primes ℓ that split or ramify in F. In the last section, we will provide precise numerical evidence to support and refine this conjecture. Together with Theorem 1.1, we have the following conjecture concerning the CM-value $\Psi(\mathcal{CM}(K_4, \psi_2))$.

Conjecture 1.5. Let Ψ be a Hilbert modular function as in Theorem 1.1 with divisor $\sum_{1\leqslant m\leqslant m_0} c(-m)T_m$. Then there exists $\alpha \in \mathcal{O}_F$ such that

$$\Psi(\mathcal{CM}(K_4, \psi_2)) = \frac{\alpha}{\alpha'}$$

and the norm of any prime ideal in $\mathcal{O}_F$ containing α is a rational prime dividing $(Dm^2 - pk^2)/4 > 0$ for some $1 \leqslant m \leqslant m_0$ and $k \in \mathbb{Z}$.

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Remark 1.6. Factorization formulas of CM-values of modular functions have appeared in both proven and conjectural forms before. When the Hilbert modular surface is the product of two modular curves, the values of Ψ at small CM points are the norms of differences of singular moduli studied by Gross and Zagier [GZ85]. When the values of Ψ are averaged over all CM points, a precise factorization formula is given in [BKY12, BY06]. When Ψ is replaced by the twisted Borcherds product, a conjectural factorization formula is given in [BY07]. In the case of Siegel modular forms, bounds for the primes appearing in the denominator of the CM-values of Igusa invariants are given in [GL12].

The structure of the paper is as follows. In § 2, we will give the necessary facts of the field extension $M/\mathbb{Q}$, the Galois representation ρ_φ and the newform f_φ, including its Petersson norm. In § 3, we will recall the construction of the big CM 0-cycle and Hilbert modular functions on X_{F_2} following [BY06], and then introduce the twisted CM 0-cycle. In § 4, we present the proof of the counting argument. Section 5 contains basic facts about harmonic Maass forms and the proof of existence and uniqueness of $\mathcal{F}_\varphi$ (Theorem 5.6). Finally, we give the proof of Theorem 1.1 in § 6 and some numerical evidence towards a conjectural form of $c^+_\varphi(n)$ in § 7.

2. Number fields and Galois representations

This section describes certain ray class fields of real quadratic fields and the complex Galois representations induced from the corresponding ray class group characters. These facts will be crucial when we study the twisted CM points on Hilbert modular surfaces in § 3 and prove the counting argument in § 4.

2.1 Number fields

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ throughout. Let $F \subset \overline{\mathbb{Q}}$ be the real quadratic field with discriminant D satisfying

$$D \equiv 1 \pmod 4 \text{ is prime and } F \text{ has class number one.} \tag{2.1}$$

Let $\mathcal{O}_F$ be the ring of integers of F and $u_F \in \mathcal{O}_F^\times$ the fundamental unit satisfying $u_F > 1$. Denote the non-trivial element in $\mathrm{Gal}(F/\mathbb{Q})$ by τ. Then $\tau(u_F) < 0$, since $D \equiv 1 \pmod 4$ is prime.

Let p be a rational prime that splits into $\mathfrak{p}\mathfrak{p}'$ in F, $\mathfrak{p}' = \tau(\mathfrak{p})$, such that

$$p \equiv 5 \pmod 8 \text{ and } \mathrm{ord}_{\mathfrak{p}}(u_F^{(p-1)/4} - 1) > 0. \tag{2.2}$$

Notice that the primes $\mathfrak{p}$ and $\mathfrak{p}'$ are distinguished by this condition. If $u_F = (a + b\sqrt{D})/2$, then the condition above is equivalent to the polynomial $X^8 - aX^4 - 1$ having a root modulo p. For fixed D, this happens for a positive proportion of primes p by the Chebotarev density theorem. Since $p \equiv 5 \pmod 8$, the second condition in (2.2) is equivalent to $(\frac{u_F}{p}) = 1$. This is then equivalent to $(\frac{p}{D})_4(\frac{D}{p})_4 = 1$ by Scholz's reciprocity law [Sch34], where $(\frac{\cdot}{\cdot})_4$ is the quartic residue symbol.

Let $\mathfrak{m} = \mathfrak{p}(1,0)$ be a modulus and $P(\mathfrak{m})$, respectively $P_{\mathfrak{m}}$, the group of principal ideals in F with a generator α such that (α) is relatively prime to $\mathfrak{p}$, respectively $\mathrm{ord}_{\mathfrak{p}}(\alpha - 1) > 0$, and $\alpha > 0$. Since F has class number one and $\mathcal{O}_F/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$, the ray class group with modulus $\mathfrak{m}$ is just $P(\mathfrak{m})/P_{\mathfrak{m}}$ and there exists $c_F \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $\mathrm{ord}_{\mathfrak{p}}(u_F - c_F) > 0$. Let $P(F)$ be the group of principal ideals in F. Then, under the map

$$\begin{aligned} F &\longrightarrow P(F) \times \{\pm 1\} \\ \alpha &\mapsto ((\alpha), \mathrm{sgn}(\alpha)), \end{aligned}$$

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whose kernel is generated by u_F, the ray class group $P(\mathfrak{m})/P_{\mathfrak{m}}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times \times \{\pm 1\}$ modulo the subgroup generated by $(c_F, 1)$ and $(-1,-1)$ and hence is cyclic. By condition (2.2), the order of c_F in $(\mathbb{Z}/p\mathbb{Z})^\times$ divides $(p-1)/4$. So, there exists a *unique* surjection

$$P(\mathfrak{m})/P_{\mathfrak{m}} \twoheadrightarrow \mathbb{Z}/4\mathbb{Z}.$$

Let $L \subset \overline{\mathbb{Q}}$ be the ray class field of F with modulus $\mathfrak{m}$ and K_8, F_4 the subfields of L corresponding to the quotient groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ of $P(\mathfrak{m})/P_{\mathfrak{m}}$, respectively. Notice that $\tau(L) \subset \overline{\mathbb{Q}}$ is the ray class field of F with modulus $\mathfrak{m}' := \mathfrak{p}'(0,1)$. Then we can apply [CDO98, Theorem 3.3] to compute the discriminants and signatures of K_8, F_4 as

$$\begin{gathered}\operatorname{disc}(K_8/F) = \mathfrak{p}^3, \quad [K_8 : \mathbb{Q}] = 8 = 4 + 2\cdot 2,\\ \operatorname{disc}(F_4/F) = \mathfrak{p}, \quad [F_4 : \mathbb{Q}] = 4 = 4 + 2\cdot 0,\\ \operatorname{disc}(K_8/\mathbb{Q}) = D^4 p^3, \quad \operatorname{disc}(F_4/\mathbb{Q}) = D^2 p.\end{gathered}$$

By Lemma 2.2 below, $\mathfrak{p}$ is totally ramified in K_8 and $\mathfrak{p}'$ is unramified. So, neither $K_8/\mathbb{Q}$ nor $F_4/\mathbb{Q}$ is Galois. Let $M \subset \overline{\mathbb{Q}}$ be the Galois closure of $K_8/\mathbb{Q}$; we have the following result about $\operatorname{Gal}(M/\mathbb{Q})$.

Proposition 2.1. *The Galois groups* $\operatorname{Gal}(M/F), \operatorname{Gal}(M/\mathbb{Q})$ *are isomorphic to* $(\mathbb{Z}/4\mathbb{Z})^2$ *and*

$$G := (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}, \tag{2.3}$$

respectively, where the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$ *acts on* $(\mathbb{Z}/4\mathbb{Z})^2$ *by*

$$\begin{gathered}(\mathbb{Z}/4\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/4\mathbb{Z})^2\\ (a,b) \mapsto (b,a).\end{gathered}$$

Proof. Fix $K_8 \cong F[X]/(g_1(X))$, where $g_1(X) = X^4 + \sum_{k=0}^{3} a_k X^k \in \mathcal{O}_F[X]$. Denote $g_2(X) := X^4 + \sum_{k=0}^{3} \tau(a_k) X^k$ and $K_8' := F[X]/(g_2(X))$, which is then the unique subfield of $\tau(L)$ satisfying $\operatorname{Gal}(\tau(K_8)/F) \cong \mathbb{Z}/4\mathbb{Z}$. Since $K_8 \cap K_8'$ is Galois over $\mathbb{Q}$ and $F_4/\mathbb{Q}$ is not Galois, the two fields K_8 and K_8' are disjoint over F. Let $M_{32} \subset \overline{\mathbb{Q}}$ be the composite of K_8 and K_8'. It is Galois over $\mathbb{Q}$ and $\operatorname{Gal}(M_{32}/F) \cong \operatorname{Gal}(K_8/F) \times \operatorname{Gal}(K_8'/K_8 \cap K_8') \cong (\mathbb{Z}/4\mathbb{Z})^2$. So, we can write $M_{32} \cong F[X,Y]/(g_1(X), g_2(Y))$ and naturally extend $\tau \in \operatorname{Gal}(F/\mathbb{Q})$ to an involution $\tau \in \operatorname{Gal}(M_{32}/\mathbb{Q})$. Thus, the following short exact sequence splits:

$$\operatorname{Gal}(M_{32}/F) \longrightarrow \operatorname{Gal}(M_{32}/\mathbb{Q}) \longrightarrow \operatorname{Gal}(F/\mathbb{Q})$$

and $\operatorname{Gal}(M_{32}/\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{Z}/4\mathbb{Z})^2$ by switching the coordinates. □

From now on, we will write $M = M_{32}$ to signify its degree and use a triple (a,b,c) to represent an element in the group $\operatorname{Gal}(M_{32}/\mathbb{Q}) \cong G$. Define $\sigma, \tau \in G$ to be[1]

$$\sigma := (1,0,0), \quad \tau := (1,0,1). \tag{2.4}$$

The group G is generated by σ and τ. Using their corresponding subgroups in $\operatorname{Gal}(M_{32}/\mathbb{Q})$, we can define some subfields of M_{32}. The subscripts of these fields indicate their degrees over $\mathbb{Q}$.

For convenience, we also include the character table of the group G and the field extension diagrams in the appendix.

Now, we will justify the ramifications and discriminants of these number fields. First, we will require the following standard lemma.

[1] This is compatible with, albeit a slight abuse of, the earlier notation $\tau \in \operatorname{Gal}(F/\mathbb{Q})$.

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Table 1. Field extensions.

M_{32}^H	Subgroup of $H \leqslant G$	Normality	Discriminant	No. of real embeddings
F_2	$\langle(2,0,0),(1,1,0),(0,0,1)\rangle$	Yes	p	2
F	$\langle(1,0,0),(0,1,0)\rangle$	Yes	D	2
$\tilde{F}_2$	$\langle(2,0,0),(1,0,1)\rangle$	Yes	Dp	2
K_4	$\langle(1,1,0),(0,0,1)\rangle$	No	Dp^3	0
$\hat{K}_4$	$\langle(2,0,0),(1,1,0)\rangle$	Yes	D^2p^2	4
$\tilde{K}_4$	$\langle(1,0,1)\rangle$	No	D^2p^3	0
F_4	$\langle(1,0,0),(0,2,0)\rangle$	No	D^2p	4
M_4	$\langle(3,1,0),(0,0,1)\rangle$	Yes	p^3	0
M_8	$\langle(1,1,0)\rangle$	Yes	D^4p^6	0
$\hat{M}_8$	$\langle(2,0,0),(0,2,0)\rangle$	Yes	D^4p^4	8
$\tilde{M}_8$	$\langle(1,3,0)\rangle$	Yes	D^4p^6	0
K_8	$\langle(1,0,0)\rangle$	No	D^4p^3	4
F_8	$\langle(0,0,1),(2,2,0)\rangle$	No	D^2p^6	0
M_{16}	$\langle(2,2,0)\rangle$	Yes	D^8p^{12}	0
K_{16}	$\langle(2,0,0)\rangle$	No	D^8p^{10}	0
F_{16}	$\langle(0,0,1)\rangle$	No	D^6p^{12}	0
M_{32}	$\langle(0,0,0)\rangle$	Yes	$D^{16}p^{24}$	0

Lemma 2.2. *Let L/K be an extension of a number field and $\mathfrak{P}$ a prime in K with characteristic p, ramification indices $e_1,\dots,e_r$ and residue field extensions of degrees $f_1,\dots,f_r$. Then the order of $\mathfrak{P}$ in the relative discriminant of L/K is at least*

$$\sum_{i=1}^{r}(e_i-1)f_i,$$

with equality if $\gcd(e_i,p)=1$ for all $1\leqslant i\leqslant r$.

Proof. This follows from the factorization of the relative different ideal [Lan94, § III.2, Proposition 8] and the formula relating the norm of the relative different to the relative discriminant [Lan94, § III.3, Proposition 14]. □

From the construction, we know that M_{32} is totally imaginary and ramified only at D and p. To calculate the discriminants and signatures of various subfields of M_{32}, it suffices to find the inertia subgroups in $\mathrm{Gal}(M_{32}/\mathbb{Q})\cong(\mathbb{Z}/4\mathbb{Z})^2\rtimes\mathbb{Z}/2\mathbb{Z}$. Let $c\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be complex conjugation. Suppose $\mathfrak{D}$ and $\mathfrak{P}$ are prime ideals in M_{32} lying above D and $\mathfrak{p}$, respectively. Let $I_{\mathfrak{D}}\leqslant\mathrm{Gal}(M_{32}/\mathbb{Q})$, respectively $I_{\mathfrak{P}}\leqslant\mathrm{Gal}(M_{32}/F)$, be the inertia subgroup of $\mathfrak{D}$, respectively $\mathfrak{P}$. Then we have the following result concerning the discriminants and the inertia subgroups.

Proposition 2.3. *The discriminant of $M_{32}/\mathbb{Q}$ is $D^{16}p^{24}$ and*

$$c\sim(2,0,0),\quad I_{\mathfrak{D}}\sim\langle(0,0,1)\rangle,\quad I_{\mathfrak{P}}\sim\langle(0,1,0)\rangle. \tag{2.5}$$

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Proof. Since F_4 and $\tau(F_4)$ are both totally real, so is their composite $\hat{M}_8$ and

$$c \in \langle (2,0,0),(0,2,0)\rangle.$$

The fields K_8/F and K_8'/F ramify at different infinite places of F. So, the conjugacy class of c in $\mathrm{Gal}(M_{32}/\mathbb{Q})$ consists of at least two elements, both fixing $\hat{M}_8$, which means that $c \sim (2,0,0)$ in G.

Since K_8/F and K_8' are unramified at primes above D, the same holds for their composite M_{32}. So, the intersection of $I_{\mathfrak{D}}$ and $\mathrm{Gal}(M_{32}/F)$ is the identity and $I_{\mathfrak{D}} = \langle (a,-a,1)\rangle$ for some $a \in \mathbb{Z}/4\mathbb{Z}$. This means that the ramification index of a prime in M_{32} above D is 2. By Lemma 2.2, the power of D dividing $\mathrm{disc}(M_{32}/\mathbb{Q})$ is exactly 16.

Since $\mathfrak{p}$ is totally ramified in K_8/F and unramified in K_8', the ramification index of $\mathfrak{p}$ in M_{32} is exactly 4 and any prime in K_8 lying above $\mathfrak{p}$ is unramified in M_{32}. This means that $I_{\mathfrak{P}} \sim \langle (0,1,0)\rangle$ in $\mathrm{Gal}(M_{32}/F)$. Applying Lemma 2.2 to $M_{32}/\mathbb{Q}$ and the prime p then finishes the proof. $\square$

From the proposition above, we could also deduce the discriminant and infinity type of each subfield of M_{32}. For example, the fields $F_2, \tilde{F}_2, F, \hat{K}_4$ are totally real, and the extensions $K_4/F_2, \tilde{K}_4/\tilde{F}_2, M_8/\hat{K}_4$ are CM extensions. Since F_2 is fixed by $(0,0,1)$, it is unramified at D and isomorphic to $\mathbb{Q}(\sqrt{p})$. By similar reasonings, $M_{32}/M_4, M_{32}/\tilde{K}_4$ and M_{32}/K_4 are unramified at primes above p and $M_4/\mathbb{Q}$ only ramifies at p and infinity. Thus, M_4 is the unique quartic subfield of $\mathbb{Q}(\zeta_p)$, where ζ_p is a pth root of unity and $\mathrm{disc}(M_4/\mathbb{Q}) = p^3$. Also, $M_{32}/\tilde{K}_4$ and F_8/K_4 are abelian and unramified with

$$\mathrm{Gal}(M_{32}/\tilde{K}_4) \cong \mathbb{Z}/8\mathbb{Z}.$$

The other discriminants in Table 1 can be verified by the same procedure.

Let $u_{\tilde{F}_2} \in \mathcal{O}_{\tilde{F}_2}^\times$ be the fundamental unit and $\tilde{\mathfrak{p}}$ the unique prime in $\tilde{F}_2$ above p. Since $\mathrm{disc}(\tilde{F}_2/\mathbb{Q}) = Dp$ is composite, the norm of $u_{\tilde{F}_2}$ is either 1 or -1. Here, we record a result concerning its effect on $\tilde{\mathfrak{p}}$ in the class group of $\tilde{F}_2$.

Lemma 2.4. *If $u_{\tilde{F}_2}$ has norm -1, then $\tilde{\mathfrak{p}}$ is not principal in $\tilde{F}_2$ and $\tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4}$ is not principal in $\tilde{K}_4$.*

Before stating the proof, we need the following general lemma.

Lemma 2.5. *Let k be a real quadratic field and K/k a CM extension such that:*

(1) *± 1 are the only roots of unity in K;*
(2) *there is a place v of k not above 2 such that K/k is ramified at v.*

Then $\mathcal{O}_k^\times \hookrightarrow \mathcal{O}_K^\times$ is an isomorphism.

Proof. Let u_K be a generator of the infinite part of $\mathcal{O}_K^\times$ and write $\overline{u_K} = \pm u_K^m$. We are done if $\overline{u_K} = u_K$. Since every complex embedding of $\overline{u_K}u_K^{-1} = \pm u_K^{m-1}$ has absolute value 1, it is a root of unity and $m = 1$. If $\overline{u_K} = -u_K$, then u_K is purely imaginary and $K \cong k[X]/(X^2+u)$, where $u := u_K\overline{u_K} \in k$ is a totally positive unit, and K/k is unramified at places away from 2. This contradicts the second assumption in the statement of the lemma. Thus, we must have $\overline{u_K} = u_K$. $\square$

Proof of Lemma 2.4. Suppose $\tilde{\mathfrak{p}} = (\tilde{\alpha})$ with $\tilde{\alpha} = (pa + b\sqrt{Dp})/2 \gg 0$ in $\tilde{F}_2$; then both $\tilde{\alpha}$ and $\tilde{\alpha}' = (pa - b\sqrt{Dp})/2$ are generators of $\tilde{\mathfrak{p}}$. So, there exists an integer r such that

$$u_{\tilde{F}_2}^{2r} = \frac{\tilde{\alpha}}{\tilde{\alpha}'} = (a\sqrt{p} + b\sqrt{D})^2.$$

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The exponent of $u_{\tilde{F}_2}$ is even, since $u_{\tilde{F}_2}$ has norm -1. But $a\sqrt{p}+b\sqrt{D}\notin \tilde{F}_2$, which is a contradiction.

Suppose $\tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4}=(\tilde{\beta})$ with $\tilde{\beta}\in\mathcal{O}_{\tilde{K}_4}$; then $(\tilde{\beta})^2=p\mathcal{O}_{\tilde{K}_4}$. By Lemma 2.5, there exists $\epsilon\in\{\pm 1\}$, $t\in\mathbb{Z}$ such that

$$\tilde{\beta}^2=\epsilon\cdot u_{\tilde{F}_2}^t\cdot p$$

and $\tilde{K}_4\cong\tilde{F}_2[X]/(X^2-\epsilon\cdot u_{\tilde{F}_2}^t\cdot p)$. Since $\tilde{K}_4$ is totally complex, $\epsilon=-1, 2\mid t$ and

$$\tilde{F}_2[X]/(X^2-\epsilon\cdot u_{\tilde{F}_2}^t\cdot p)\cong\tilde{F}_2[X]/(X^2+p).$$

However, $\tilde{K}_4$ does not contain a subfield isomorphic to $\mathbb{Q}[X]/(X^2+p)$, which is a contradiction. □

Remark 2.6. By the same reasoning, the unique prime in K_4 above p is not principal either.

2.2 Genus field and genus character

Given a finite extension of number fields K/k, the genus field K_{gen} is the composition of K and the maximal subfield of the Hilbert class field of K that is abelian over k. The case when $k=\mathbb{Q}$ and K/k is imaginary quadratic was studied by Gauss. For any prime in K, its factorization in K_{gen} is determined by its relative norm in k. In [Sta76], the notion of relative genus field was introduced to study a similar phenomenon.

Let K/k be a finite extension of number fields and $M/K, k_1/k, M_1/k$ normal extensions such that M_1 contains k_1 and M. Also, let

$$K_1:=Kk_1,\quad K_2:=M\cap K_1,\quad k_2:=K_2\cap k_1=M\cap k_1.$$

They all fit into the following field extension diagram.

$$\begin{tikzcd}
 & & M_1 \arrow[dd, dash] \\
 & M \arrow[ur, dash] \arrow[dd, dash] & \\
 & & K_1 \arrow[dd, dash] \\
 & K_2 \arrow[ur, dash] \arrow[dd, dash] & \\
K \arrow[ur, dash] \arrow[dd, dash] & & k_1 \\
 & k_2 \arrow[ur, dash] & \\
k \arrow[ur, dash] & &
\end{tikzcd}$$

The field K_2 is called the *genus field* of M/K relative to k_1/k. If k_2/k is normal, then K_2 is the composite of K and the maximal subfield of k_1 contained in M. For any character χ of $\mathrm{Gal}(M_1/k)$, denote its restriction to $\mathrm{Gal}(M_1/K)$ by $\tilde{\chi}$. Then $\tilde{\chi}$ is called a *genus character* of M/K relative to k_1/k if χ factors through $\mathrm{Gal}(k_2/k)$. Let $\mathfrak{p}$ be a prime in k and $\mathfrak{P}$ be any prime in K lying above $\mathfrak{p}$. Then part of the main theorem in [Sta76] gives us the following theorem.

THEOREM 2.7 [Sta76, Theorem]. *Suppose $\mathfrak{p}$ does not ramify in M_1.*

(1) *For any character χ of* $\mathrm{Gal}(M_1/k)$,

$$\tilde{\chi}(\mathfrak{P})=\chi(\mathrm{Nm}_{K/k}(\mathfrak{P})).$$

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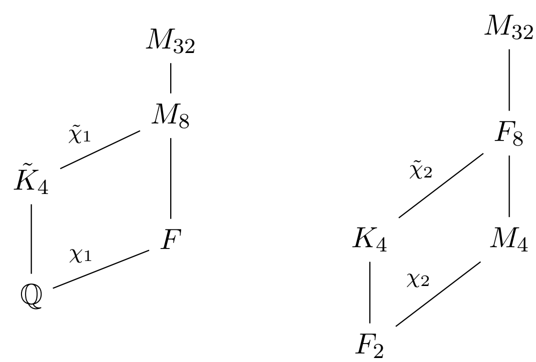
(2) *Suppose* $\mathrm{Nm}_{K/k}(\mathfrak{P}) = \mathfrak{p}^f$ *for some positive integer* f. *Then the splitting behavior of* $\mathfrak{p}$ *in* k_2 *and* f *determines the splitting behavior of* $\mathfrak{P}$ *in the genus field* K_2.

The picture simplifies significantly if M contains k_1 and is normal over k, in which case $M = M_1, K_2 = K_1$ and $k_2 = k_1$. In the cases we are interested in, let

$$\begin{aligned} &\text{(i):} \quad k = \mathbb{Q}, \quad K = \tilde{K}_4, \quad k_1 = k_2 = F, \quad K_1 = K_2 = M_8, \quad M = M_1 = M_{32}, \\ &\text{(ii):} \quad k = F_2, \quad K = K_4, \quad k_1 = k_2 = M_4, \quad K_1 = K_2 = F_8, \quad M = M_1 = M_{32}. \end{aligned}$$

The field extension diagram simplifies as follows with χ_j the non-trivial quadratic characters.



Theorem 2.7 now gives us the following simple consequence.

Proposition 2.8. *Let* $\tilde{\mathfrak{L}}$ *be a prime ideal in* $\tilde{K}_4$ *lying above a rational prime* ℓ *unramified in* M_8. *Then* $\tilde{\mathfrak{L}}$ *is inert in* M_8 *if and only if* $\mathrm{Nm}_{\tilde{K}_4/\mathbb{Q}}(\tilde{\mathfrak{L}}) = \ell$ *and* $(\frac{\ell}{D}) = (\frac{\ell}{p}) = -1$. *In this case, the ideal* $\ell\mathcal{O}_{F_2}$ *is a prime ideal and splits completely in* M_8. *Furthermore, any prime ideal* $\mathfrak{L}$ *in* K_4 *lying above such* ℓ *is inert in* F_8.

Proof. Applying Theorem 2.7 to case (i) gives us

$$\tilde{\chi}_1(\tilde{\mathfrak{L}}) = \chi_1(\mathrm{Nm}_{\tilde{K}_4/\mathbb{Q}}\tilde{\mathfrak{L}}) = \chi_1(\ell^f),$$

where f is the residue class degree of $\tilde{\mathfrak{L}}$. Then $\tilde{\mathfrak{L}}$ is inert in M_8 if and only if $\tilde{\chi}_1(\tilde{\mathfrak{L}}) = -1$, which happens if and only if $f = 1$ and $\chi_1(\ell) = (\frac{\ell}{D}) = -1$. In this case, ℓ splits in $\tilde{F}_2 \subset \tilde{K}_4$, which implies that $(\frac{\ell}{Dp}) = 1$ and $(\frac{\ell}{p}) = -1$. Also, the prime in M_8 above $\tilde{\mathfrak{L}}$ has residue class degree 2. Thus, ℓ is inert in F_2 and there are four primes in M_8 above it.

Recall that $M_4/\mathbb{Q}$ is the quartic subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. Since ℓ is inert in F_2, its decomposition group in $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is the whole group and it is inert in $\mathbb{Q}(\zeta_p)$. Applying Theorem 2.7 to case (ii) gives us

$$\tilde{\chi}_2(\mathfrak{L}) = \chi_2(\mathrm{Nm}_{K_4/F_2}(\mathfrak{L})) = \chi_2(\ell) = -1$$

and $\mathfrak{L}$ is inert in F_8. $\square$

2.3 Galois representation and modular forms

In this section, we will study the complex Galois representations attached to the number fields in § 2.1 and calculate the Petersson norm of the associated holomorphic weight-one newform.

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Let φ be a character of $P(\mathfrak{m})/P_{\mathfrak{m}}$ of order 4. Then, by class field theory, one can view it as a character of $\mathrm{Gal}(M_{32}/F)$ defined by

$$\begin{aligned} \varphi : \mathrm{Gal}(M_{32}/F) &\longrightarrow \mathbb{C}^{\times} \\ (a, b, 0) &\mapsto e^{\pi i b/2}. \end{aligned} \tag{2.6}$$

Denote the induced representation and the associated projective representations by ρ_φ and $\tilde{\rho}_\varphi$, respectively. In $\mathrm{Gal}(M_{32}/\mathbb{Q})$, complex conjugation corresponds to the conjugacy class $(2, 0, 0)$. Thus, ρ_φ is *odd* as

$$\det(\rho_\varphi(2, 0, 0)) = -1.$$

By the conductor formula for induced representation, we know that the conductor of ρ_φ is

$$\mathrm{disc}(F) \cdot \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{p}) = Dp.$$

From the character table of G in the appendix, we see that $\rho_\varphi \cong \rho_0$ is *irreducible*. An element $(a, b, c) \in G$ is in the kernel of $\tilde{\rho}_\varphi$ if and only if $a = b$ and $c = 0$. Thus, the subfield of M_{32} fixed by the kernel of $\tilde{\rho}_\varphi$ in G is M_8 and

$$\mathrm{Im}(\tilde{\rho}_\varphi) \cong \mathrm{Gal}(M_8/\mathbb{Q}) \cong D_8.$$

The kernel of $\det(\rho_\varphi)$ contains the commutator subgroup of G, which is generated by $(1, 3, 0)$. So, $\det(\rho_\varphi)$ factors through $\mathrm{Gal}(M_{32}/\tilde{M}_8)$ and is a character of

$$\mathrm{Gal}(\tilde{M}_8/\mathbb{Q}) \cong \mathrm{Gal}(M_4/\mathbb{Q}) \times \mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Notice that the first isomorphism above is canonical, and we can use it to write

$$\det(\rho_\varphi) = \chi_D \phi_p, \tag{2.7}$$

where $\chi_D : \mathrm{Gal}(F/\mathbb{Q}) \longrightarrow \mathbb{C}^{\times}$ is the quadratic Dirichlet character and $\phi_p : \mathrm{Gal}(M_4/\mathbb{Q}) \longrightarrow \mathbb{C}^{\times}$ is a character of order 4 satisfying

$$\phi_p(\ell) = \begin{cases} \varphi(\mathfrak{l})\varphi(\bar{\mathfrak{l}}) & \text{if } (\ell) = \mathfrak{l}\bar{\mathfrak{l}} \text{ in } \mathcal{O}_F, \\ \varphi(\ell) & \text{if } (\ell) \text{ is inert in } \mathcal{O}_F \end{cases} \tag{2.8}$$

for all primes $\ell \nmid Dp$. The following result shows that ρ_φ is the only odd, irreducible Galois representation with such conductor and determinant.

Proposition 2.9. *Let D, p be primes satisfying conditions (2.1), (2.2) and χ_D, ϕ_p the characters as above. Then any odd and irreducible representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ with conductor Dp and determinant $\chi_D\phi_p$ is isomorphic to ρ_φ. Equivalently, the spaces of cusp forms $S_1(Dp, \chi_D\phi_p)$ and $S_1(Dp, \chi_D\overline{\phi_p})$ are both one dimensional over $\mathbb{C}$.*

Proof. For a prime ℓ, let I_ℓ be the inertia subgroup in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $\mathrm{cond}(\rho) = Dp$ is square free, the representation ρ is unramified at $\ell \nmid Dp$ and tamely ramified at $\ell \mid Dp$. This implies that there exists $\chi_\ell : I_\ell \longrightarrow \mathbb{C}^{\times}$ non-trivial for $\ell \mid Dp$ such that

$$\rho\,|_{I_\ell} \cong \tilde{\rho}\,|_{I_\ell} \cong \mathbb{1} \oplus \chi_\ell,$$

where $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$ is the projective representation associated to ρ. Since $\det(\rho) = \chi_D\phi_p$, the order of χ_ℓ is divisible by 2 for $\ell \mid Dp$ and the image of $\tilde{\rho}$ contains two distinct subgroups

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of index 2. They correspond to the two real quadratic fields $F=\mathbb{Q}(\sqrt{D})$ and $F_2=\mathbb{Q}(\sqrt{p})$ contained in $\overline{\mathbb{Q}}^{\ker(\tilde{\rho})}$. This rules out the possibilities of $\mathrm{im}(\tilde{\rho})\cong A_4, S_4$ or A_5. So, ρ is isomorphic to an induced representation. From the conductor formula and its determinant, we know that ρ could only be isomorphic to $\mathrm{Ind}_F^{\mathbb{Q}}(\varphi')$ for some ray class group character φ' having the same modulus as φ. Thus, the characters φ' and φ, which differ by some character of $\mathrm{Cl}(F)$, are the same, since $\mathrm{Cl}(F)$ is trivial by condition (2.1).

The equivalent result is an immediate consequence of the theorem of Deligne–Serre [DS75] and that the map $f(z)\mapsto\overline{f(\overline{z})}$ is an isomorphism between $S_1(Dp,\chi_D\phi_p)$ and $S_1(Dp,\chi_D\overline{\phi_p})$. $\square$

We denote the weight-one newform associated to ρ_φ by f_φ, which was defined in (1.2). Its Fourier coefficients $c_\varphi(n)$ are multiplicative and given by

$$c_\varphi(n)=\sum_{\mathfrak{a}\subset\mathcal{O}_F,\mathrm{Nm}(\mathfrak{a})=n}\varphi(\mathfrak{a}).$$

With this, it is easy to check that $c_\varphi(n)$ satisfies the conditions

$$\begin{aligned}\chi_D(n)c_\varphi(n)&=c_\varphi(n)\text{ for all }n\text{ relatively prime to }D,\\ \phi_p(n)\overline{c_\varphi(n)}&=c_\varphi(n)\text{ for all }n\text{ relatively prime to }p.\end{aligned}\tag{2.9}$$

Let $\rho_{\overline{\varphi}}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\longrightarrow\mathbb{C}$ be the representation induced from $\overline{\varphi}$, the complex conjugate of φ. Then it is isomorphic to the dual representation $\rho_\varphi^\vee$ and the tensor product $\rho_\varphi\otimes\rho_{\overline{\varphi}}$ is isomorphic to $\mathbb{1}\oplus\mathrm{ad}(\rho_\varphi)$, where $\mathrm{ad}(\rho_\varphi)$ is the adjoint representation. From the character table, we have

$$\mathrm{ad}(\rho_\varphi)\cong\chi_D\oplus\mathrm{Ind}_{F_2}^{\mathbb{Q}}\varphi_2\cong\chi_D\oplus\mathrm{Ind}_{\tilde{F}_2}^{\mathbb{Q}}\tilde{\varphi}_2,\tag{2.10}$$

where φ_2 and $\tilde{\varphi}_2$ are the non-trivial quadratic characters of $\mathrm{Gal}(K_4/F_2)$ and $\mathrm{Gal}(\tilde{K}_4/\tilde{F}_2)$, respectively. This means that the L-function associated to $\rho_\varphi\otimes\rho_{\overline{\varphi}}$ factors as

$$L(\rho_\varphi\otimes\rho_{\overline{\varphi}},s)=L_F(s)\frac{L_{K_4}(s)}{L_{F_2}(s)}=L_F(s)\frac{L_{\tilde{K}_4}(s)}{L_{\tilde{F}_2}(s)},$$

where $L_*(s)$ is the Dedekind zeta function of the number field $*$.

By the standard Rankin–Selberg method (see e.g. [DL15, Proposition 3.1]), we have

$$\langle f_\varphi,f_\varphi\rangle=\frac{Dp}{2\pi^2}\mathrm{Res}_{s=1}L(\rho_\varphi\otimes\rho_{\overline{\varphi}},s).\tag{2.11}$$

Since K_4 and $\tilde{K}_4$ are non-Galois quartic fields, we have $w_{K_4}=w_{\tilde{K}_4}=2$. Furthermore, p is totally ramified in both K_4 and $\tilde{K}_4$. So, Lemma 2.5 implies that $\mathcal{O}_{\tilde{K}_4}^\times=\mathcal{O}_{\tilde{F}_2}^\times,\mathcal{O}_{K_4}^\times=\mathcal{O}_{F_2}^\times$. Combining this with the analytic class number formula yields the following proposition.

Proposition 2.10. *Let $f_\varphi\in S_1(Dp,\chi_D\phi_p)$ be as defined in (1.2). Then we have*

$$\langle f_\varphi,f_\varphi\rangle=\frac{2h_{K_4}}{h_{F_2}}\log u_F=\frac{2h_{\tilde{K}_4}}{h_{\tilde{F}_2}}\log u_F,\tag{2.12}$$

where h_* is the class number of the number field $*$ and $u_F>0$ is the fundamental unit of F.

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3. Twisted CM 0-cycle and Hilbert modular functions

In this section, we will first review the definition of CM 0-cycles on Hilbert modular surfaces following [BY06]. Then, using the character φ defined in (2.6), we will construct twisted CM 0-cycles. Finally, we will recall the Hilbert modular forms studied in [BY06], which are Borcherds lifts of elliptic modular functions. The factorizations of their values at the CM 0-cycles, which are defined over $\mathbb{Q}$, are given in [BY06]. We will express their values at the twisted CM 0-cycles as an infinite sum in the same way as in [BY06]. This will be used in proving the main theorem in § 6.

3.1 CM 0-cycle

For a number field L, let $I(L)$ and $P(L)$ denote the groups of fractional ideals and principal ideals, respectively. Then $\mathrm{Cl}(L) := I(L)/P(L)$ is the class group of L. If L is totally positive, let $P^+(L) \leqslant P(L)$ be the subgroup consisting of principal ideals with a totally positive generator and $\mathrm{Cl}^+(L) := I(L)/P^+(L)$ be the narrow class group. For a subset $S \subset L$, let S^+ be the subset of totally positive elements in S. Given an extension L'/L, define the subgroups $I_0(L') \leqslant I(L')$ and $\mathrm{Cl}_0(L') \leqslant \mathrm{Cl}(L')$ by

$$\begin{aligned} I_0(L') &:= \{\mathfrak{b}' \in I(L') : \mathrm{Nm}_{L'/L}(\mathfrak{b}') \in P^+(L)\}, \\ \mathrm{Cl}_0(L') &:= I_0(L')/P(L'). \end{aligned}$$

From now on, we will resume the notation from § 2 and recall the construction of CM points on Hilbert modular surfaces in [BY06, § 3]. Let $\mathcal{O}_{F_2} \subset F_2$ be the ring of integers, $\Gamma(\mathcal{O}_{F_2}) \leqslant \mathrm{SL}_2(F_2)$ the principal congruence subgroup and $X_{F_2} = X(\mathcal{O}_{F_2}) := \Gamma(\mathcal{O}_{F_2})\backslash\mathbb{H}^2$ the open Hilbert modular surface. It is one component of the moduli space classifying abelian surfaces with real multiplication by $\mathcal{O}_{F_2}$.

Recall that $\sigma, \tau \in G$ are defined in (2.4). We will use the same letters to denote the corresponding cosets in $\mathrm{Gal}(M_{32}/\mathbb{Q})/\mathrm{Gal}(M_{32}/\mathbb{Q}) \cong \mathrm{Gal}(M_8/\mathbb{Q})$. Let $\Sigma = \{1, \sigma\}$ be a CM type of K_4/F_2 with values in M_8. Fix a non-zero, totally imaginary element $\xi_\Sigma \in K_4$ such that $\Sigma(\xi_\Sigma) = (\xi_\Sigma, \sigma(\xi_\Sigma)) \in \mathcal{H}^2$. Let $\mathfrak{d}_{K_4/F_2} \subset K_4$ be the relative different of K_4/F_2 and define

$$\mathfrak{f}_{\mathfrak{a},\Sigma} := \xi_\Sigma \mathfrak{d}_{K_4/F_2} \mathfrak{a}\overline{\mathfrak{a}} \cap F_2$$

for any fractional ideal $\mathfrak{a} \subset K_4$. When $[\mathfrak{a}] \in \mathrm{Cl}_0(K_4)$, the ideal $\mathfrak{f}_{\mathfrak{a},\Sigma} = (r_{\mathfrak{a},\Sigma})$ is principal. By [BY06, Lemma 3.1], the data $(\mathfrak{a}, r_{\mathfrak{a},\Sigma})$ gives an abelian surface of CM type Σ and hence a CM point in the space $X(\mathcal{O}_{F_2})$.

Two pairs $(\mathfrak{a}, r_{\mathfrak{a},\Sigma})$ and $(\mathfrak{a}', r_{\mathfrak{a}',\Sigma})$ are equivalent, i.e. give isomorphic abelian surfaces, if there exists $\alpha \in K_4$ such that

$$\mathfrak{a}' = (\alpha)\mathfrak{a}, \quad r_{\mathfrak{a}',\Sigma} = r_{\mathfrak{a},\Sigma}\alpha\overline{\alpha}.$$

Let $[\mathfrak{a}, r_{\mathfrak{a},\Sigma}]$ denote the equivalence class of such pairs. The kernel of the forgetful map from the set of such equivalence classes to $\mathrm{Cl}_0(K_4)$ is $\mathcal{O}_{F_2}^{\times,+}/\mathrm{Nm}_{K_4/F_2}(\mathcal{O}_{K_4}^\times)$. Since $\mathrm{disc}(F_2)$ is a prime and K_4/F_2 is non-Galois, the fundamental unit of F_2 has norm -1 and Lemma 2.5 implies that this forgetful map is bijective. For $\mathcal{A} = [\mathfrak{a}] \in \mathrm{Cl}_0(K_4)$, write $\mathfrak{a} = \mathcal{O}_{F_2}\alpha + \mathcal{O}_{F_2}\beta$ such that $\alpha/\beta \in K_4$ satisfies

$$\Sigma\left(\frac{\alpha}{\beta}\right) := \left(\frac{\alpha}{\beta}, \sigma\left(\frac{\alpha}{\beta}\right)\right) \in \mathcal{H}^2. \tag{3.1}$$

Then the $\mathrm{SL}_2(\mathcal{O}_{F_2})$-orbit of $\Sigma(\alpha/\beta)$ depends only on the class of $\mathfrak{a}$ and gives the corresponding point on $X_{F_2}(\mathbb{C})$. By the theory of complex multiplication, this point is defined over $\overline{\mathbb{Q}}$ and we

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denote it by $Z_{\mathcal{A},\Sigma} \in X_{F_2}(\overline{\mathbb{Q}})$. Define the untwisted CM 0-cycle $\mathcal{CM}(K_4, \mathcal{O}_{F_2})$ by

$$\begin{aligned} \mathcal{CM}(K_4, \Sigma, \mathcal{O}_{F_2}) &= \sum_{\mathcal{A} \in \mathrm{Cl}_0(K_4)} Z_{\mathcal{A},\Sigma}, \\ \mathcal{CM}(K_4, \mathcal{O}_{F_2}) &= \sum_{j=0}^{3} \mathcal{CM}(K_4, \sigma^j\Sigma, \mathcal{O}_{F_2}), \end{aligned} \tag{3.2}$$

where the set $\sigma^j\Sigma := \{\sigma^j, \sigma^{j+1}\}$ is another CM type for $0 \leqslant j \leqslant 3$.

Under the CM type Σ, the CM extension $\tilde{K}_4/\tilde{F}_2$ is the reflex field of K_4/F_2 and $\tilde{\Sigma} = \{1, \sigma^{-1}\}$ the reflex CM type. Let $\mathrm{Nm}_{\tilde{\Sigma}}$ be the type norm defined on $I(\tilde{K}_4)$ by

$$\begin{aligned} \mathrm{Nm}_{\tilde{\Sigma}} : I(\tilde{K}_4) &\longrightarrow I_0(K_4), \\ \tilde{\mathfrak{b}} &\mapsto ((\tilde{\mathfrak{b}}\sigma^{-1}(\tilde{\mathfrak{b}}))M_8) \cap \tilde{K}_4. \end{aligned}$$

Notice that $\mathrm{Nm}_{K_4/F_2}(\mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{b}})) = \mathrm{Nm}_{\tilde{K}_4/\mathbb{Q}}(\tilde{\mathfrak{b}})\mathcal{O}_{F_2} \in I_0(K_4)$. The type norm $\mathrm{Nm}_{\Sigma} : I(K_4) \longrightarrow I_0(\tilde{K}_4)$ is defined similarly. They induce the following maps on class groups:

$$\begin{aligned} \mathrm{Nm}_{\tilde{\Sigma}} : \mathrm{Cl}(\tilde{K}_4) &\longrightarrow \mathrm{Cl}_0(K_4), \\ \mathrm{Nm}_{\Sigma} : \mathrm{Cl}_0(K_4) &\longrightarrow \mathrm{Cl}_0(\tilde{K}_4). \end{aligned} \tag{3.3}$$

Example 3.1. Let $\mathcal{L}$ be an unramified prime in M_8 such that the associated Frobenius element $\mathrm{Frob}_{\mathcal{L}}$ is conjugate to τ. Then the subfield of M_8 fixed by the decomposition group of $\mathcal{L}$ is a conjugate of $\tilde{K}_4$. So, $\mathcal{L}$ lies above a prime $\tilde{\mathfrak{L}}$ in $\tilde{K}_4$, which has residue field degree 1 and is inert in M_8. Denote its norm in $\mathbb{Q}$ by ℓ. By Proposition 2.8, ℓ is inert in F and F_2 and $\ell\mathcal{O}_{F_2}$ splits completely in M_8. In this case, the type norm of $\tilde{\mathfrak{L}}$ is one of the two prime ideals in K_4 above the inert prime ℓ in F_2.

Concerning the images of the type norm maps, we have the following results, which are analogues of [BY06, Lemma 5.3].

Proposition 3.2. *The type norm maps in (3.3) are surjective. Furthermore, the sizes of their respective kernels are $h_{\tilde{F}_2}$ and $h^+_{\tilde{F}_2}/h_{\tilde{F}_2}$.*

Proof. By [BY06, Lemma 5.3], the composition $\mathrm{Nm}_{\tilde{\Sigma}} \circ \mathrm{Nm}_{\Sigma}$, respectively $\mathrm{Nm}_{\Sigma} \circ \mathrm{Nm}_{\tilde{\Sigma}}$, is the square map on $\mathrm{Cl}_0(K_4)$, respectively $\mathrm{Cl}_0(\tilde{K}_4)$. So, it suffices to prove the surjectivity of $\mathrm{Nm}_{\tilde{\Sigma}}$, respectively Nm_{Σ}, onto the 2-part of $\mathrm{Cl}_0(K_4)$, respectively $\mathrm{Cl}_0(\tilde{K}_4)$.

By [Oka00, Lemma 17], given a CM extension K/k with the 2-rank of $\mathrm{Cl}(k)$ being zero, the 2-rank of $\mathrm{Cl}(K)$ is one less than the number of primes in k that ramify in K. Since $\mathrm{disc}(F_2) = p$ is an odd prime, the 2-part of $\mathrm{Cl}(F_2)$ is trivial and the 2-part of $\mathrm{Cl}_0(K_4)$ is isomorphic to the 2-part of $\mathrm{Cl}(K_4)$. Since $\mathrm{disc}(K_4) = Dp^3$ and D is a prime, there are two primes in F_2 that ramify in K_4. So, the 2-rank of $\mathrm{Cl}(K_4)$ is 1 and the 2-part of $\mathrm{Cl}(K_4)$ is isomorphic to $\mathbb{Z}/2^t\mathbb{Z}$ for some positive integer t. Recall that F_8/K_4 is unramified. So, $t \geqslant 1$ and

$$\mathrm{Gal}(F_8/K_4) \cong \mathrm{Cl}(K_4)/2\mathrm{Cl}(K_4) \cong \mathrm{Cl}_0(K_4)/2\mathrm{Cl}_0(K_4).$$

Let $\tilde{\mathfrak{L}}$ be a prime in $\tilde{K}_4$ as in Example 3.1. Then $\mathfrak{L} := \mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{L}})$ is a prime in K_4 lying above the inert prime (ℓ) in F_2 and hence inert in F_8 by Proposition 2.8. This means that $[\mathfrak{L}] \in \mathrm{Cl}(K_4)/2\mathrm{Cl}(K_4)$ is non-trivial. Set

$$\tilde{\mathfrak{b}} := \tilde{\mathfrak{L}}^{h_{F_2}} \in \mathrm{Cl}(\tilde{K}_4).$$

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Then $\mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{b}}) \in I_0(K_4)$ and the class $[\mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{b}})]$ is non-trivial in $\mathrm{Cl}_0(K_4)/2\mathrm{Cl}_0(K_4)$, since h_{F_2} is odd. Together with $\mathrm{Nm}_{\tilde{\Sigma}} : \mathrm{Cl}_0(\tilde{K}_4) \twoheadrightarrow 2\mathrm{Cl}_0(K_4)$, we know that $\mathrm{Nm}_{\tilde{\Sigma}}$ in (3.3) is surjective. The size of the kernel can be calculated from

$$\#\mathrm{Cl}_0(\tilde{K}_4) \cdot \frac{h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} = \frac{h_{\tilde{K}_4}}{h_{\tilde{F}_2}} = \frac{h_{K_4}}{h_{F_2}} = \#\mathrm{Cl}_0(K_4) \cdot \frac{h^+_{F_2}}{h_{F_2}} \tag{3.4}$$

and the fact that $h_{F_2} = h^+_{F_2}$.

For Nm_{Σ}, let $u_{\tilde{F}_2}$ be the fundamental unit of $\tilde{F}_2$. Since the 2-part of $\mathrm{Cl}_0(K_4)$ is isomorphic to $\mathbb{Z}/2^t\mathbb{Z}$ and $\mathrm{Nm}_{\tilde{\Sigma}} \circ \mathrm{Nm}_{\Sigma}$ is the square map, the image of $\mathbb{Z}/2^t\mathbb{Z}$ under Nm_{Σ} in $\mathrm{Cl}_0(\tilde{K}_4)$ and the 2-part of $\mathrm{Cl}_0(\tilde{K}_4)$ both have size at least 2^{t-1}. When the norm of $u_{\tilde{F}_2}$ is 1, we have $h^+_{\tilde{F}_2} = 2h_{\tilde{F}_2}$ and the size of the 2-part of $\mathrm{Cl}_0(\tilde{K}_4)$ is 2^{t-1} by (3.4). This means that the 2-part of $\mathrm{Cl}_0(\tilde{K}_4)$ is isomorphic to $\mathbb{Z}/2^{t-1}\mathbb{Z}$ and $\mathrm{Nm}_{\Sigma} : \mathrm{Cl}_0(K_4) \longrightarrow \mathrm{Cl}_0(\tilde{K}_4)$ is a two-to-one surjection. When the norm of $u_{\tilde{F}_2}$ is -1, we have $\#\mathrm{Cl}_0(K_4) = \#\mathrm{Cl}_0(\tilde{K}_4)$. Let $\tilde{\mathfrak{p}}$, respectively $\mathfrak{P}$, be the unique prime in $\tilde{F}_2$, respectively K_4, above p. Since p is totally ramified in $\tilde{F}_2$ and K_4, we have

$$\mathrm{Nm}_{\Sigma}(\mathfrak{P}) = \tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4}.$$

By Lemma 2.4, the class $[\tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4}]$ is non-trivial, which implies that $[\mathfrak{P}]$ is non-trivial in $\mathrm{Cl}_0(K_4)$. Since $\mathfrak{P}^2 = \mathrm{Nm}_{K_4/F_2}(\mathfrak{P}) = \sqrt{p}\mathcal{O}_{F_2}$, the classes $[\tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4}]$ and $[\mathfrak{P}]$ have order 2 in $\mathrm{Cl}_0(\tilde{K}_4)$ and $\mathrm{Cl}_0(K_4)$, respectively. Then the image of the generator of $\mathbb{Z}/2^t\mathbb{Z}$ has order 2^t in $\mathrm{Cl}_0(\tilde{K}_4)$ and the map Nm_{Σ} is an isomorphism between the 2-parts of $\mathrm{Cl}_0(K_4)$ and $\mathrm{Cl}_0(\tilde{K}_4)$. □

Via the type norm $\mathrm{Nm}_{\tilde{\Sigma}}$, the group $I(\tilde{K}_4)$ acts on $[\mathfrak{a}, r_{\mathfrak{a},\Sigma}]$ by

$$\sigma_{\tilde{\mathfrak{b}}}[\mathfrak{a}, r_{\mathfrak{a},\Sigma}] := [\mathfrak{a}\mathrm{Nm}_{\tilde{\Sigma}}\tilde{\mathfrak{b}}, r_{\mathfrak{a},\Sigma}N\tilde{\mathfrak{b}}]. \tag{3.5}$$

In the same way, it acts on $Z_{\mathcal{A},\Sigma}$. Let $H(\tilde{K}_4) \leqslant I(\tilde{K}_4)$ be the subgroup containing the fractional ideals $\tilde{\mathfrak{b}}$ satisfying

$$\mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{b}}) = \mu\mathcal{O}_{K_4}, \quad N\tilde{\mathfrak{b}} = \mu\overline{\mu}$$

for some $\mu \in K_4^{\times}$, where $N\tilde{\mathfrak{b}} := \#(\mathcal{O}_{\tilde{K}_4}/\tilde{\mathfrak{b}})$. The CM ideal class group $\mathcal{CC}(\tilde{K}_4, \tilde{\Sigma})$ is defined by $I(\tilde{K}_4)/H(\tilde{K}_4)$. Notice that it is a quotient group of $\mathrm{Cl}(\tilde{K}_4)$. By Proposition 3.2, the group $\mathcal{CC}(\tilde{K}_4, \tilde{\Sigma})$ acts on the set $\{[\mathfrak{a}, r_{\mathfrak{a},\Sigma}] : [\mathfrak{a}] \in \mathrm{Cl}_0(K_4)\}$ transitively and faithfully.

Let $\tilde{M} \subset \overline{\mathbb{Q}}$ be the Hilbert class field of $\tilde{K}_4$ and $\tilde{M}_{\mathcal{CC}} \leqslant \tilde{M}$ the subfield corresponding to the subgroup $H(\tilde{K}_4)/P(\tilde{K}_4) \leqslant \mathrm{Cl}(\tilde{K}_4)$. For any $\mathcal{A} \in \mathrm{Cl}_0(K_4)$, the CM point $Z_{\mathcal{A},\Sigma}$ is defined over $\tilde{M}_{\mathcal{CC}}$. Given a class $\tilde{\mathcal{A}} \in H(\tilde{K}_4)/P(\tilde{K}_4)$, let $\tilde{\mathfrak{L}}$ be a prime in $\mathcal{O}_{\tilde{K}_4}$ representing it. Suppose $\tilde{\mathfrak{L}}$ lies above the rational prime ℓ not dividing Dp. If $\mathrm{Nm}_{\tilde{K}_4/\mathbb{Q}}(\tilde{\mathfrak{L}}) = \ell^f$ with $f \geqslant 2$, then $\tilde{\mathfrak{L}}$ splits in M_8 by Proposition 2.8. If $f = 1$, then we can write $\mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{L}}) = (\mu)$ with $\mu = a + b\sqrt{-\Delta} \in F_2(\sqrt{-\Delta}) \cong K_4$ satisfying

$$a^2 + b^2\Delta = \ell,$$

since $\tilde{\mathfrak{L}} \in H(\tilde{K}_4)$. From the discriminants of F_2 and K_4, we know that $\Delta/\sqrt{p} \in \mathcal{O}_{F_2}$. By considering the equation above in $\mathcal{O}_{F_2}/\sqrt{p}\mathcal{O}_{F_2} \cong \mathbb{Z}/p\mathbb{Z}$, we see that $(\frac{\ell}{p}) = 1$. Again, by Proposition 2.8, the ideal $\tilde{\mathfrak{L}}$ splits in M_8. Thus, the element $\sigma_{\tilde{\mathcal{A}}} \in \mathrm{Gal}(\tilde{M}/\tilde{K}_4)$ associated to $\tilde{\mathcal{A}}$ via the Artin map is contained in $\mathrm{Gal}(\tilde{M}/M_8)$ and we have proved the following lemma.

Lemma 3.3. *The field $\tilde{M}_{\mathcal{CC}}$ contains M_8.*

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3.2 Twisted CM 0-cycle

Now, we are ready to construct the twisted CM 0-cycle using the character φ in (2.6). The group $\mathrm{Gal}(M_{32}/\tilde{F}_2) \cong \langle \tau, \sigma^2 \rangle$ is a generalized dihedral group with the relation

$$\tau^8 = (\sigma^2)^2 = \tau^3\sigma^2\tau\sigma^2 = \mathrm{Id}.$$

Since $\tilde{K}_4/\tilde{F}_2$ is a CM extension, the action of σ^2 is complex conjugation. Recall that $\rho_\varphi := \mathrm{Ind}_F^{\mathbb{Q}}(\varphi)$ is the odd, irreducible representation in § 2.3. Since $M_{32}/\tilde{K}_4$ is abelian with $\mathrm{Gal}(M_{32}/\tilde{K}_4) \cong \langle \tau \rangle \cong \mathbb{Z}/8\mathbb{Z}$, the restriction $\rho_\varphi|_{\mathrm{Gal}(M_{32}/\tilde{K}_4)}$ is a *reducible* representation and can be written as

$$\rho_\varphi|_{\mathrm{Gal}(M_{32}/\tilde{K}_4)} \cong \tilde{\psi} \oplus \tilde{\psi}^{\sigma^2}, \tag{3.6}$$

where $\tilde{\psi}(\tau)$ is an eighth root of unity ζ_8 and

$$\tilde{\psi}^{\sigma^2}(\tau) := \tilde{\psi}(\sigma^2\tau(\sigma^2)^{-1}) = \tilde{\psi}(\tau^5).$$

The characters $\tilde{\psi}$ and φ are closely related as follows.

Proposition 3.4. *The restriction of $\tilde{\psi}$ to $\mathrm{Gal}(M_{32}/M_8)$ has the following decomposition:*

$$\mathrm{Gal}(M_{32}/M_8) \hookrightarrow \mathrm{Gal}(M_{32}/F) \twoheadrightarrow \mathrm{Gal}(K_8/F) \xrightarrow{\varphi} \mathbb{C}^\times. \tag{3.7}$$

Proof. Let $(a, a, 0) \in \mathrm{Gal}(M_{32}/M_8)$ for some $a \in \mathbb{Z}/4\mathbb{Z}$. Then its image under $\mathrm{Gal}(M_{32}/M_8) \hookrightarrow \mathrm{Gal}(M_{32}/F) \twoheadrightarrow \mathrm{Gal}(K_8/F)$ is a. Also, the matrix $\rho_\varphi(a, a, 0) = \left(\begin{smallmatrix} \varphi(a) & \\ & \varphi(a) \end{smallmatrix}\right)$ is a constant multiple of the diagonal matrix and hence is stable under conjugation. On the other hand, since $\tilde{\psi}(a, a, 0) = \tilde{\psi}^{\sigma^2}(a, a, 0)$, (3.6) tells us that

$$\rho_\varphi(a, a, 0) = \begin{pmatrix} \tilde{\psi}(a, a, 0) & \\ & \tilde{\psi}(a, a, 0) \end{pmatrix}.$$

Thus, we have $\tilde{\psi}(a, a, 0) = \varphi(a)$. □

Define the character $\tilde{\psi}_2$ by

$$\tilde{\psi}_2 := \tilde{\psi}^{-1}\tilde{\psi}^{\sigma^2} = \tilde{\psi}(\tilde{\psi}^{\sigma^2})^{-1}, \tag{3.8}$$

which is the quadratic character with kernel $\mathrm{Gal}(M_{32}/M_8)$. Since $M_{32}/\tilde{K}_4$ is unramified, we can treat $\tilde{\psi}$ and $\tilde{\psi}_2$ as characters of $\mathrm{Cl}(\tilde{K}_4)$. The composite of the type norm maps $\mathrm{Nm}_\Sigma \circ \mathrm{Nm}_{\tilde{\Sigma}}$ measures the difference between the classes of an ideal $\mathfrak{a}$ and its complex conjugate $\overline{\mathfrak{a}}$. Thus, it is natural to expect the following result.

Proposition 3.5. *The character $\tilde{\psi}_2$ has the following decomposition:*

$$\mathrm{Cl}(\tilde{K}_4) \xrightarrow{\mathrm{Nm}_{\tilde{\Sigma}}} \mathrm{Cl}_0(K_4) \xrightarrow{\mathrm{Nm}_\Sigma} \mathrm{Cl}_0(\tilde{K}_4) \xrightarrow{\tilde{\psi}} \mathbb{C}^\times. \tag{3.9}$$

Proof. Let $[\tilde{\mathfrak{L}}] \in \mathrm{Cl}(\tilde{K}_4)$ be a class with $\tilde{\mathfrak{L}}$ a prime in $\tilde{K}_4$. Then we have

$$\mathrm{Nm}_\Sigma \circ \mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{L}}) = (\tilde{\mathfrak{L}}^2\sigma(\tilde{\mathfrak{L}}\mathcal{O}_{M_8})\sigma^{-1}(\tilde{\mathfrak{L}}\mathcal{O}_{M_8})) \cap \tilde{K}_4$$

from the definition of type norm. Since $\#(\mathcal{O}_{\tilde{K}_4}/\tilde{\mathfrak{L}}) = \tilde{\mathfrak{L}}\overline{\tilde{\mathfrak{L}}}\sigma(\tilde{\mathfrak{L}})\sigma^{-1}(\tilde{\mathfrak{L}})$, the class of $\mathrm{Nm}_\Sigma \circ \mathrm{Nm}_{\tilde{\Sigma}}(\tilde{\mathfrak{L}})$ in $\mathrm{Cl}_0(\tilde{K}_4)$ is $[\tilde{\mathfrak{L}}\overline{\tilde{\mathfrak{L}}}^{-1}]$, which depends only on the class of $\tilde{\mathfrak{L}}$ in $\mathrm{Cl}(\tilde{K}_4)$. Let $\mathrm{Frob}_{\tilde{\mathfrak{L}}}, \mathrm{Frob}_{\overline{\tilde{\mathfrak{L}}}} \in$

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$\mathrm{Gal}(M_{32}/\tilde{K}_4)$ be the Frobenius elements associated to $\tilde{\mathfrak{L}}$ and $\overline{\tilde{\mathfrak{L}}}$, respectively. Then, as elements in $\mathrm{Gal}(M_{32}/\tilde{F}_2)$, they are conjugates under σ^2. So, we have

$$\begin{aligned}\tilde{\psi} \circ \mathrm{Nm}_{\Sigma} \circ \mathrm{Nm}_{\tilde{\Sigma}}([\tilde{\mathfrak{L}}]) = \tilde{\psi}([\tilde{\mathfrak{L}}\overline{\tilde{\mathfrak{L}}}^{-1}]) &= \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})\tilde{\psi}(\mathrm{Frob}_{\overline{\tilde{\mathfrak{L}}}})^{-1} \\ &= \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})\tilde{\psi}^{\sigma^2}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})^{-1} = \tilde{\psi}_2([\tilde{\mathfrak{L}}]).\end{aligned}$$

□

Now, we view $\tilde{\psi}$ as a character of $\mathrm{Cl}(\tilde{K}_4)$ and define $\psi_2 : \mathrm{Cl}_0(K_4) \longrightarrow \mathbb{C}^\times$ by

$$\psi_2 := \tilde{\psi} \circ \mathrm{Nm}_{\Sigma}. \tag{3.10}$$

Then Proposition 3.5 is equivalent to $\tilde{\psi}_2 = \psi_2 \circ \mathrm{Nm}_{\tilde{\Sigma}}$. Since the type norm $\mathrm{Nm}_{\tilde{\Sigma}}$ is surjective and $\#\mathrm{Cl}(K_4)/\#\mathrm{Cl}_0(K_4) = h^+_{F_2}$ is odd, one can extend ψ_2 to a unique quadratic character of $\mathrm{Cl}(K_4)$, which we also denote by ψ_2. Then it has kernel $\mathrm{Gal}(M_{32}/F_8)$, since F_8/K_4 is unramified and the 2-part of $\mathrm{Cl}(K_4)$ is cyclic.

Let $\mathcal{CM}(K_4, \Sigma, \psi_2)$ be the twisted CM 0-cycle defined in (1.4). For an arbitrary class $\mathcal{A}_0 \in \mathrm{Cl}_0(K_4)$, the surjectivity of $\mathrm{Nm}_{\tilde{\Sigma}}$ in Proposition 3.2 enables us to write

$$\mathcal{CM}(K_4, \Sigma, \psi_2) = \frac{1}{h_{\tilde{F}_2}} \sum_{\tilde{\mathcal{A}} \in \mathrm{Cl}(\tilde{K}_4)} \tilde{\psi}_2(\tilde{\mathcal{A}})\sigma_{\tilde{\mathcal{A}}}(Z_{\mathcal{A}_0,\Sigma}).$$

By Lemma 3.3, the character $\tilde{\psi}_2$ factors through $\mathcal{CC}(\tilde{K}_4, \tilde{\Sigma})$. So, $\mathcal{CM}(K_4, \Sigma, \psi_2)$ is defined over M_8 and not trivially zero. Remark 3.5 in [BY06] then implies that $\mathcal{CM}(K_4, \psi_2)$ is defined over F, the real quadratic subfield of M_8 fixed by σ.

3.3 Hilbert modular forms

Now we will recall some results on Borcherds lifts and Hilbert modular forms following [BY06].

In the notation of the previous section, let χ_p be the quadratic Dirichlet character modulo p. Define the rational quadratic space V by

$$V := \left\{ A = \begin{pmatrix} a & \lambda \\ \sigma(\lambda) & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Q}, \lambda \in F_2 \right\}, \tag{3.11}$$

which the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_{F_2})$ acts on via

$$\gamma \cdot A := \sigma(\gamma) A \gamma^t.$$

Consider the lattice

$$L = \left\{ A = \begin{pmatrix} a & \lambda \\ \sigma(\lambda) & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}, \lambda \in \mathfrak{d}_{F_2} \right\} \subset V,$$

where $\mathfrak{d}_{F_2} = \sqrt{p}\mathcal{O}_{F_2}$ is the different of F_2. For a positive integer m, denote

$$L_m = \{A \in L : \det(A) = m/p\}.$$

The subset

$$T_m = \bigcup_{\left(\begin{smallmatrix} a & \lambda \\ \sigma(\lambda) & b \end{smallmatrix}\right) \in L_m/\{\pm 1\}} \{(z_1, z_2) \in \mathcal{H}^2 \mid a z_1 z_2 + \lambda z_1 + \sigma(\lambda) z_2 + b = 0\} \tag{3.12}$$

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defines an $\mathrm{SL}_2(\mathcal{O}_{F_2})$-invariant analytic divisor on $\mathcal{H}^2$ and descends to an algebraic divisor on the Hilbert modular surface X_{F_2}. This is the Hirzebruch–Zagier divisor of discriminant m, which was first studied in [HZ76]. Notice that $L_m = \emptyset$ and $T_m = 0$ whenever $\chi_p(m) = -1$.

Let $M_0^{!,+}(p, \chi_p) \subset M_0^!(p, \chi_p)$ be the subspace consisting of modular functions $f(z) = \sum_{n \gg -\infty} c(f, m) q^m$ satisfying $c(f, m) = 0$ whenever $\chi_p(m) = -1$. Let δ_p be the function defined in (1.9) (with D replaced by p). If $\delta_p(m)c(m) \in \mathbb{Z}$ for all $m < 0$, then there exists a meromorphic Hilbert modular form $\Psi_f(z_1, z_2)$ for $\mathrm{SL}_2(\mathcal{O}_{F_2})$ with weight $c(f, 0)$ and divisor

$$T_f = \sum_{m \geqslant 1} \delta_p(m) c(f, -m) T_m$$

by [BB03, Theorem 9]. This Hilbert modular form $\Psi_f(z_1, z_2)$ is called the *Borcherds lift* of f. It is normalized integral and its Fourier expansion is given in [BY06, Theorem 2.4(iii)]. Suppose $c(f, 0) = 0$; then $\Psi_f(z_1, z_2)$ is a Hilbert modular function and its value can be expressed in terms of an automorphic Green's function as follows.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $(z_1, z_2) \in \mathcal{H}^2 \backslash T_m$, define the automorphic Green's function for T_m by

$$\Phi_m(z_1, z_2, s) := \sum_{\left(\begin{smallmatrix} a & \lambda \\ \sigma(\lambda) & b \end{smallmatrix}\right) \in L_m} Q_{s-1}\left(1 + \frac{|a z_1 z_2 + \lambda z_2 + \sigma(\lambda) z_1 + b|^2}{2 y_1 y_2 m/p}\right), \tag{3.13}$$

where $z_j = x_j + i y_j$ and $Q_{s-1}(t)$ is the Legendre function of the second kind defined by

$$\begin{aligned} Q_{s-1}(t) &= \int_0^\infty \left(t + \sqrt{t^2 - 1} \cosh u\right)^{-s} du, \qquad \mathrm{Re}(s) > 1, \quad t > 1, \\ Q_0(t) &= \frac{1}{2} \log\left(1 + \frac{2}{t - 1}\right). \end{aligned} \tag{3.14}$$

As s approaches 1, the Green's function $\Phi_m(z_1, z_2, s)$ has a simple pole with residue $R(m)$ independent of z_1 or z_2. Define the regularized Green's function $\Phi_m(z_1, z_2)$ for the divisor T_m as

$$\Phi_m(z_1, z_2) := \lim_{s \to 1} \left(\Phi_m(z_1, z_2, s) - \frac{R(m)}{s - 1}\right). \tag{3.15}$$

By [BY06, Theorem 2.8], the logarithm of $|\Psi_f(z_1, z_2)|$ is related to the regularized automorphic Green's function by

$$2 \log |\Psi_f(z_1, z_2)| = \mathcal{C}(f) - \sum_{m > 0} \delta_p(m) c(f, -m) \Phi_m(z_1, z_2). \tag{3.16}$$

Here, $\mathcal{C}(f)$ is an explicit constant depending only on f and both sides are invariant under the action of $\mathrm{SL}_2(\mathcal{O}_{F_2})$. Since $M_0^{!,+}(p, \chi_p)$ has an integral basis, (3.16) can be extended to hold for any $f \in M_0^{!,+}(p, \chi_p)$.

For $\mathcal{A} \in \mathrm{Cl}_0(K_4)$, let $Z_{\mathcal{A},\Sigma}$ be the associated CM point on X_{F_2}. Using a relationship between the lattice $L_m \subset L$ and the number field $\tilde{K}_4$, we can express the value $\Phi_m(Z_{\mathcal{A},\Sigma}, s)$ in terms of the arithmetic of $\tilde{K}_4$ and K_4. First, choose a representative $\mathfrak{a} \subset \mathcal{O}_{K_4}$ in the class of $\mathcal{A}$ and write $\mathfrak{a} = \mathcal{O}_{F_2} \alpha + \mathcal{O}_{F_2} \beta$ such that α/β satisfies (3.1). Then define two $\mathbb{Q}$-quadratic forms on $\tilde{K}_4$ by

$$Q^-(\rho) = \mathrm{Tr}_{\tilde{F}_2/\mathbb{Q}} \frac{\rho \overline{\rho}}{\sqrt{Dp}}, \quad Q^+(\rho) = \mathrm{Tr}_{\tilde{F}_2/\mathbb{Q}} \rho \overline{\rho} \tag{3.17}$$

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and the map $\rho_{\alpha,\beta} : V \longrightarrow \tilde{K}_4$ by

$$\begin{aligned}\rho_{\alpha,\beta} : V &\longrightarrow \tilde{K}_4\\ A &\mapsto (\sigma(\alpha)), \sigma(\beta))A\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.\end{aligned}$$

A key fact here is that $\rho_{\alpha,\beta}$ is a $\mathbb{Q}$-isometry between the quadratic spaces $(\tilde{K}_4, -(N\mathfrak{f}_0/N\mathfrak{a})Q^-)$ and $(V, \det)$ by [BY06, Proposition 4.3]. The image of L_m under $\rho_{\alpha,\beta}$ could also be described precisely. Since $\mathrm{disc}(K_4) = Dp^3$ and $D \equiv 1 \pmod 4$ is a split prime in F_2, [BY06, condition (4.20)] is satisfied and [BY06, Proposition 4.8] implies that

$$\rho_{\alpha,\beta}(L_m) = \left\{\rho \in \mathfrak{d}^{-1}_{\tilde{K}_4/\tilde{F}_2}\mathrm{Nm}_\Sigma(\mathfrak{a}) : \frac{\rho\overline{\rho}}{N\mathfrak{a}} = \frac{pk - m\sqrt{Dp}}{2p} \in d^{-1,+}_{\tilde{K}_4/\tilde{F}_2}\right\}, \tag{3.18}$$

where $\mathfrak{d}_{\tilde{K}_4/\tilde{F}_2} \subset \mathcal{O}_{\tilde{K}_4}$ and $d_{\tilde{K}_4/\tilde{F}_2} \subset \mathcal{O}_{\tilde{F}_2}$ are the relative different and relative discriminant of $\tilde{K}_4/\tilde{F}_2$, respectively. Since $\mathrm{disc}(\tilde{K}_4) = D^2p^3$ and $\mathrm{disc}(\tilde{F}_2) = Dp$, the relative different $\mathfrak{d}_{\tilde{K}_4/\tilde{F}_2}$ is $\tilde{\mathfrak{P}}$, the unique prime in $\tilde{K}_4$ above p, and $d_{\tilde{K}_4/\tilde{F}_2} = \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}(\mathfrak{d}_{\tilde{K}_4/\tilde{F}_2})$.

For $A = \left(\begin{smallmatrix} a & \lambda \\ \sigma(\lambda) & b\end{smallmatrix}\right) \in V$ and $z := \alpha/\beta \in K_4$, define

$$\begin{aligned}\rho &:= \rho_{\alpha,\beta}(A),\\ \mu(z, A) &:= \frac{\rho\overline{\rho}}{N\mathfrak{a}} = \frac{Q^+(\rho) + \sqrt{Dp}Q^-(\rho)}{2N\mathfrak{a}},\\ d_A(z, \sigma(z)) &:= 1 + \frac{|az\sigma(z) + \lambda\sigma(z) + \sigma(\lambda)z + b|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(\sigma(z))\det(A)} = 1 + \frac{2\rho\overline{\rho}}{4\,\mathrm{Im}(z)\,\mathrm{Im}(\sigma(z))\det(A)}\\ &= \frac{Q^+(\rho)}{\sqrt{Dp}}\frac{1}{N\mathfrak{a}\det A}.\end{aligned}$$

Then we can write

$$\begin{aligned}\Phi_m(Z_{\mathcal{A},\Sigma}, s) &= \sum_{A\in L_m} Q_{s-1}(d_A(z, \sigma(z)))\\ &= \sum_{\mu=(pk-m\sqrt{Dp})/2p\in d^{-1,+}_{\tilde{K}_4/\tilde{F}_2}} Q_{s-1}\left(\frac{pk}{m\sqrt{Dp}}\right)\sum_{\substack{A\in L_m\\ \mu(z,A)=\mu}} 1.\end{aligned} \tag{3.19}$$

Since $\tilde{K}_4/\mathbb{Q}$ is non-Galois, ± 1 are the only roots of unity in $\tilde{K}_4$. Thus, the following map is a two-to-one surjection:

$$\begin{aligned}\rho_{\alpha,\beta}(L_m) &\longrightarrow \{\tilde{\mathfrak{b}} \subset \mathfrak{d}_{\tilde{K}_4/\tilde{F}_2} : [\tilde{\mathfrak{b}}] = [\mathfrak{d}^2_{\tilde{K}_4/\tilde{F}_2}][\mathrm{Nm}_\Sigma(\mathfrak{a})]^{-1}\}\\ \rho &\mapsto \rho\mathfrak{d}^2_{\tilde{K}_4/\tilde{F}_2}(\mathrm{Nm}_\Sigma(\mathfrak{a}))^{-1}.\end{aligned}$$

From this, we can deduce that

$$\begin{aligned}\#\{A \in L_m : \mu(z, A) = \mu\} &= \#\{\rho \in \mathfrak{d}^{-1}_{\tilde{K}_4/\tilde{F}_2}\mathrm{Nm}_\Sigma(\mathfrak{a}) : \rho\overline{\rho} = \mu N\mathfrak{a}\}\\ &= 2 \cdot \#\{\tilde{\mathfrak{b}} \in [\mathfrak{d}^2_{\tilde{K}_4/\tilde{F}_2}][\mathrm{Nm}_\Sigma(\mathfrak{a})]^{-1} \text{ integral} : \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}\tilde{\mathfrak{b}} = p\mu\}.\end{aligned}$$

Let $\mathfrak{P} \subset \mathcal{O}_{K_4}$ be the unique prime above p. Then $[\mathfrak{P}]^2$ is the identity in $\mathrm{Cl}(K_4)$, $\mathrm{Nm}_\Sigma(\mathfrak{P}) = \tilde{\mathfrak{P}}^2 = \mathfrak{d}^2_{\tilde{K}_4/\tilde{F}_2}$ and we have

$$\#\{A \in L_m : \mu(z, A) = \tilde{\mu}/p\} = 2 \cdot \#\{\tilde{\mathfrak{b}} \in [\mathrm{Nm}_\Sigma(\mathfrak{P}\mathfrak{a})]^{-1} \text{ integral} : \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}\tilde{\mathfrak{b}} = \tilde{\mu}\} \tag{3.20}$$

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for any $\tilde{\mu} = (pk - m\sqrt{Dp})/2 \in \mathfrak{d}^+_{\tilde{K}_4/\tilde{F}_2}$. Summing these together with a twist by ψ_2 over all the $[\mathfrak{a}] \in \mathrm{Cl}_0(K_4)$ gives us

$$\sum_{[\mathfrak{a}]\in \mathrm{Cl}_0(K_4)} \sum_{\substack{A\in L_m \\ \mu(z,A)=\tilde{\mu}/p}} \psi_2(\mathfrak{a}) = 2 \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_0(K_4) \\ \tilde{\mathfrak{b}}\subset \mathfrak{d}_{\tilde{K}_4/\tilde{F}_2} \\ [\tilde{\mathfrak{b}}]=[\mathrm{Nm}_\Sigma(\mathfrak{P}\mathfrak{a})]^{-1} \\ \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}\tilde{\mathfrak{b}}=\tilde{\mu}}} \psi_2(\mathfrak{a}) = 2\psi_2(\mathfrak{P}) \sum_{\substack{[\mathfrak{a}']\in \mathrm{Cl}_0(K_4) \\ \tilde{\mathfrak{b}}\subset \mathcal{O}_{\tilde{K}_4} \\ [\tilde{\mathfrak{b}}]=[\mathrm{Nm}_\Sigma(\mathfrak{a}')]^{-1} \\ \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}\tilde{\mathfrak{b}}=\tilde{\mu}}} (\tilde{\psi}\circ \mathrm{Nm}_\Sigma)(\mathfrak{a}')$$
$$= 2\psi_2(\mathfrak{P}) \frac{h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \sum_{\substack{\tilde{\mathfrak{b}}\subset \mathcal{O}_{\tilde{K}_4} \\ \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}\tilde{\mathfrak{b}}=\tilde{\mu}}} \tilde{\psi}(\tilde{\mathfrak{b}}) = \frac{2h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \psi_2(\mathfrak{P}) c_{\tilde{\psi}}(\tilde{\mu}).$$

The last two steps follow from the surjectivity of $\mathrm{Nm}_\Sigma : \mathrm{Cl}_0(K_4) \longrightarrow \mathrm{Cl}_0(\tilde{K}_4)$, whose kernel has size $h^+_{\tilde{F}_2}/h_{\tilde{F}_2}$, and the definition of $c_{\tilde{\psi}}(\tilde{\mu})$ in (4.1). Thus, the value of the Green's function $\Phi_m(z_1, z_2, s)$ at the twisted CM 0-cycle $\mathcal{CM}(K_4, \Sigma, \psi_2)$ is

$$\Phi_m(\mathcal{CM}(K_4, \Sigma, \psi_2), s) = \frac{2h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \psi_2(\mathfrak{P}) \sum_{\tilde{\mu}=(pk-m\sqrt{Dp})/2\in d^+_{\tilde{K}_4/\tilde{F}_2}} Q_{s-1}\left(\frac{pk}{m\sqrt{Dp}}\right) c_{\tilde{\psi}}(\tilde{\mu}).$$

This is the analogue of [BY06, Theorem 5.1] for the twisted CM 0-cycle. Similarly, for another CM type $\Sigma' := \sigma^3\Sigma = \{1, \sigma^{-1}\}$, the value of $\Phi_m(z_1, z_2, s)$ at $\mathcal{CM}(K_4, \Sigma', \psi_2)$ is

$$\Phi_m(\mathcal{CM}(K_4, \Sigma', \psi_2), s) = \frac{2h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \psi_2(\mathfrak{P}) \sum_{\tilde{\mu}'=(pk+m\sqrt{Dp})/2\in d^+_{\tilde{K}_4/\tilde{F}_2}} Q_{s-1}\left(\frac{pk}{m\sqrt{Dp}}\right) c_{\tilde{\psi}}(\tilde{\mu}')$$
$$= \Phi_m(\mathcal{CM}(K_4, \Sigma, \psi_2), s).$$

The second equality follows from Proposition 4.1, which implies that $c_{\tilde{\psi}}(\tilde{\mu}) = c_{\tilde{\psi}}(\sigma(\tilde{\mu}))$. Since ψ_2 is a non-trivial character, the value of the regularized Green's function $\Phi_m(z_1, z_2)$, which was defined in (3.15), at the twisted CM 0-cycle $\mathcal{CM}(K_4, \phi_2)$ can be expressed as

$$\Phi_m(\mathcal{CM}(K_4, \psi_2)) = \frac{8h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \psi_2(\mathfrak{P}) \lim_{s\to 1} \sum_{\tilde{\mu}=(pk-m\sqrt{Dp})/2\in d^+_{\tilde{K}_4/\tilde{F}_2}} Q_{s-1}\left(\frac{pk}{m\sqrt{Dp}}\right) c_{\tilde{\psi}}(\tilde{\mu}).$$

Combining this with (3.16) yields

$$\log|\Psi_f(\mathcal{CM}(K_4, \psi_2))| = -\frac{4\psi_2(\mathfrak{P})h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \sum_{m\geqslant 1} \delta_p(m) c(f, -m) \lim_{s\to 1} \sum_{\tilde{\mu}=(pk-m\sqrt{Dp})/2\in d^+_{\tilde{K}_4/\tilde{F}_2}} Q_{s-1}\left(\frac{pk}{m\sqrt{Dp}}\right) c_{\tilde{\psi}}(\tilde{\mu}). \tag{3.21}$$

Note that $\log|(C\cdot\Psi_f)(\mathcal{CM}(K_4, \psi_2))| = \log|\Psi_f(\mathcal{CM}(K_4, \psi_2))|$ for any non-zero constant C, since ψ_2 is a non-trivial character.

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4. Counting result

Recall that $\tilde{\psi}$ is the character defined in (3.6) and can be viewed as a character of $\mathrm{Cl}(\tilde{K}_4)$. For any integral ideal $\mathcal{I}$ in $\tilde{F}_2$, define the ideal counting function $c_{\tilde{\psi}}(\mathcal{I})$ by

$$c_{\tilde{\psi}}(\mathcal{I}) := \sum_{\substack{\tilde{\mathfrak{b}} \subset \mathcal{O}_{\tilde{K}_4} \\ \mathrm{Nm}_{\tilde{K}_4/\tilde{F}_2}(\tilde{\mathfrak{b}}) = \mathcal{I}}} \tilde{\psi}(\tilde{\mathfrak{b}}). \tag{4.1}$$

Notice that if $\mathcal{I} = \mathcal{I}_1\mathcal{I}_2$ with $\mathcal{I}_1, \mathcal{I}_2$ relatively prime in $\tilde{F}_2$, then

$$c_{\tilde{\psi}}(\mathcal{I}) = c_{\tilde{\psi}}(\mathcal{I}_1)c_{\tilde{\psi}}(\mathcal{I}_2). \tag{4.2}$$

For $\mathcal{I} = (\tilde{\mu})$ principal, the quantity $c_{\tilde{\psi}}(\tilde{\mu})$ can be thought of as the $\tilde{\mu}$th Fourier coefficient of a Hilbert modular form with parallel weight one associated to the Galois representation

$$\rho_{\tilde{\psi},\tilde{F}_2} := \mathrm{Ind}_{\tilde{K}_4}^{\tilde{F}_2} \tilde{\psi} : \mathrm{Gal}(M_{32}/\tilde{F}_2) \longrightarrow \mathrm{GL}_2(\mathbb{C}). \tag{4.3}$$

On the other hand, we can compare characters to obtain

$$\rho_{\tilde{\psi},\tilde{F}_2} \cong \rho_\varphi|_{\mathrm{Gal}(M_{32}/\tilde{F}_2)}, \tag{4.4}$$

where the right-hand side is related to the Doi–Naganuma lift of f_φ. The goal of this section is to establish a precise relationship between $c_{\tilde{\psi}}$ and c_φ through Proposition 4.1, which will be crucial in proving Theorem 1.1.

Proposition 4.1. *For any totally positive element $\tilde{\mu} = (k - m\sqrt{Dp})/2 \in \tilde{F}_2^+$, we have*

$$c_{\tilde{\psi}}(\tilde{\mu}) = \frac{\mathrm{ord}_{\tilde{\mathfrak{D}}}(\tilde{\mu}) + 1}{\mathrm{ord}_D(\gcd(k,m)) + 1} \sum_{d \mid \gcd(k,m)} c_\varphi\left(\frac{k^2 - m^2 Dp}{4d^2}\right)\overline{\phi_p(d)}, \tag{4.5}$$

where $D\mathcal{O}_{\tilde{F}_2} = \tilde{\mathfrak{D}}^2$. In particular, when $\mathrm{ord}_D(k) > \mathrm{ord}_D(n)$,

$$c_{\tilde{\psi}}(\tilde{\mu}) = 2 \sum_{d \mid \gcd(k,m)} c_\varphi\left(\frac{k^2 - m^2 Dp}{4d^2}\right)\overline{\phi_p(d)}. \tag{4.6}$$

Proof. First, we can factor $(\tilde{\mu})$ as

$$(\tilde{\mu}) = \prod_{j=1}^{2} \tilde{\mathfrak{l}}_j^{r_j} \prod_{j=3}^{M_1} (\tilde{\mathfrak{l}}_j^{a_j} \sigma(\tilde{\mathfrak{l}}_j)^{b_j}) \prod_{j=M_1+1}^{M} \tilde{\mathfrak{l}}^{r_j},$$

where $\tilde{\mathfrak{l}}_1 = \tilde{\mathfrak{p}}, \tilde{\mathfrak{l}}_2 = \tilde{\mathfrak{D}}$. Each $\tilde{\mathfrak{l}}_j$ is above a distinct rational prime ℓ_j, which is split in $\tilde{F}_2$ for $3 \leqslant j \leqslant M_1$ and is inert in $\tilde{F}_2$ for $M_1 + 1 \leqslant j \leqslant M$. In this notation, the norm of $\tilde{\mu}$ factors as

$$\mathrm{Nm}(\tilde{\mu}) = \prod_{j=1}^{2} \ell_j^{r_j} \prod_{j=3}^{M_1} \ell_j^{a_j+b_j} \prod_{j=M_1+1}^{M} \ell_j^{2r_j}, \quad \gcd(k,m) = \prod_{j=1}^{M} \ell_j^{t_j}.$$

The exponents t_j satisfy

$$t_j = \begin{cases} \lfloor r_j/2 \rfloor, & 1 \leqslant j \leqslant 2, \\ \min(a_j, b_j), & 3 \leqslant j \leqslant M_1, \\ r_j, & M_1 + 1 \leqslant j \leqslant M. \end{cases} \tag{4.7}$$

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By (4.2) and the multiplicativity of c_φ, the two sides of (4.5) can be expressed as

$$c_{\tilde{\psi}}(\tilde{\mu}) = \prod_{j=1}^{2} c_{\tilde{\psi}}(\tilde{\mathfrak{l}}_j^{r_j}) \prod_{j=3}^{M_1} c_{\tilde{\psi}}(\tilde{\mathfrak{l}}_j^{a_j}) c_{\tilde{\psi}}(\sigma(\tilde{\mathfrak{l}}_j)^{b_j}) \prod_{j=M_1+1}^{M} c_{\tilde{\psi}}(\ell_j^{r_j}), \tag{4.8}$$

$$\begin{aligned}\sum_{d\mid\gcd(k,n)} c_\varphi\left(\frac{k^2-m^2Dp}{4d^2}\right)\overline{\phi_p(d)} \;=\; &\prod_{j=1}^{2}\left(\sum_{\nu_j=0}^{t_j} c_\varphi(\ell_j^{r_j-2\nu_j})\overline{\phi_p(\ell_j^{\nu_j})}\right)\\ &\cdot\prod_{j=3}^{M_1}\left(\sum_{\nu_j=0}^{t_j} c_\varphi(\ell_j^{a_j+b_j-\nu_j})\overline{\phi_p(\ell_j^{\nu_j})}\right)\\ &\cdot\prod_{j=M_1+1}^{M}\left(\sum_{\nu_j=0}^{t_j} c_\varphi(\ell_j^{2(r_j-\nu_j)})\overline{\phi_p(\ell_j^{\nu_j})}\right).\end{aligned} \tag{4.9}$$

Thus, it is enough to prove (4.5) by considering each prime ℓ_j separately. For convenience, we will drop the subscript j in $\tilde{\mathfrak{l}}_j, \ell_j, r_j, \nu_j, a_j$ and b_j from now on.

When $\ell \nmid Dp$, let $\mathcal{L}$ be a prime in M_{32} above $\tilde{\mathfrak{l}}$ and $\mathrm{Frob}_{\mathcal{L},\tilde{F}_2} \in \mathrm{Gal}(M_{32}/\tilde{F}_2)$ be the associated Frobenius element. By (4.4), the following identity between power series in $\overline{\mathbb{Q}}[\![X]\!]$ holds:

$$\sum_{r\geqslant 0} c_{\tilde{\psi}}(\tilde{\mathfrak{l}}^r)X^r = \det(1-\rho_{\tilde{\psi},\tilde{F}_2}(\mathrm{Frob}_{\mathcal{L},\tilde{F}_2})X)^{-1} = \det(1-\rho_\varphi(\mathrm{Frob}_{\mathcal{L},\tilde{F}_2})X)^{-1}. \tag{4.10}$$

There are three cases depending on the splitting behavior of ℓ in $\tilde{F}_2$.

Case 1: ℓ is inert in $\tilde{F}_2$. Let $\mathrm{Frob}_{\mathcal{L}} \in \mathrm{Gal}(M_{32}/\mathbb{Q})$ be the Frobenius element associated to $\mathcal{L}$. Then, for all $b \in M_{32}$,

$$\mathrm{Frob}_{\mathcal{L}}^2(b) \equiv b^{2\ell} \pmod{\mathcal{L}}.$$

Since ℓ is inert in $\tilde{F}_2$, $\mathrm{Frob}_{\mathcal{L}}^2$ is conjugate to $\mathrm{Frob}_{\mathcal{L},\tilde{F}_2}$ in $\mathrm{Gal}(M_{32}/\mathbb{Q})$ and hence also in $\mathrm{Gal}(M_{32}/\tilde{F}_2)$. Then (4.10) implies that

$$\sum_{r\geqslant 0} c_{\tilde{\psi}}(\ell^r)X^r = \det(1-\rho_{\tilde{\psi},\tilde{F}_2}(\mathrm{Frob}_{\mathcal{L},\tilde{F}_2})X)^{-1} = \det(1-\rho_\varphi(\mathrm{Frob}_{\mathcal{L}}^2)X)^{-1}. \tag{4.11}$$

Let α, β be the two eigenvalues of $\rho_\varphi(\mathrm{Frob}_{\mathcal{L}})$. Then α^2, β^2 are the eigenvalues of $\rho_{\tilde{\psi},\tilde{F}_2}(\mathrm{Frob}_{\mathcal{L},\tilde{F}_2})$ and

$$\alpha\beta = \det(\rho_\varphi(\mathrm{Frob}_{\mathcal{L}})) = \chi_D(\ell)\phi_p(\ell) = \left(\frac{Dp}{\ell}\right)\left(\frac{p}{\ell}\right)\phi_p(\ell) = -\overline{\phi_p(\ell)}.$$

So, we can write

$$\begin{aligned}\sum_{\nu=0}^{r} c_\varphi(\ell^{2r-2\nu})\overline{\phi_p(\ell)}^{\nu} &= \sum_{\nu=0}^{r}\left(\sum_{j=0}^{2(r-\nu)}\alpha^j\beta^{2r-2\nu-j}\right)(-\alpha\beta)^\nu = \sum_{\nu=0}^{r}(-1)^\nu\sum_{w=\nu}^{2r-\nu}\alpha^w\beta^{2r-w}\\ &= \sum_{w=0}^{2r}\alpha^w\beta^{2r-w}\sum_{\nu=0}^{\min(w,2r-w)}(-1)^\nu = \sum_{\substack{w=0\\ w\text{ even}}}^{2r}\alpha^w\beta^{2r-w}\\ &= \sum_{u=0}^{r}(\alpha^2)^u(\beta^2)^{r-u} = c_{\tilde{\psi}}(\ell^r).\end{aligned}$$

The last step follows from the first equality in (4.11).

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Case 2: ℓ splits in $\tilde{F}_2$. In this case, $\mathrm{Frob}_{\mathcal{L},\tilde{F}_2} \in \mathrm{Gal}(M_{32}/\tilde{F}_2)$ is also the Frobenius element associated to $\mathcal{L}$ in $\mathrm{Gal}(M_{32}/\mathbb{Q})$ and (4.10) implies that for any $r \geqslant 0$,

$$c_{\tilde{\psi}}(\tilde{\mathfrak{l}}^r) = c_{\tilde{\psi}}(\sigma(\tilde{\mathfrak{l}})^r) = c_\varphi(\ell^r).$$

Recall from (4.7) that $t = \min(a, b)$. Then Lemma 4.2 implies that

$$c_{\tilde{\psi}}(\tilde{\mathfrak{l}}^a)c_{\tilde{\psi}}(\sigma(\tilde{\mathfrak{l}})^b) = c_\varphi(\ell^{a+b-t})c_\varphi(\ell^t) = \sum_{\nu=0}^{t} c_\varphi(\ell^{a+b-2\nu})\overline{\phi_p(\ell)}^{\nu}.$$

Case 3: ℓ ramifies in $\tilde{F}_2$. If $\ell = p$, then $\tilde{\mathfrak{p}}\mathcal{O}_{\tilde{K}_4} = \tilde{\mathfrak{P}}^2$ ramifies. Let $\mathrm{Frob}_{\tilde{\mathfrak{P}}} \in \mathrm{Gal}(M_{32}/\tilde{K}_4)$ be the Frobenius element associated to $\tilde{\mathfrak{P}}$. Since $(\frac{p}{D}) = 1$, the ideal $\tilde{\mathfrak{P}}\mathcal{O}_{M_8}$ splits and $\mathrm{Frob}_{\tilde{\mathfrak{P}}} \in \mathrm{Gal}(M_{32}/M_8)$. Write $p\mathcal{O}_F = \mathfrak{p}\tau(\mathfrak{p})$ with $\tau(\mathfrak{p})$ unramified in K_8/F. Since the norm of $\tilde{\mathfrak{P}}$ is p, the element $\mathrm{Frob}_{\tilde{\mathfrak{P}}}$ is sent to the Frobenius element of $\tau(\mathfrak{p})$ in $\mathrm{Gal}(K_8/F)$ under the map $\mathrm{Gal}(M_{32}/M_8) \hookrightarrow \mathrm{Gal}(M_{32}/F) \twoheadrightarrow \mathrm{Gal}(K_8/F)$. Combining this with Proposition 3.4 yields

$$c_{\tilde{\psi}}(\tilde{\mathfrak{p}}^r) = \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{P}}})^r = \varphi(\mathrm{Frob}_{\tau(\mathfrak{p})})^r = c_\varphi(p^r) = \sum_{\nu=0}^{t} c_\varphi(p^{r-\nu})\overline{\phi_p(p^\nu)}.$$

If $\ell = D$, then $\tilde{\mathfrak{l}} = \tilde{\mathfrak{D}}$ splits completely in M_8, since there are four primes in M_8 above D. So, we can write $\tilde{\mathfrak{D}}\mathcal{O}_{\tilde{K}_4} = \tilde{\mathfrak{L}}\overline{\tilde{\mathfrak{L}}}, D\mathcal{O}_F = \mathfrak{D}^2$ with corresponding Frobenius elements $\mathrm{Frob}_{\tilde{\mathfrak{L}}}$, $\mathrm{Frob}_{\overline{\tilde{\mathfrak{L}}}} \in \mathrm{Gal}(M_{32}/M_8)$ and $\mathrm{Frob}_{\mathfrak{D}} \in \mathrm{Gal}(K_8/F)$. Notice that $\sigma^2(\tilde{\mathfrak{L}}) = \overline{\tilde{\mathfrak{L}}}$ and

$$\mathrm{Frob}_{\overline{\tilde{\mathfrak{L}}}} = \sigma^2 \,\mathrm{Frob}_{\tilde{\mathfrak{L}}}\, \sigma^{-2} = \mathrm{Frob}_{\tilde{\mathfrak{L}}}.$$

By the same reasoning as in the case $\ell = p$, we have $\tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}}) = \varphi(\mathrm{Frob}_{\mathfrak{D}}) = c_\varphi(D)$ and

$$\begin{aligned} c_{\tilde{\psi}}(\tilde{\mathfrak{D}}^r) &= \sum_{\nu=0}^{r} \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})^\nu \tilde{\psi}(\mathrm{Frob}_{\overline{\tilde{\mathfrak{L}}}})^{r-\nu} = \sum_{\nu=0}^{r} \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})^\nu \tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})^{r-\nu} \\ &= (r+1)\tilde{\psi}(\mathrm{Frob}_{\tilde{\mathfrak{L}}})^r = (r+1)\varphi(\mathrm{Frob}_{\mathfrak{D}})^r = (r+1)c_\varphi(D^r). \end{aligned}$$

On the other hand, $\phi_p(D) = \varphi(D) = \varphi(\mathfrak{D}^2) = c_\varphi(D^2) \in \{\pm 1\}$ and

$$\sum_{\nu=0}^{t} c_\varphi(D^{r-2\nu})\overline{\phi_p(D)}^{\nu} = (t+1)c_\varphi(D^r).$$

Notice that $r + 1 = 2(t+1)$ when $\mathrm{ord}_D(k) > \mathrm{ord}_D(n)$, since

$$r = \begin{cases} 2t, & \mathrm{ord}_D(k) \leqslant \mathrm{ord}_D(n), \\ 2t+1, & \mathrm{ord}_D(k) > \mathrm{ord}_D(n). \end{cases}$$

Finally, combining the three cases together and comparing (4.8) with (4.9) produces (4.5). $\square$

Lemma 4.2. *Let ℓ be a prime that splits in $\tilde{F}_2$. Then, for all $0 \leqslant 2t \leqslant r$,*

$$c_\varphi(\ell^{r-t}) \cdot c_\varphi(\ell^t) = \sum_{\nu=0}^{t} c_\varphi(\ell^{r-2\nu})\overline{\phi_p(\ell)}^{\nu}. \tag{4.12}$$

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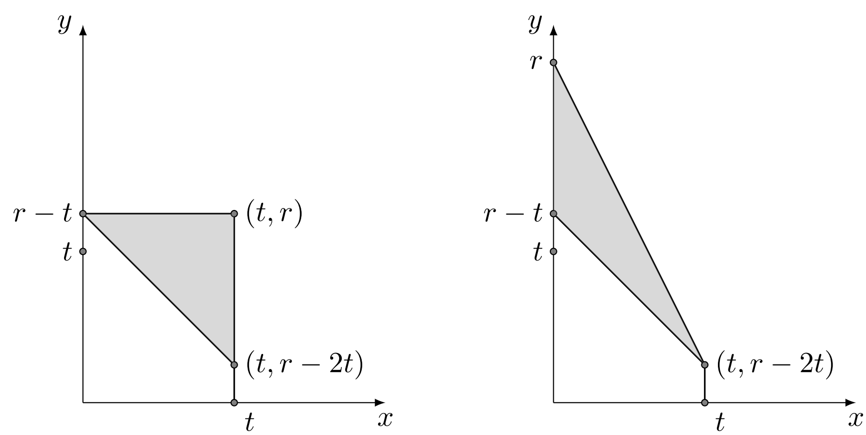
Remark 4.3. When $r = 2$ and $t = 1$, (4.12) follows from the fact that the eigenvalue of f_φ under the ℓth Hecke operator is $c_\varphi(\ell)$.

Proof. Let $\mathrm{Frob}_\ell \in \mathrm{Gal}(M_{32}/\mathbb{Q})$ be an element in the conjugacy class of the Frobenius element associated to ℓ, and α, β be the eigenvalues of $\rho_\varphi(\mathrm{Frob}_\ell)$. Since ℓ splits in F_2, we have

$$\alpha\beta = \det(\rho_\varphi(\mathrm{Frob}_\ell)) = \chi_D(\ell)\phi_p(\ell) = \left(\frac{Dp}{\ell}\right)\left(\frac{p}{\ell}\right)\phi_p(\ell) = \overline{\phi_p(\ell)}.$$

So, the left- and right-hand sides of (4.12) become

$$\mathrm{LHS} = \left(\sum_{j_1=0}^{r-t} \alpha^{j_1}\beta^{r-t-j_1}\right) \cdot \left(\sum_{j_2=0}^{t} \alpha^{j_2}\beta^{t-j_2}\right) = \sum_{j_1=0}^{r-t}\sum_{j_2=0}^{t} \alpha^{j_1+j_2}\beta^{r-j_1-j_2} = \sum_{u=0}^{r} \alpha^u\beta^{r-u}C_L(u),$$
$$\mathrm{RHS} = \sum_{\nu=0}^{t}\left(\sum_{j=0}^{r-2\nu} \alpha^{j}\beta^{r-2\nu-j}\right)(\alpha\beta)^\nu = \sum_{\nu=0}^{t}\sum_{j=0}^{r-2\nu} \alpha^{j+\nu}\beta^{r-\nu-j} = \sum_{u=0}^{r} \alpha^u\beta^{r-u}C_R(u),$$
$$C_L(u) := \#\{(x,y) \in \mathbb{Z}^2 : 0 \leqslant x \leqslant t, 0 \leqslant y \leqslant r-t, x+y = u\},$$
$$C_R(u) := \#\{(x,y) \in \mathbb{Z}^2 : 0 \leqslant x \leqslant t, 0 \leqslant y \leqslant r-2x, x+y = u\}.$$



When $0 \leqslant u \leqslant r-t$, it is clear from the picture above that $C_L(u) = C_R(u)$. When $r-t+1 \leqslant u \leqslant r$, we also have $C_L(u) = C_R(u)$, since the involution

$$(x,y) \leftrightarrow (t-x, 2x+y-t)$$

switches the two shaded regions while preserving $x+y$. Thus, (4.12) holds. $\square$

5. Harmonic Maass forms

In this section, we will review some background information on harmonic Maass forms following [BF04]. Then we will prove Theorem 5.6 concerning the properties of a unique preimage of f_φ under the differential operator ξ_1 defined in (1.1).

Let $k \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\nu : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a character. It is also a character of the congruence subgroup $\Gamma_0(N)$ via $\nu\left(\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)\right) := \nu(d)$. Recall that the weight-k hyperbolic Laplacian Δ_k and differential operator ξ_k are defined in (1.1). We call a real-analytic function $\mathcal{F} : \mathcal{H} \to \mathbb{C}$ a *harmonic Maass form* of weight k, level N and character ν if the following are satisfied:

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(i) $(\mathcal{F} \mid_k \gamma)(z) = \nu(\gamma)\mathcal{F}(z)$ for all $\gamma \in \Gamma_0(N)$;
(ii) $\Delta_k(\mathcal{F}) = 0$;
(iii) the function $\mathcal{F}(z)$ is only allowed to have polar-type singularities at the cusps of $\Gamma_0(N)$.

Let $H_k(N,\nu)$ be the space of harmonic Maass forms of weight k, level N and character ν, whose image under ξ_k is a cusp form. Denote by $M_k^!(N,\nu), M_k(N,\nu)$ and $S_k(N,\nu)$ the usual subspaces of weakly holomorphic modular forms, holomorphic modular forms and cusp forms. Every $\mathcal{F} \in H_k(N,\nu)$ can be written canonically as the sum of a holomorphic part and a non-holomorphic part

$$\mathcal{F}(z) = \tilde{f}(z) + f^*(z),$$

where $\tilde{f}(z)$ has the following Fourier expansion at the cusp infinity:

$$\tilde{f}(z) = \sum_{n \gg -\infty} c^+(n) q^n.$$

If the cusp form $f(z) := \xi_k(\mathcal{F}(z)) \in S_{2-k}(N,\overline{\nu})$ has Fourier expansion $\sum_{n\geqslant 1} c(n)q^n$ at infinity, then the non-holomorphic part $f^*(z)$ looks like

$$f^*(z) = -\sum_{n\geqslant 0} \overline{c(n)}\beta_k(n,y)q^{-n}$$

at the cusp infinity, where, for $n > 0$,

$$\beta_k(n,y) := \int_y^\infty e^{-4\pi n t} t^{-k}\, dt \tag{5.1}$$

is the incomplete Gamma function after a suitable change of variables.

Property (ii) and (1.1) give the following map:

$$\xi_k : H_k(N,\nu) \longrightarrow S_{2-k}(N,\overline{\nu}), \tag{5.2}$$

whose kernel is $M_k^!(N,\nu)$.

Proposition 5.1. *The map (5.2) is surjective.*

A proof of this proposition for vector-valued modular forms on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is given in [BF04] using Serre duality. The same argument is also applicable to the weight-one case. Using the relationship between vector-valued and scalar-valued modular forms (see e.g. [BB03]), the surjectivity of ξ_k can be translated to the setting of scalar-valued modular forms. Since ξ_k commutes with the slash operator, one could add nebentypus character by imposing conditions on harmonic Maass forms on the congruence subgroup $\Gamma_1(N)$.

In [BF04], Bruinier and Funke introduced a pairing between $\mathcal{F} = \tilde{f} + f^* \in H_k(N,\nu)$ and $g \in S_{2-k}(N,\overline{\nu})$, which is given by the Petersson inner product $\langle g, \xi_k(\mathcal{F})\rangle$. Using Stokes' theorem, they expressed this pairing in terms of the Fourier coefficients of g and the principal parts of $\tilde{f}$ at various cusps of $\Gamma_0(N)$ (see [BF04, Proposition 3.5] for vector-valued modular forms). When N is odd and square free, the cusps of $\Gamma_0(N)$ are indexed by the divisors of N. For $d \mid N$, let $\sigma_d \in \mathrm{GL}_2(\mathbb{R})$ be a scaling matrix sending the cusp ∞ to the cusp $1/d$. Then this pairing can be expressed as

$$\{g, \mathcal{F}\} := \langle g, \xi_k(\mathcal{F})\rangle = \text{Constant term of } \left(\sum_{d\mid N} (\tilde{f}\cdot g) \mid_2 \sigma_d\right). \tag{5.3}$$

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Proposition 5.2. *The pairing defined in (5.3) is a perfect pairing between $S_{2-k}(N,\overline{\nu})$ and $H_k(N,\nu)/M^!_k(N,\nu)$.*

Proof. By Proposition 5.1, the pairing is non-degenerate in $g \in S_{2-k}(N,\overline{\nu})$. On the other hand, suppose there exists $\{a_d(-n) \in \mathbb{C} : n \geqslant 1, d \mid N\}$ such that $a_d(-n) = 0$ for n sufficiently large and

$$\sum_{d|N}\sum_{n\geqslant 1} c(g\ |_{2-k}\ \sigma_d, n)a_d(-n) = 0 \tag{5.4}$$

for all $g \in S_{2-k}(N,\overline{\nu})$. When $k < 1$, one could use Poincaré series to explicitly construct $\mathcal{F} \in M^!_k(N,\nu)$ such that

$$(\mathcal{F}\ |_k\ \sigma_d)(z) = \sum_{n\geqslant 1} a_d(-n)q^{-n} + O(1)$$

for all $d \mid N$, and the pairing is perfect, since $S_{2-k}(N,\overline{\nu})$ is a finite-dimensional vector space.

Let $\Delta(z) \in S_{12}$ be the unique cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. When $k = 1$, let $S^{\Delta}_{13}(N,\overline{\nu})$ be the image of the map

$$\begin{aligned} S_1(N,\overline{\nu}) &\longrightarrow S_{13}(N,\overline{\nu}) \\ g(z) &\mapsto g(z)\Delta(z), \end{aligned}$$

which is a subspace of $S_{13}(N,\overline{\nu})$. The coefficients $\{a_d(-n) : n \geqslant 1, d \mid N\}$ satisfy (5.4) for all $g(z) \in S_1(N,\overline{\nu})$, which can be written as

$$\sum_{d|N}\sum_{n\geqslant 1} c\bigg(\frac{(h\ |_{13}\ \sigma_d)(z)}{\Delta(z)}, n\bigg)a_d(-n) = 0$$

for all $h \in S^{\Delta}_{13}(N,\overline{\nu})$. Since $h \in S_{13}(N,\overline{\nu})$ is in the subspace $S^{\Delta}_{13}(N,\overline{\nu})$ if and only if $c((h\ |_{13}\ \sigma_d)(z)/\Delta(z), n) = 0$ for all $n \leqslant 0, d \mid N$, we could find $\{a_d(n) \in \mathbb{C} : n \geqslant 0, d \mid N\}$ such that

$$\sum_{d|N}\sum_{n\in\mathbb{Z}} c\bigg(\frac{(h\ |_{13}\ \sigma_d)(z)}{\Delta(z)}, n\bigg)a_d(-n) = 0$$

for all $h \in S_{13}(N,\overline{\nu})$. By the perfect pairing for $k = -11$, there exists $\mathcal{G}(z) \in M^!_{-11}(N,\nu)$ such that

$$(\mathcal{G}\ |_{-11}\ \sigma_d)(z) = \frac{\sum_{n\in\mathbb{Z}} a_d(n)q^n}{\Delta(z)}$$

for all $d \mid N$. Then $\mathcal{F}(z) := \mathcal{G}(z)\Delta(z) \in M^!_1(N,\nu)$ has the desired principal part $\sum_{n\geqslant 1} a_d(-n)q^{-n} + O(1)$ at the cusp $1/d$ for all $d \mid N$. □

For certain characters ν, one could use various projection operators to decompose the space $H_k(N,\nu)$ into various eigenspaces under the Atkin–Lehner involutions. The pairing on these eigenspaces can then be expressed in terms of the Fourier coefficients at the cusp infinity. For a prime $\ell \mid N$, let $W_\ell = \left(\begin{smallmatrix} \ell\alpha & \beta \\ N & \ell \end{smallmatrix}\right)$ be the Atkin–Lehner involution and $U_\ell := \sum_{\lambda=1}^{\ell} \left(\begin{smallmatrix} 1 & \lambda \\ & \ell \end{smallmatrix}\right)$ the U-operator.

Write $N = N'\ell'$ and $\nu = \nu_{N'}\nu_\ell$, where $\nu_{N'}$ and ν_ℓ have conductors N' and ℓ, respectively. If $\nu_{N'}$ is a quadratic character, define

$$\mathrm{pr}^{\varepsilon}_{\nu_\ell}(\mathcal{F})(z) := \frac{1}{2}\bigg(\varepsilon\frac{\overline{\nu_{N'}(\ell)\nu_\ell(-N')}}{G(\nu_\ell)}(\mathcal{F}\ |_k\ U_\ell W_\ell)(z) + \mathcal{F}(z)\bigg) \tag{5.5}$$

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for $\varepsilon = \pm 1$ and $\mathcal{F}(z) = \sum_{n \in \mathbb{Z}} c(\mathcal{F}, n, y) q^n \in H_1(N, \nu)$. Here, $G(\nu_\ell)$ is the Gauss sum associated to the character ν_ℓ. Then, as a generalization of [BB03, § 3], we have the following lemma.

Lemma 5.3 [Li13, Proposition 2.4.1]. *Using the same notation as above, the following are equivalent:*

(1) *for all $n \in \mathbb{Z}$ relatively prime to ℓ,*

$$\nu_\ell(n) c(\mathcal{F}, n, y) = -\varepsilon c(\mathcal{F}, n, y); \tag{5.6}$$

(2) $\mathrm{pr}^{\varepsilon}_{\nu_\ell}(\mathcal{F}) = 0$;

(2)′ $\mathrm{pr}^{-\varepsilon}_{\nu_\ell}(\mathcal{F}) = \mathcal{F}$;

(3)

$$\mathcal{F} \mid_k W_\ell = -\varepsilon \frac{\nu_\ell(N')}{G(\nu_\ell)} \mathcal{F} \mid_k U_\ell. \tag{5.7}$$

If ν_ℓ has order greater than 2 and $\nu_{N'}$ is a quadratic character, there is a similar result. Let

$$\mathrm{pr}^{\varepsilon}_{\nu_\ell}(\mathcal{F})(z) := \frac{1}{2}\left(\varepsilon \frac{\overline{\nu_{N'}(\ell)\nu_\ell(-N')}}{G(\nu_\ell)} (\mathcal{F} \mid_k U_\ell W_\ell)(z) + \mathcal{F}^c(z)\right) \tag{5.8}$$

for $\varepsilon = \pm 1$. Here, $\mathcal{F}^c(z) := \overline{\mathcal{F}(\overline{z})} \in H_1(N, \overline{\nu})$ and $\nu_{N'}$ being quadratic are necessary, since W_ℓ sends $H_1(N, \nu)$ to $H_1(N, \overline{\nu})$. Then we have the following analogue of Lemma 5.3.

Lemma 5.4 [Li13, Proposition 2.4.2]. *Using the same notation as above, the following are equivalent:*

(1) *for all $n \in \mathbb{Z}$ relatively prime to ℓ,*

$$\nu_\ell(n) \overline{c(\mathcal{F}, n, y)} = -\varepsilon c(\mathcal{F}, n, y); \tag{5.9}$$

(2) $\mathrm{pr}^{\varepsilon}_{\nu_\ell}(\mathcal{F}) = 0$;

(2)′ $\mathrm{pr}^{-\varepsilon}_{\nu_\ell}(\mathcal{F}) = \mathcal{F}^c$;

(2)″ $\mathrm{pr}^{-\varepsilon}_{\nu_\ell}(i\mathcal{F}) = 0$;

(3)

$$\mathcal{F} \mid_k W_\ell = -\varepsilon \frac{\nu_\ell(N')}{G(\overline{\nu_\ell})} \mathcal{F}^c \mid_k U_\ell. \tag{5.10}$$

Note that the subspace of forms in $H_1(N, \nu)$ satisfying one of the conditions in Lemma 5.4 is a *real* vector space. When ν_ℓ and $\nu_{N'}$ are both quadratic, the definitions (5.5) and (5.8) agree when $\mathcal{F} \in H_1(N, \nu)$ satisfies $\mathcal{F} = \mathcal{F}^c$. Since the projection operators are defined using slash operators, they commute with each other and ξ_k as follows.

Lemma 5.5 [Li13, Proposition 2.4.4]. *Let $\ell, \ell' \mid N$ be distinct primes and $\varepsilon, \varepsilon' \in \{\pm 1\}$. Suppose $\nu_{N/\ell'}(\cdot) = (\frac{\cdot}{N/\ell'})$ and $\nu_{\ell'}$ has order greater than 2. Then the projection operators satisfy the following properties:*

$$\begin{aligned} \mathrm{pr}^{-\varepsilon}_{\nu_\ell} \circ \mathrm{pr}^{\varepsilon}_{\nu_\ell} &= \mathrm{pr}^{-\varepsilon}_{\nu_{\ell'}} \circ \mathrm{pr}^{\varepsilon}_{\nu_{\ell'}} = 0, \\ \mathrm{pr}^{\varepsilon}_{\nu_\ell} \circ \mathrm{pr}^{\varepsilon'}_{\nu_{\ell'}} &= \mathrm{pr}^{\varepsilon'}_{\nu_{\ell'}} \circ \mathrm{pr}^{\varepsilon}_{\nu_\ell}, \\ \mathrm{pr}^{\varepsilon}_{\nu_\ell} \circ \xi_k &= \xi_k \circ \mathrm{pr}^{\varepsilon\nu_\ell(-1)}_{\nu_\ell}, \\ \mathrm{pr}^{\varepsilon}_{\nu_{\ell'}} \circ \xi_k &= \xi_k \circ \mathrm{pr}^{\varepsilon\nu_{\ell'}(-1)}_{\nu_{\ell'}}. \end{aligned} \tag{5.11}$$

Let D, p, χ_D, ϕ_p and f_φ be the same as in § 2.3. In this setting, there are two Eisenstein series in $M_1(Dp, \chi_D\phi_p)$ linearly independent over $\mathbb{C}$ given by [DS05, § 4.8]

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$$E_1^{\mathbb{1},\chi_D\phi_p}(z) := L(0,\chi_D\phi_p) + 2\sum_{n\geqslant 1}\bigg(\sum_{m\mid n,\ m>0}\chi_D(m)\phi_p(m)\bigg)q^n,$$
$$E_1^{\chi_D,\phi_p}(z) := 2\sum_{n\geqslant 1}\bigg(\sum_{m\mid n,\ m>0}\chi_D(n/m)\phi_p(m)\bigg)q^n.$$

Here, $L(s,\chi_D\phi_p)$ is the Dirichlet L-function associated to the character $\chi_D\phi_p$. Since $\chi_D\phi_p$ is odd, the special value $L(0,\chi_D\phi_p)$ is a non-zero multiple of $L(1,\chi_D\phi_p)$ via the functional equation and hence it does not vanish [Was97, Theorem 4.9]. Define $E_1(z)\in M_1(Dp,\chi_D\phi_p)$ by

$$E_1(z) := E_1^{\mathbb{1},\chi_D\phi_p}(z) + E_1^{\chi_D,\phi_p}(z) = L(0,\chi_D\phi_p) + 4q + O(q^2). \tag{5.12}$$

From its explicit Fourier coefficients, one could verify that

$$\mathrm{pr}^-_{\chi_D}(E_1) = \mathrm{pr}^-_{\phi_p}(E_1) = 0. \tag{5.13}$$

Proposition 2.9 tells us that the space $S_1(Dp,\chi_D\phi_p)$ is spanned over $\mathbb{C}$ by f_φ. Together with Lemmas 5.3 and 5.4, (2.9) implies that

$$\mathrm{pr}^+_{\chi_D}(f_\varphi) = \mathrm{pr}^+_{\overline{\phi_p}}(f^c_\varphi) = f_\varphi. \tag{5.14}$$

So, for any $\mathcal{F} = \tilde{f} + f^* \in H_1(Dp,\chi_D\overline{\phi_p})$ satisfying

$$\mathrm{pr}^+_{\chi_D}(\mathcal{F}) = \mathrm{pr}^+_{\overline{\phi_p}}(\mathcal{F}) = \mathcal{F},$$

the pairing $\{f_\varphi,\mathcal{F}\}$ becomes

$$\langle f_\varphi, \xi_1(\mathcal{F})\rangle = \{f_\varphi,\mathcal{F}\} = \sum_{n\in\mathbb{Z}}(c_\varphi(n)c^+(-n) + \overline{c_\varphi(pn)c^+(-pn)})\delta_D(n), \tag{5.15}$$

where $\tilde{f}(z) = \sum_{n\gg -\infty} c^+(n)q^n$ and $\delta_D(n)$ is defined in (1.9). Note that the sum on the right-hand side is a finite sum. Since the pairing is perfect, any non-trivial solution of the $c^+(n)$ to the equation

$$\sum_{n\in\mathbb{Z}}(c_\varphi(n)c^+(-n) + \overline{c_\varphi(pn)c^+(-pn)})\delta_D(n) = 0$$

forms the principal part of a modular form $\mathcal{F}\in M^!_1(Dp,\chi_D\overline{\phi_p})$.

By the perfect pairing in Proposition 5.2, the space $H_1(Dp,\chi_D\overline{\phi_p})/M^!_1(Dp,\chi_D\overline{\phi_p})$ is spanned over $\mathbb{C}$ by a harmonic Maass form $\mathcal{F}_\varphi(z)$ satisfying $\xi_1(\mathcal{F}_\varphi) = f_\varphi$. Using the projection operators, we can choose a unique representative $\mathcal{F}_\varphi\in H_1(Dp,\chi_D\overline{\phi_p})$ to study.

Theorem 5.6. *There exists a unique harmonic Maass form $\mathcal{F}_\varphi \in H_1(Dp,\chi_D\overline{\phi_p})$ with holomorphic part*

$$\tilde{f}_\varphi(z) = c^+_\varphi(-1)q^{-1} + c^+_\varphi(0) + \sum_{\substack{n\geqslant 2\\ \chi_D(n)\neq -1}} c^+_\varphi(n)q^n,$$
$$c^+_\varphi(-1) = \langle f_\varphi, f_\varphi\rangle = \frac{2h_{\tilde{K}_4}}{h_{\tilde{F}_2}}\log u_F,$$
$$c^+_\varphi(0) = -\frac{2\langle f_\varphi, f_\varphi\rangle}{L(0,\chi_D\phi_p)}$$

such that $\xi_1(\mathcal{F}_\varphi) = f_\varphi$ and $|(\mathcal{F}_\varphi \mid W_p)(z)|$ has exponential decay as $y\to\infty$.

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Proof. Given two such harmonic Maass forms, denote their difference by $\mathcal{D}(z)$. It is holomorphic and $O(q^2)$ at the cusp infinity. Furthermore, it satisfies $\mathrm{pr}^-_{\chi_D}(\mathcal{D}(z)) = 0$. By Lemma 5.3 and the exponential decay of $(\mathcal{F}_\varphi \mid W_p)(z)$, the form $\mathcal{D}(z)$ vanishes at the other cusps of $\Gamma_0(Dp)$ and hence is a cusp form. Proposition 2.9 tells us that the only cusp form in $S_1(Dp, \chi_D\overline{\phi_p})$ is $\overline{f_\varphi(\overline{z})} = q + O(q^2)$, which implies that $\mathcal{D}(z) = 0$.

To show the existence of such $\mathcal{F}_\varphi$, we begin with any $\mathcal{F}(z) \in H_1(Dp, \chi_D\overline{\phi_p})$ satisfying $\xi_1(\mathcal{F}) = f_\varphi$ by Proposition 5.1. Applying Lemma 5.5 and $\phi_p(-1) = -1$ to (5.14), we can replace $\mathcal{F}$ with $(\mathrm{pr}^+_{\chi_D} \circ \mathrm{pr}^-_{\phi_p})(\mathcal{F}^c)$ to make sure that

$$\mathrm{pr}^-_{\chi_D}(\mathcal{F}) = \mathrm{pr}^+_{\overline{\phi_p}}(\mathcal{F}) = 0.$$

Suppose the holomorphic part of $\mathcal{F}$, denoted by $\tilde{f}$, has the Fourier expansion

$$\tilde{f}(z) = \sum_{\substack{n \geqslant -n_0 \\ \chi_D(n) \neq -1}} c^+(n) q^n$$

at the cusp infinity. By the perfect pairing between the one-dimensional $\mathbb{C}$ vector spaces $H_1(Dp, \chi_D\overline{\phi_p})/M_1^!(Dp, \chi_D\overline{\phi_p})$ and $S_1(Dp, \chi_D\phi_p)$, we can take $n_0 = 1$. Equation (5.15) then reduces to

$$c^+(-1) = \langle f_\varphi, f_\varphi\rangle.$$

By (5.13) and (5.14), we know that

$$\mathrm{pr}^-_{\chi_D}(E_1^c) = \mathrm{pr}^-_{\chi_D}(f_\varphi^c) = 0.$$

Since the constant term of E_1^c is non-zero, subtracting appropriate multiples of E_1^c and f_φ^c from $\mathcal{F}$ guarantees that the holomorphic part of $\mathcal{F} \mid W_p$ is $O(q)$ and $c^+(1) = 0$. Since $\beta_1(n, y) q^{-n}$ decays exponentially for all $n \geqslant 1$, the form $(f \mid W_p)(z)$ also decays exponentially. Finally, (5.15) with f_φ replaced by E_1 becomes

$$4c^+(-1) + 2L(0, \chi_D\phi_p)c^+(0) = 0,$$

which yields $c^+(0) = -2\langle f_\varphi, f_\varphi\rangle / L(0, \chi_D\phi_p)$. □

6. Proof of Theorem 1.1

In the same notation as § 5, let $\mathcal{F}_\varphi(z) \in H_1(Dp, \chi_D\overline{\phi_p})$ be the unique harmonic Maass form in Theorem 5.6 with Fourier expansion

$$\mathcal{F}_\varphi(z) = \sum_{n \in \mathbb{Z}} \hat{c}_\varphi(n, y) q^n = c_\varphi^+(-1) q^{-1} + c_\varphi^+(0) + \sum_{n \geqslant 2} c_\varphi^+(n) q^n - \sum_{n \geqslant 1} \overline{c_\varphi(n)} \beta_1(n, y) q^{-n}.$$

To prove Theorem 1.1, we will first construct a modular form $\Xi_\varphi(z) \in M_2(p, \chi_p)$ from $\mathcal{F}_\varphi$ and calculate its Fourier expansion. This will be the replacement of the cusp form in [BY06, Theorem 8.1].

Let $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ be the Jacobi theta function. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, let $\tilde{\gamma} = [\gamma, j(\gamma, z)] \in \tilde{\Gamma}_0(4)$ be the element in the metaplectic cover of $\Gamma_0(4)$, where $j(\gamma, z) := \theta(\gamma z)/\theta(z)$. Define the

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character $\chi' : (\mathbb{Z}/4p)^\times \longrightarrow \mathbb{C}^\times$ by

$$\chi'(\cdot) = \left(\frac{-4}{\cdot}\right)\phi_p(\cdot). \tag{6.1}$$

Let $\tilde{W}_D := \left[\left(\begin{smallmatrix} D\alpha_D & \beta_D \\ 4Dp & D \end{smallmatrix}\right), D^{-1/4}\sqrt{4Dpz+D}\right], \tilde{W}_p := \left[\left(\begin{smallmatrix} p\alpha_p & \beta_p \\ 4Dp & p \end{smallmatrix}\right), p^{-1/4}\sqrt{4Dpz+p}\right]$ be the Atkin–Lehner involutions. Consider the real-analytic function $\mathcal{F}_\varphi(4z)\theta(pz)$. It is a modular form of weight $3/2$, level $4Dp$ and character $\chi'(\cdot)(\frac{\cdot}{D})$.

Define the function $\tilde{\Omega}_\varphi(z)$ by

$$\begin{aligned} \tilde{\Omega}_\varphi(z) &:= D^{-1/4}((\mathcal{F}_\varphi(4z)\theta(pz)) \mid_{3/2} \tilde{W}_D + (\mathcal{F}_\varphi(4z)\theta(pz)) \mid_{3/2} \tilde{U}_D)(z) \\ &= \sum_{n\in\mathbb{Z}} c(\tilde{\Omega}_\varphi, n, y) q^n. \end{aligned} \tag{6.2}$$

By [Li13, Lemma 2.3.6], it is a real-analytic modular form of weight $3/2$, level $4p$ and character χ'. Using the calculations

$$\begin{aligned} ((\mathcal{F}_\varphi(4z)\theta(pz)) \mid_{3/2} \tilde{W}_D)(z) &= p^{-1/4}\left(\mathcal{F}_\varphi \mid_1 \begin{pmatrix} D\alpha_D & 4\beta_D \\ Dp & D \end{pmatrix}\right)(4z) \cdot \left(\theta \mid_{1/2} \left[\begin{pmatrix} p & \\ & 1 \end{pmatrix}, p^{-1/4}\right] \tilde{W}_D\right)(z) \\ &= D^{1/4}\left(\frac{1}{D}\sum_{j=0}^{D-1} \mathcal{F}_\varphi\left(\frac{4z+j}{D}\right)\right) \cdot \theta(Dpz), \\ ((\mathcal{F}_\varphi(4z)\theta(pz)) \mid_{3/2} \tilde{U}_D)(z) &= D^{1/4}\left(\frac{1}{D}\sum_{j=0}^{D-1} \mathcal{F}_\varphi\left(4\left(\frac{z+j}{D}\right)\right)\theta\left(p\left(\frac{z+j}{D}\right)\right)\right), \end{aligned}$$

one can write $c(\tilde{\Omega}_\varphi, n, y)$ as

$$\begin{aligned} c(\tilde{\Omega}_\varphi, n, y) &= \sum_{k\in\mathbb{Z}} \hat{c}_\varphi\left(\frac{Dn - pk^2}{4}, \frac{4y}{D}\right)\delta_D(k) = a_\varphi(n) + a^*_\varphi(n, y), \\ a^*_\varphi(n, y) &:= -\sum_{k\in\mathbb{Z}} \overline{c_\varphi\left(\frac{pk^2 - Dn}{4}\right)}\beta_1\left(\frac{pk^2 - Dn}{D}4\pi y\right)\delta_D(k) \end{aligned} \tag{6.3}$$

with $a_\varphi(n)$ defined in (1.8).

Since $D, p \equiv 1 \pmod 4$, the coefficient $c(\tilde{\Omega}_\varphi, n, y)$ vanishes whenever $n \equiv 2, 3 \pmod 4$. This means that $\tilde{\Omega}_\varphi(z)$ is in the Kohnen plus space of weight $3/2$, level $4p$ and character χ'. By [Li13, Lemmas 2.3.3 and 2.3.4], the function $\tilde{\Omega}_\varphi \mid_{3/2} \tilde{W}_p$ equals

$$((\mathcal{F}_\varphi \mid W_p U_D)(4z)\theta(Dz) + ((\mathcal{F}_\varphi \mid W_p)(4z)\theta(pz)) \mid_{3/2} \tilde{U}_D$$

up to a non-zero constant factor. Similar calculations and the exponential decay of $\mathcal{F}_\varphi \mid W_p$ imply that $\tilde{\Omega}_\varphi \mid_{3/2} \tilde{W}_p$ is contained in Kohnen's plus space and has exponential decay at the cusp infinity. Since β_1 decays exponentially, the function $\tilde{\Omega}_\varphi$ is $2c^+_\varphi(0) + O(q)$ at the cusp infinity and $O(q)$ at all other cusps of $\Gamma_0(p)$. Let $E_{3/2,p} \in M_{3/2}(4p, \chi')$ be the unique Eisenstein series in Kohnen's plus space such that $E_{3/2,p}(z)$ is $1 + O(q)$ at infinity and $O(q)$ at all other cusps. Its existence follows from the dimension formula of half-integral weight Eisenstein series [CO77].

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Then $\tilde{\Omega}_\varphi(z) - 2c^+_\varphi(0)E_{3/2,p}(z)$ is $O(q)$ at all cusps of $\Gamma_0(4p)$ and we can apply the holomorphic projection operator π_{hol} to it and obtain

$$\Omega_\varphi(z) := \pi_{\mathrm{hol}}(\tilde{\Omega}_\varphi(z) - 2c^+_\varphi(0)E_{3/2,p}(z)) \in S_{3/2}(4p, \chi') \tag{6.4}$$

in Kohnen's plus space.

In [Koh82], Kohnen studied the space of half-integral weight cusp forms satisfying Kohnen's plus space condition and showed that the Shimura lifting maps it isomorphically to the space of cusp forms as Hecke modules when the character is quadratic and the level is odd and square free. After many people's work [MRV90, Pei82, Tsu99, Ued93, van83], the Shimura lifting has been generalized to modular forms of weight $k+1/2$ for all integers $k \geqslant 1$, odd level and arbitrary character. In our situation, we can apply the Shimura lifting to $\Omega_\varphi(z)$ (in the notation of [Koh82, Theorem 2]) and define

$$\Xi_\varphi(z) := \mathscr{S}_{1,1,p,\phi_p}(\Omega_\varphi(z) + 2c^+_\varphi(0)E_{3/2,p}(z)) \in M_2(p, \chi_p). \tag{6.5}$$

The following lemma calculates the Fourier coefficients of Ξ_φ.

Lemma 6.1. *The modular form $\Xi_\varphi(z)$ has the Fourier expansion $\Xi_\varphi(z) = \sum_{n \geqslant 0} c(\Xi_\varphi, m)q^m$ at infinity, where*

$$\begin{aligned} c(\Xi_\varphi, m) &= \begin{cases} L(0,\phi_p)c^+_\varphi(0), & m = 0, \\ b_\varphi(m) - c_\varphi(p)b'_\varphi(m), & m \geqslant 1, p \nmid m, \end{cases} \\ b'_\varphi(m) &:= \lim_{s\to 1} \sum_{k'\in\mathbb{Z}} \left(\sum_{d \mid \gcd(k',m)} \phi_p(d) \overline{c_\varphi\left(\frac{(pk')^2 - pDm^2}{4d^2}\right)\delta_D(k')} \right) 2Q_{s-1}\left(\frac{pk'}{\sqrt{pD}m}\right), \end{aligned} \tag{6.6}$$

$b_\varphi(m)$ is defined in (1.7) and $Q_{s-1}(t)$ is the Legendre function of the second kind in (3.14).

Proof. Let $c(\Omega_\varphi, n)$ and $c(E_{3/2,p}, n)$ be the nth Fourier coefficients of Ω_φ and $E_{3/2,p}$, respectively. By the definition of holomorphic projection [Stu80], the coefficient $c(\Omega, n)$ comes from the inner product between $\tilde{\Omega}_\varphi - 2c^+_\varphi(0)E_{3/2,p}$ and the nth Poincaré series of weight $3/2$, level $4p$ and character χ' defined by taking the limit of the following function:

$$\mathcal{P}_n(z,s) := \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(4p)} \overline{\chi'(d)} j(\gamma, z)^{-3} e^{2n\pi i \gamma z} \mathrm{Im}(\gamma z)^{(s-1)/2}$$

as s approaches 1. Since $\tilde{\Omega}_\varphi - 2c^+_\varphi(0)E_{3/2,p}$ decays exponentially at all cusps of $\Gamma_0(4p)$, one can switch the limit in s and inner product. After applying Rankin–Selberg unfolding, we obtain

$$\begin{aligned} c(\Omega_\varphi, n) &= \frac{(4\pi n)^{1/2}}{\Gamma(1/2)} \left\langle \tilde{\Omega}_\varphi - 2c^+_\varphi(0)c(E_{3/2,p}, n), \lim_{s\to 0} \mathcal{P}_{n,s} \right\rangle \\ &= \lim_{s\to 1} \left(a_\varphi(n) - 2c^+_\varphi(0)c(E_{3/2,p}, n) + \frac{(4\pi n)^{1/2}}{\Gamma(1/2)} \int_0^\infty a^*_\varphi(n,y) e^{-4\pi n y} y^{s/2} \frac{dy}{y} \right) \\ &= a_\varphi(n) - 2c^+_\varphi(0)c(E_{3/2,p}, n) - c_\varphi(p)a'_\varphi(n), \end{aligned}$$

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where $a'_\varphi(n)$ is defined by

$$a'_\varphi(n) := \lim_{s\to 1}\sum_{k\in\mathbb{Z}} \overline{c_\varphi\bigg(\frac{(pk)^2-pDm}{4}\bigg)}\delta_D(k)\varrho_{s-1}\bigg(\frac{pk^2}{Dm}-1\bigg),$$
$$\varrho_s(\mu) := \int_1^\infty \frac{du}{(\mu u+1)^{(1+s)/2}}u, \quad \mu>0.$$

Here, we have used the multiplicative property $c_\varphi(pn) = c_\varphi(p)c_\varphi(n)$ for all $n\in\mathbb{N}$. With the following comparisons (see [GZ85, § 7] for similar arguments):

$$\varrho_0(\mu) = 2Q_0(\sqrt{\mu+1}),$$
$$Q_{s-1}(\sqrt{\mu+1}) - \frac{s\Gamma(s)^2}{2^{2-s}\Gamma(2s)}\varrho_{s-1}(\mu) = O(\mu^{-1/2-s/2}),$$

we can substitute $\varrho_{s-1}(pk^2/Dm-1)$ with $2Q_{s-1}(pk/\sqrt{pDm})$ in the limit as s goes to 1 and obtain

$$\begin{aligned} c(\Omega_\varphi,n) + 2c_\varphi^+(0)c(E_{3/2,p},n) &= a_\varphi(n) - c_\varphi(p)a'_\varphi(n),\\ a'_\varphi(n) &= \lim_{s\to 1}\sum_{k\in\mathbb{Z}}\overline{c_\varphi\bigg(\frac{(pk)^2-pDn}{4}\bigg)}\delta_D(k)2Q_{s-1}\bigg(\frac{pk}{\sqrt{pDn}}\bigg). \end{aligned} \tag{6.7}$$

Now, given $g(z) = \sum_{n\geqslant 0} c(g,n)q^n \in M_{3/2}(4p,\chi')$ satisfying Kohnen's plus space condition, its Shimura lift $\mathscr{S}_{1,1,p,\phi_p}(g)\in M_2(p,\chi_p)$ has the shape (see [Koh82, Theorem 2(ii)])

$$\mathscr{S}_{1,1,p,\phi_p}(g)(z) := \frac{L(0,\phi_p)c(g,0)}{2} + \sum_{m\geqslant 1}\bigg(\sum_{d\mid m}\phi_p(d)c\bigg(g,\frac{m^2}{d^2}\bigg)\bigg)q^m. \tag{6.8}$$

Applying this to (6.5) yields $c(\Xi_\varphi,0) = L(0,\phi_p)a_\varphi(0)/2 = L(0,\phi_p)c_\varphi^+(0)$ and

$$\begin{aligned} c(\Xi_\varphi,m) &= \sum_{d\mid m}\phi_p(d)\bigg(a_\varphi\bigg(\frac{m^2}{d^2}\bigg) - c_\varphi(p)a'_\varphi\bigg(\frac{m^2}{d^2}\bigg)\bigg) = b_\varphi(m) - c_\varphi(p)\sum_{d\mid m}\phi_p(d)a'_\varphi\bigg(\frac{m^2}{d^2}\bigg)\\ &= b_\varphi(m) - c_\varphi(p)\lim_{s\to 1}\sum_{d\mid m}\sum_{k\in\mathbb{Z}}\phi_p(d)\overline{c_\varphi\bigg(\frac{(pkd)^2-pDm^2}{4d^2}\bigg)}\delta_D(k)2Q_{s-1}\bigg(\frac{pkd}{\sqrt{pD}m}\bigg)\\ &= b_\varphi(m) - c_\varphi(p)\lim_{s\to 1}\sum_{k'\in\mathbb{Z}}\sum_{d\mid\gcd(k',m)}\phi_p(d)\overline{c_\varphi\bigg(\frac{(pk')^2-pDm^2}{4d^2}\bigg)}\delta_D(k')2Q_{s-1}\bigg(\frac{pk'}{\sqrt{pD}m}\bigg)\\ &= b_\varphi(m) - c_\varphi(p)b'_\varphi(m) \end{aligned}$$

when $p\nmid m$. □

Proof of Theorem 1.1. Let $\Psi(z_1,z_2)$ be a normalized integral Hilbert modular function with divisor

$$\sum_{\substack{m\geqslant 1\\ \gcd(pD,m)=1}} c(-m)T_m.$$

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By [BY06, Theorem 2.8], there exists a modular function $f(z)=\sum_{m\geqslant 1}c(-m)q^{-m}+O(q)\in M_0^{!,+}(p,\chi_p)$ such that $\Psi=C\cdot\Psi_f$ for some non-zero constant C. By Proposition 4.1, we know that

$$c_{\tilde{\psi}}\left(\frac{pk'-m\sqrt{Dp}}{2}\right)=\sum_{d|\gcd(k',m)}\phi_p(d)\overline{c_\varphi\left(\frac{(pk')^2-pDm^2}{4d^2}\right)}\delta_D(k')$$

when $D\nmid k'$ and $k'>m\sqrt{Dp}/p$. Furthermore, $\psi_2(\mathfrak{P})=\varphi(\mathfrak{p}')^2=c_\varphi(p)^2$. Thus, we can rewrite (3.21) as

$$\begin{aligned}\log|\Psi(\mathcal{CM}(K_4,\psi_2))| &= \log|(C\cdot\Psi_f)(\mathcal{CM}(K_4,\psi_2))|=\log|\Psi_f(\mathcal{CM}(K_4,\psi_2))|\\ &= -\frac{c_\varphi(p)^2h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}}\sum_{m\geqslant 1}c(-m)b'_\varphi(m).\end{aligned}$$

Using the perfect pairing between $S_2(p,\chi_p)$ and $M_0^{!,+}(p,\chi_p)$, we can deduce that

$$0=\sum_{m\geqslant 1}c(-m)c(\Xi_\varphi,m)=\sum_{m\geqslant 1}c(-m)b_\varphi(m)-c_\varphi(p)\sum_{m\geqslant 1}c(-m)b'_\varphi(m).$$

Putting together the two equations above gives us (1.6). $\square$

7. Numerical calculations and conjectures

When $D=29, p=5$ and $\phi_5(2)=i$, the newform $f_\varphi\in S_1(145,\chi_{29}\phi_5)$ has the Fourier expansion

$$f_\varphi(z)=q+iq^4+iq^5+(-i-1)q^7-iq^9+(-i+1)q^{13}-q^{16}+O(q^{20}).$$

In this case, we have

$$\begin{gathered}u_F=\tfrac{5+\sqrt{29}}{2},\quad L(0,\chi_D\phi_p)=-2-2i,\quad \mathfrak{p}=(\tfrac{3+\sqrt{29}}{2}),\quad c_\varphi(p)=i,\\ h_{\tilde{F}_2}=h^+_{\tilde{F}_2},\quad h_{F_2}=1,\quad h_{K_4}=2,\quad \langle f_\varphi,f_\varphi\rangle=4\log u_F.\end{gathered}$$

Let $\mathcal{F}_\varphi$ be the harmonic Maass form in Theorem 5.6 and $c^+_\varphi(n)$ the nth Fourier coefficients of its holomorphic part.

Using the modularity of $\mathcal{F}_\varphi(z)$, one could substitute in various $z\in\mathcal{H}$ and numerically compute the coefficients $c^+_\varphi(n)$ using linear algebra. This idea goes back at least to Hejhal [HD83] and has been implemented in [BSV06] to compute the Fourier coefficients of Maass cusp forms. Using the computer program SAGE [Ste12], we have numerically computed $c^+_\varphi(n)$ for $n\leqslant 1000$ with precision at least 10^{-20}. The numerically computed coefficients

$$c^+_\varphi(-1)=6.5889245585484\ldots,\quad c^+_\varphi(0)=1.64723114637110\ldots-i\cdot 1.64723114637110\ldots$$

agree with the values given by Theorem 5.6

$$c^+_\varphi(-1)=4\log u_F,\quad c^+_\varphi(0)=(1-i)\log u_F.$$

Let $\Psi(z_1,z_2)$ be a normalized integral Hilbert modular function on X_{F_2} with divisor T_6-2T_1. It is the Borcherds lift of a weakly holomorphic modular function of level 5 and nebentypus character $(\frac{\cdot}{5})$, whose Fourier expansion has the form $q^{-6}-2q^{-1}+242q+O(q^2)$ (see [BB03, BY06]). Its value at the untwisted CM 0-cycle $\mathcal{CM}(K_4,\mathcal{O}_{F_2})$ is an integer and has the factorization

$$\Psi(\mathcal{CM}(K_4,\mathcal{O}_{F_2}))=2^{64}\cdot 3^{28}\cdot 17^4\cdot 181^4\cdot 241^4.$$

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In comparison, its numerical value at $\mathcal{CM}(K_4,\psi_2)$ is an algebraic number in F and seems to have the factorization

$$\Psi(\mathcal{CM}(K_4,\psi_2)) = \left(\frac{\pi_{181}}{\pi'_{181}}\right)^4 \cdot \left(\frac{\pi_{241}}{\pi'_{241}}\right)^4 \cdot u_F^{16}, \tag{7.1}$$

where $\pi_{181} = (1+5\sqrt{29})/2, \pi_{241} = (35+3\sqrt{29})/2$ each generate a prime ideal in F above 181 and 241, respectively.

The other side of (1.6) is

$$\begin{aligned}\text{RHS of (1.6)} &= -i(b_\varphi(6) - 2b_\varphi(1)) = -i(a_\varphi(36) + ia_\varphi(9) - ia_\varphi(4) + a_\varphi(1)) + 2ia_\varphi(1)\\ &= -2i(c_\varphi^+(261) + c_\varphi^+(256) + c_\varphi^+(241) + c_\varphi^+(216) + c_\varphi^+(181) + c_\varphi^+(136)\\ &\quad + c_\varphi^+(81) + c_\varphi^+(16)) + 2(c_\varphi^+(64) + c_\varphi^+(54) + c_\varphi^+(34) + c_\varphi^+(4)) - 2(c_\varphi^+(29)\\ &\quad + c_\varphi^+(24) + c_\varphi^+(9)) + 2ic_\varphi^+(6).\end{aligned}$$

Numerically, the coefficients $c_\varphi^+(n)$ appearing in the sum above are given by

$$\begin{aligned}&c_\varphi^+(4) = c_\varphi^+(6) = c_\varphi^+(24) = c_\varphi^+(29) = c_\varphi^+(34) = c_\varphi^+(81) = c_\varphi^+(261) = 0,\\ &c_\varphi^+(9) = -ic_\varphi^+(16) = c_\varphi^+(54) = c_\varphi^+(64) = -ic_\varphi^+(136) = ic_\varphi^+(216) = ic_\varphi^+(256) = 4\log u_F,\\ &c_\varphi^+(181) = 2i\log\left|\frac{\pi_{181}}{\pi'_{181}}\right| + 4i\log u_F, \quad c_\varphi^+(241) = 2i\log\left|\frac{\pi_{241}}{\pi'_{241}}\right|.\end{aligned}$$

This agrees with Theorem 1.1.

Let $\Lambda_F := \mathbb{Z}[i]\cdot\log u_F \subset \mathbb{C}$ be a lattice. For a rational prime ℓ, let $\mathfrak{l}$ be a prime in F above ℓ and $\pi_\mathfrak{l} \in \mathcal{O}_F$ a generator. Define the quantity $\mathscr{C}_\varphi^+(\ell) \in \mathbb{C}/\Lambda$ by

$$\mathscr{C}_\varphi^+(\ell) := (\overline{\varphi}(\mathfrak{l}) - \overline{\varphi}(\mathfrak{l}'))\log\left|\frac{\pi_\mathfrak{l}}{\pi'_\mathfrak{l}}\right| + \Lambda. \tag{7.2}$$

Note that a choice of $\pi_\mathfrak{l}$ corresponds to a lift of $\mathscr{C}_\varphi^+(\ell)$ to $\mathbb{C}$. Numerical calculations suggest the following refinement of Conjecture 1.3.

Conjecture 7.1. For all rational primes ℓ, the image of $c_\varphi^+(\ell)$ in $\mathbb{C}/\Lambda$ is $\mathscr{C}_\varphi^+(\ell)$.

Under this conjecture, there should exist $b(\pi_\mathfrak{l}) \in \mathbb{Z}[i]$ such that

$$c_\varphi^+(\ell) = (\overline{\varphi}(\mathfrak{l}) - \overline{\varphi}(\mathfrak{l}'))\log\left|\frac{\pi_\mathfrak{l}}{\pi'_\mathfrak{l}}\right| + b(\pi_\mathfrak{l})\log u_F.$$

In the following table, we have listed the values of π_ℓ and $b(\pi_\ell)$ for all primes $\ell \leqslant 100$ satisfying $(\frac{\ell}{29}) \neq -1$.

ℓ	5	7	13	23	29
$(\pi_\mathfrak{l}, b(\pi_\mathfrak{l}))$	$(\frac{3+\sqrt{29}}{2}, -2)$	$(6+\sqrt{29}, 0)$	$(\frac{49+9\sqrt{29}}{2}, 0)$	$(\frac{11+\sqrt{29}}{2}, 0)$	$(\sqrt{29}, 0)$
ℓ	53	59	67	71	83
$(\pi_\mathfrak{l}, b(\pi_\mathfrak{l}))$	$(13+2\sqrt{29}, 0)$	$(28+5\sqrt{29}, 8)$	$(\frac{23+3\sqrt{29}}{2}, 0)$	$(10+\sqrt{29}, 0)$	$(\frac{19+\sqrt{29}}{2}, 0)$

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Applying the ℓth Hecke operator T_ℓ to $\tilde{f}_\varphi$ yields the following relationship:

$$\kappa_{D,p} \cdot (T_\ell \tilde{f}_\varphi - \overline{c_\varphi(\ell)}\tilde{f}_\varphi - c_\varphi^+(\ell) f_{\overline{\varphi}}) \in \Lambda_F((q)),$$

where $\kappa_{D,p} := |L(0, \chi_D \phi_p)|^2 \in \mathbb{Z}$. From this relationship, one could deduce the following formula for $c_\varphi^+(n)$ from Conjecture 7.1:

$$\kappa_{D,p} \cdot c_\varphi^+(n) = \kappa_{D,p} \cdot \sum_{\ell^r \| n} \overline{c_\varphi\left(\frac{n}{\ell^r}\right) J_\varphi(\ell^{r-1})} \cdot \mathscr{C}_\varphi^+(\ell) \in \mathbb{C}/\Lambda_F, \tag{7.3}$$

where $\sum_{n \geqslant 1} J_\varphi(n) n^{-s} := (\sum_{n \geqslant 1} c_\varphi(n) n^{-s})^2$.

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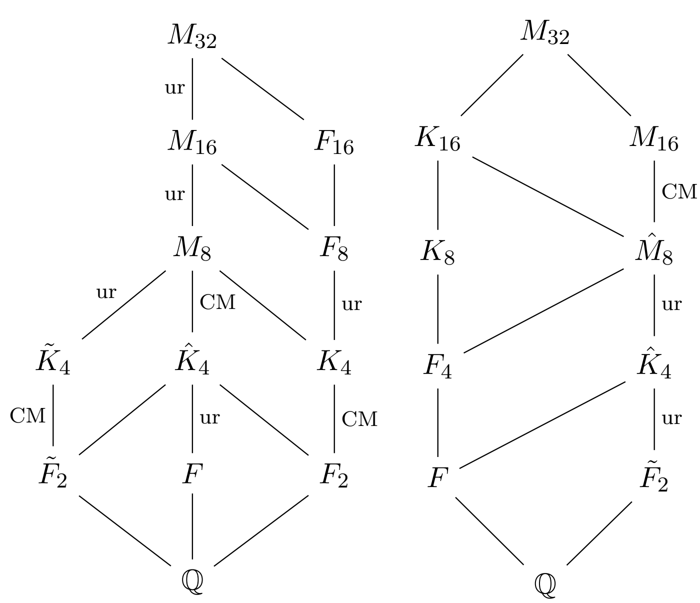
Appendix. Field extension diagrams and character table



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C	$(0,0,0)$	$(1,0,0)$	$(2,0,0)$	$(3,0,0)$	$(1,1,0)$	$(2,1,0)$	$(3,1,0)$	$(2,2,0)$	$(3,2,0)$	$(3,3,0)$	$(0,0,1)$	$(1,0,1)$	$(2,0,1)$	$(3,0,1)$
$\#C$	1	2	2	2	1	2	2	1	2	1	4	4	4	4
ϕ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_1	1	i	-1	$-i$	-1	$-i$	1	1	i	-1	1	i	-1	$-i$
ϕ_1^c	1	$-i$	-1	i	-1	i	1	1	$-i$	-1	1	$-i$	-1	i
χ_p	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1
χ_D	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_{Dp}	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
ϕ_3	1	i	-1	$-i$	-1	$-i$	1	1	i	-1	-1	$-i$	1	i
ϕ_3^c	1	$-i$	-1	i	-1	i	1	1	$-i$	-1	-1	i	1	$-i$
ρ_0	2	$1+i$	0	$1-i$	$2i$	$-1+i$	0	-2	$-1-i$	$-2i$	0	0	0	0
ρ_0^c	2	$1-i$	0	$1+i$	$-2i$	$-1-i$	0	-2	$-1+i$	$2i$	0	0	0	0
ρ_1	2	$-1+i$	0	$-1-i$	$-2i$	$1+i$	0	-2	$1-i$	$2i$	0	0	0	0
ρ_1^c	2	$-1-i$	0	$-1+i$	$2i$	$1-i$	0	-2	$1+i$	$-2i$	0	0	0	0
ϱ_0	2	0	2	0	-2	0	-2	2	0	-2	0	0	0	0
ϱ_1	2	0	-2	0	2	0	-2	2	0	2	0	0	0	0

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