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Abstract

We develop a theory of unbounded derived categories of quasi-coherent sheaves on algebraic stacks. In particular, we show that these categories are compactly generated by perfect complexes for stacks that either have finite stabilizers or are local quotient stacks. We also extend Toën and Antieau–Gepner's results on derived Azumaya algebras and compact generation of sheaves on linear categories from derived schemes to derived Deligne–Mumford stacks. These are all consequences of our main theorem: compact generation of a presheaf of triangulated categories on an algebraic stack is local for the quasi-finite flat topology.

Introduction

Our first main result is the following.

Theorem A. *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$, of $\mathcal{O}_X$-modules with quasi-coherent cohomology, is compactly generated by a single perfect complex. Moreover, for every quasi-compact open subset $U \subseteq X$, there exists a compact perfect complex with support exactly $X\backslash U$.*

This generalizes the results of Bondal and Van den Bergh [BVdB03, Theorem 3.1.1] for schemes and Toën [Toë12, Corollary 5.2] for Deligne–Mumford stacks admitting coarse moduli spaces (i.e., X has finite inertia). While we can show that every compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ is a perfect complex (Lemma 4.4), a subtlety in Theorem A is that the converse does not always hold in positive characteristic. Indeed, if X is not tame, then the structure sheaf is perfect but not compact.

Theorem A, as well as the theory developed to establish it, have been used to classify the thick tensor ideals in $\mathsf{D}_{\mathrm{qc}}(X)^c$ [Hal16] (generalizing work of [Kri09, DM12]), to resolve the telescope conjecture for algebraic stacks [HR17] (extending [Ant14]), and for results on dg-enhancements [CS16, BLS16].

Extending Theorem A to certain stacks with infinite stabilizer groups is our second main result. We briefly recall a notion from [Ryd15, § 2]: an algebraic stack X is of *s-global type* if étale-locally it is the quotient of a quasi-affine scheme by GL_N for some N.

Theorem B. *Let X be an algebraic stack of s-global type. If X is of equicharacteristic zero (i.e., it is a $\mathbb{Q}$-stack), then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a countable set of perfect complexes. Moreover, for every quasi-compact open subset $U \subseteq X$, there exists a compact perfect complex with support exactly $X\backslash U$.*

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Stacks of s-global type are frequently encountered in practice. Sumihiro's theorem [Sum74] and its recent generalization by Brion [Bri15] show that many quotient stacks are of s-global type (Proposition 9.1). Thus, we have the following corollary (see Corollary 9.2 for an amplification).

Corollary. *Let X be a variety over a field k of characteristic zero. Let G be an affine algebraic k-group acting on X. If X is either (a) normal or (b) seminormal and quasi-projective, then $\mathsf{D}(\mathsf{QCoh}_G(X)) = \mathsf{D}_{\mathrm{qc}}([X/G])$ is compactly generated. Moreover, for every G-invariant open subset $U \subseteq X$, there exists a perfect G-equivariant complex with support exactly $X \backslash U$.*

A key advantage of Theorem B over previous results (e.g., [BFN10, Corollary 3.22] and [Lie04, Proposition 2.2.4.13]) is its applicability to a much wider class of stacks. The main result of [AHR15] implies that algebraic stacks of finite type over a field with affine diagonal and linearly reductive stabilizers at closed points (e.g., stacks with a good moduli space) are of s-global type [AHR15, Theorem 2.25]. In general, it is not possible to generate $\mathsf{D}_{\mathrm{qc}}(X)$ by a single perfect complex (e.g., $X = B\mathbb{G}_m$).

Theorems A and B are both consequences of a general result that we now describe. Let β be a cardinal. Let X be an algebraic stack. We say that X satisfies the *β-Thomason condition* if:

(i) $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a set of cardinality $\leqslant \beta$; and
(ii) for every quasi-compact open subset $U \subseteq X$, there is a compact perfect complex supported on the complement.

We say that X satisfies the *Thomason condition* if it satisfies the β-Thomason condition for some β. Our main results are that a very large class of stacks satisfy the Thomason condition. In order to prove these results, however, we found it necessary to consider the following refinement of the Thomason condition.

We say that X is *β-crisp* if the β-Thomason condition is satisfied for every étale localization of X (Definition 8.1). If X is β-crisp and β is finite, then X is compactly generated by a single perfect complex. Hence, Theorems A and B are implied by the following result.

Theorem C. *Let $p\colon X' \to X$ be a morphism of quasi-compact and quasi-separated algebraic stacks that is representable, separated, quasi-finite, locally of finite presentation, and faithfully flat. If X' is β-crisp, then X is β-crisp.*

Theorem C is proved using the technique of quasi-finite flat dévissage for algebraic stacks, due to the second author [Ryd11], together with some descent results for compact generation. In §§ 5–6, these descent results are stated in great generality, for presheaves of triangulated categories, without requiring monoidal or linear structures. This allows us to establish compact generation in other contexts (see Theorem 6.9, § 9 and further below in the Introduction). Along the way, we will review and develop foundational material for unbounded derived categories on algebraic stacks.

We also wish to point out that for schemes the fppf and quasi-finite flat topologies coincide, but for algebraic stacks they differ. Moreover, compact generation is not fppf local for algebraic stacks. Indeed, $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ when k is of characteristic $p > 0$ has no compact objects besides 0; thus, it is not compactly generated, even though it is so fppf-locally [HNR14, Proposition 3.1]. In particular, Theorem C (as well as its generalizations to other contexts in this article), which is about quasi-finite local compact generation, can be viewed as the correct generalization of fppf local compact generation results to algebraic stacks.

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Azumaya algebras and the cohomological Brauer group

Our work is strongly influenced by Toën's excellent paper [Toë12] on derived Azumaya algebras and generators of twisted derived categories. In [Toë12], Toën showed that compact generation of certain linear categories on derived schemes is an fppf-local question. The salient example is the *derived category of twisted sheaves* $\mathsf{D}(\mathsf{QCoh}^\alpha(X))$, where the twisting is given by a Brauer class α of $H^2(X, \mathbb{G}_m)$. A compact generator of $\mathsf{D}(\mathsf{QCoh}^\alpha(X))$ gives rise to a *derived Azumaya algebra*: the endomorphism algebra of the generator [Toë12, Proposition 4.6]. More classically, a twisted vector bundle that is generating gives rise to an Azumaya algebra $\mathcal{A}$.

The Brauer group $\mathrm{Br}(X)$ classifies Azumaya algebras $\mathcal{A}$ up to Morita equivalence, that is, up to equivalence of the category of modules $\mathsf{Mod}(\mathcal{A})$. Moreover, $\mathsf{Mod}(\mathcal{A}) \simeq \mathsf{QCoh}^\alpha(X)$ for a unique element α in the cohomological Brauer group $\mathrm{Br}'(X) := H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$. Existence of twisted vector bundles thus answers the question of whether $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ is surjective.

Constructing twisted vector bundles, or equivalently Azumaya algebras, is difficult. Indeed, the question is not local as vector bundles rarely extend over open immersions. When X is affine, or the separated union of two affines, Gabber proved in his thesis that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ [Gab81] or, equivalently, that $\mathsf{D}(\mathsf{QCoh}^\alpha(X))$ is compactly generated by a twisted vector bundle [Lie04, Theorem 2.2.3.3]. The state of the art is also due to Gabber: twisted vector bundles exist if X is quasi-projective [deJ03].

Compact objects of the derived category are typically easier to construct as we may extend them over open immersions using Thomason's localization theorem (Corollary 3.13). With this technique, Lieblich proved that $\mathsf{D}(\mathsf{QCoh}^\alpha(X))$ is compactly generated when X is any quasi-compact and quasi-separated scheme [Lie04, Corollary 2.2.4.14]. Lieblich has also studied twisted vector bundles in great detail and obtained a number of arithmetic applications.

In § 9, we prove that compact generation is quasi-finite flat local for twisted derived categories. In particular, we prove that on a quasi-compact algebraic stack with quasi-finite and separated diagonal every twisted derived category has a compact generator (Example 9.3). We thus establish a derived analogue of $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for such stacks, extending the results of Toën [Toë12] and Antieau and Gepner [AG14].

Sheaves of linear categories on derived Deligne–Mumford stacks

Although we work with non-derived schemes and stacks, our methods are strong enough to deduce similar results for derived (and spectral) Deligne–Mumford stacks (Example 9.4). Indeed, if X is a derived Deligne–Mumford stack, then the small étale topos of X is equivalent to the small étale topos of the non-derived 0-truncation $\pi_0 X$. Thus, (local) compact generation of a presheaf of triangulated categories on X can be studied on $\pi_0 X$.

Sometimes results for stacks can be deduced from schemes using a similar approach: if $\pi\colon X \to X_{\mathrm{cms}}$ is a coarse moduli space, then a presheaf $\mathcal{T}$ of triangulated categories on X induces a presheaf $\pi_*\mathcal{T}$ of triangulated categories on X_{cms}. If $\pi_*\mathcal{T}$ is locally compactly generated, then it is enough to show that compact generation is local on X_{cms} to deduce compact generation of $\mathcal{T}(X)$. This is how Toën extended his result to Deligne–Mumford stacks admitting a coarse moduli scheme [Toë12, Corollary 5.2].

Perfect and compact objects

As we already have mentioned, some care has to be taken since perfect objects are not necessarily compact. The perfect objects are the *locally compact* objects or, equivalently, the dualizable objects. If X is a quasi-compact and quasi-separated algebraic stack, then the following conditions are equivalent (Remark 4.6):

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– every perfect object of $\mathsf{D}_{\mathrm{qc}}(X)$ is compact;
– the structure sheaf $\mathcal{O}_X$ is compact;
– there exists an integer d_0 such that for all quasi-coherent sheaves M on X, the cohomology groups $H^d(X, M)$ vanish for all $d > d_0$; and
– the derived global section functor $\mathsf{R}\Gamma\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{Ab})$ commutes with small coproducts.

We say that a stack is *concentrated* when it satisfies the conditions above.

In [HR15, Theorem B], we give a complete list of the group schemes G/k such that BG is concentrated: every linear group and certain non-affine groups in characteristic zero but only the linearly reductive groups in positive characteristic. Note that [HR15, Theorem A] also gives many examples of classifying stacks that are not concentrated, yet compactly generated.

Drinfeld and Gaitsgory have proved that noetherian algebraic stacks with affine stabilizer groups in characteristic zero are concentrated [DG13, Theorem 1.4.2]. This is generalized in [HR15, Theorem C] to positive characteristic. In particular, a stack with finite stabilizers is concentrated if and only if it is tame.

Perfect stacks

Ben-Zvi *et al.* introduced the notion of a *perfect* (derived) stack in [BFN10]. In our context, an algebraic stack X is perfect if and only if it has affine diagonal, it is concentrated, and its derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated [BFN10, Proposition 3.9]. A direct consequence of our main theorems and [HR15, Theorem C] is that the following classes of algebraic stacks are perfect:

(i) quasi-compact tame Deligne–Mumford stacks with affine diagonal; and
(ii) $\mathbb{Q}$-stacks of s-global type with affine diagonal.

The affine diagonal assumption is needed only because it is required in the definition of a perfect stack. It is useful though: if X is perfect, then $\mathsf{D}(\mathsf{QCoh}(X)) = \mathsf{D}_{\mathrm{qc}}(X)$ by [HNR14].

In the terminology of Lurie [Lur11a, Definition 8.14], an algebraic stack is perfect if it has quasi-affine diagonal, is concentrated, and $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated. Thus, in Lurie's terminology, we have shown that:

(i) quasi-compact tame Deligne–Mumford stacks with quasi-compact and separated diagonals; and
(ii) $\mathbb{Q}$-stacks of s-global type

are perfect.

Coherence

Compact generation is extremely useful and we will illustrate this with a simple application; also the origin of this paper. Let A be a commutative ring and $\mathsf{Mod}(A)$ the category of A-modules. A functor $F\colon \mathsf{Mod}(A) \to \mathsf{Ab}$ is *coherent* if there exists a homomorphism of A-modules $M_1 \to M_2$ together with isomorphisms

$$F(N) \cong \operatorname{coker}(\operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N))$$

natural in N. This definition is due to Auslander [Aus66], who initiated the study of coherent functors. Hartshorne [Har98] studied in detail coherent functors when A is noetherian and M_1 and M_2 are finitely generated and obtained some very nice applications to classical algebraic geometry. For background material on coherent functors, we refer the reader to Hartshorne's

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article. Recently, the first author has used coherent functors to prove 'cohomology and base change' for algebraic stacks [Hal14] and to give a new criterion for algebraicity of a stack [Hal17].

Using the compact generation results of this article, we can give a straightforward proof of the following theorem (combine Theorem A with Corollary 4.16).

Theorem D. *Let A be a noetherian ring and let $\pi\colon X \to \operatorname{Spec} A$ be a proper morphism of algebraic stacks with finite diagonal. If $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, then the functor*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

is coherent.

Theorem D generalizes a result of the first author for algebraic spaces [Hal14, Theorem E], which was proved using a completely different argument. The first author has also proved a non-noetherian and infinite-stabilizer variant of Theorem D at the expense of assuming that $\mathcal{G}$ has flat cohomology sheaves over S [Hal14, Theorem C].

Related results

The first proof that $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated when X is a quasi-compact separated scheme appears to be due to Neeman [Nee96, Proposition 2.5] although he attributed the ideas to Thomason [TT90]. Bondal and Van den Bergh [BVdB03, Theorem 3.1.1] adapted the proof to deal with quasi-separated schemes and noted that there is a single compact generator. Lipman and Neeman further refined the result by giving an effective bound on the existence of maps from the compact generator [LN07, Theorem 4.2]. As noted by Ben-Zvi *et al.*, the proof of Bondal and Van den Bergh readily extends to derived schemes [BFN10, Proposition 3.19].

In [Lur11a, Theorem 6.1] and [Lur11b, Theorem 1.5.10], Lurie proved that compact generation is étale local on $\mathbb{E}_\infty$-algebras and on spectral algebraic spaces for quasi-coherent stacks (sheaves of linear ∞-categories). Lurie used *scallop decompositions*, which are a special type of étale neighborhoods (or Nisnevich squares). Unfortunately, scallop decompositions do not exist for algebraic stacks. This is what necessitates our stronger inductive assumption, β-crispness, for our local-global principle, Theorem C. Indeed, to apply Thomason's localization theorem it is necessary to establish the existence of compact objects with prescribed support. On affine schemes (which appear in the scallop decompositions) this is done using Koszul complexes, cf. Bökstedt and Neeman [BN93, Proposition 6.1]. This is the basis for our induction and also used in all previous proofs, e.g., [Toë12, Lemma 4.10] and [AG14, Proposition 6.9].

Drinfeld and Gaitsgory [DG13, Theorem 8.1.1] proved that on an algebraic stack of finite type over a field of characteristic zero with affine stabilizers, the derived category of D-modules is compactly generated. They remarked that compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$ is much subtler and open in general [DG13, 0.3.3].

Antieau [Ant14] has considered local-global results for the telescope conjecture. Some of these are generalized in [HR17].

Krishna [Kri09] has considered the K-theory and G-theory for tame Deligne–Mumford stacks with the resolution property admitting projective coarse moduli schemes (i.e., projective stacks).

Future extensions

In [LN07, Theorem 4.1], Lipman and Neeman proved that pseudo-coherent complexes can be approximated arbitrarily well by perfect complexes on a quasi-compact and quasi-separated scheme. Local approximability by perfect complexes is essentially the definition of pseudo-coherence, so this is a local-global result in the style of Theorem C. This result has been extended

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to algebraic spaces in [Sta, 08HH] and we expect that it can be extended to stacks with quasi-finite diagonal using the methods of this paper amplified with t-structures. Similarly, we expect that there is an effective bound on the compact generator in Theorem A as in [LN07, Theorem 4.2].

Contents of this paper

In §§ 1–2, we recall and develop some generalities on unbounded derived categories of quasi-coherent sheaves on stacks and concentrated morphisms; working in the unbounded derived category is absolutely essential for this range of mathematics. Unfortunately, some important foundational notions, such as concentrated morphisms, had not been considered in the literature before.

In § 3, we recall the concept of compact objects and Thomason's localization theorem for triangulated categories.

In § 4, we address fundamental results on perfect and compact objects in the derived categories of quasi-coherent sheaves on algebraic stacks. Using this, we establish a general projection formula for stacks, tor-independent base change, and finite flat duality. We also prove Theorem D assuming Theorem A.

In §§ 5–6, we introduce presheaves of triangulated categories and Mayer–Vietoris triangles. We also prove our main result on descent of compact generation (Theorem 6.9).

In § 7, we introduce the β-resolution property, which gives a convenient method to keep track of the number of vector bundles needed for generating the derived category of a stack with the resolution property.

In § 8, we introduce compact generation with supports and β-crispness and relate these to Koszul complexes.

In § 9, we prove the main theorems.

Notations and assumptions

For an abelian category $\mathcal{A}$, denote by $\mathsf{D}(\mathcal{A})$ its unbounded derived category. For a complex $M \in \mathsf{D}(\mathcal{A})$, denote its ith cohomology group by $\mathcal{H}^i(M)$. For a sheaf of rings A on a topos E, denote by $\mathsf{Mod}(A)$ (respectively $\mathsf{QCoh}(A)$) the category of A-modules (respectively the category of quasi-coherent A-modules). If the sheaf of rings A on the topos E is implicit, it will be convenient to denote $\mathsf{Mod}(A)$ as $\mathsf{Mod}(E)$ and $\mathsf{D}(\mathsf{Mod}(A))$ as $\mathsf{D}(E)$.

For algebraic stacks, we adopt the conventions of the *Stacks Project* [Sta]. This means that algebraic stacks are stacks over the big fppf site of some scheme, admitting a smooth, representable, surjective morphism from a scheme (note that there are no separation hypotheses here). A morphism of algebraic stacks is *quasi-separated* if its diagonal and double diagonal are represented by quasi-compact morphisms of algebraic spaces.

For a scheme X, denote its underlying topological space by $|X|$. For a scheme X and a point $x \in |X|$, denote by $\kappa(x)$ the residue field at x.

Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks. Then, for any other 1-morphism of algebraic stacks $g\colon Z \to Y$, we denote by $f_Z\colon X_Z \to Z$ the pullback of f by g.

1. Quasi-coherent sheaves on algebraic stacks

In this section we review derived categories of quasi-coherent sheaves on algebraic stacks. For generalities on unbounded derived categories on ringed topoi, we refer the reader to [KS06, § 18.6]. In [KS06, § 18.6], a morphism of ringed topoi is assumed to have a left-exact inverse image; we will not make this assumption, but instead indicate explicitly when it does and does not hold.

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Let X be an algebraic stack. Let $\mathsf{Mod}(X)$ (respectively $\mathsf{QCoh}(X)$) denote the abelian category of $\mathcal{O}_X$-modules (respectively quasi-coherent $\mathcal{O}_X$-modules) on the lisse-étale topos of X [LM00, 12.1]. Let $\mathsf{D}(X)$ (respectively $\mathsf{D}_{\mathrm{qc}}(X)$) denote the unbounded derived category of $\mathsf{Mod}(X)$ (respectively the full subcategory of $\mathsf{D}(X)$ with cohomology in $\mathsf{QCoh}(X)$). Superscripts such as $+$, $-$, $\geqslant n$, and b decorating $\mathsf{D}(X)$ and $\mathsf{D}_{\mathrm{qc}}(X)$ are to be interpreted as usual.

If X is a Deligne–Mumford stack (e.g., a scheme or an algebraic space), then there is an associated small étale topos, which we denote as $X_{\text{ét}}$. There is a natural morphism of ringed topoi $\mathrm{res}_X\colon X_{\text{lis-ét}} \to X_{\text{ét}}$. Let $\mathsf{Mod}(X_{\text{ét}})$ (respectively $\mathsf{QCoh}(X_{\text{ét}})$) denote the abelian category of $\mathcal{O}_{X_{\text{ét}}}$-modules (respectively quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules). The restriction of $(\mathrm{res}_X)_*\colon \mathsf{Mod}(X) \to \mathsf{Mod}(X_{\text{ét}})$ to $\mathsf{QCoh}(X)$ is fully faithful with essential image $\mathsf{QCoh}(X_{\text{ét}})$ [LM00, Proposition 13.2.3]. Let $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ denote the triangulated category $\mathsf{D}_{\mathsf{QCoh}(X_{\text{ét}})}(\mathsf{Mod}(X_{\text{ét}}))$. Then the natural functor $\mathsf{R}(\mathrm{res}_X)_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ is an equivalence of categories [LM00, Proposition 12.10.1]. If X is a scheme, then the corresponding statement for the Zariski topos also holds [LM00, Lemme 13.1.5].

1.1 Hypercoverings and simplicial sites

We now recall the relationship between the unbounded derived categories of quasi-coherent sheaves on an algebraic stack and those on a smooth hypercovering (i.e., cohomological descent). Our approach follows [Ols07] and [LO08]. Let X be an algebraic stack and let $p_\bullet\colon U_\bullet \to X$ be a smooth hypercovering by algebraic spaces. Typically, we will take $U_\bullet$ to be the 0-coskeleton associated to a smooth covering $p_0\colon U_0 \to X$, where U_0 is an algebraic space. In plainer language, U_n is the $(n+1)$th fiber product of U_0 over X and the simplicial structure (i.e., face and degeneracy maps) comes from the various projections and diagonals between the U_n as n varies.

The simplicial algebraic space $U_\bullet$ gives rise to two semi-simplicial topoi: $U^+_{\bullet,\text{lis-ét}}$ and $U^+_{\bullet,\text{ét}}$. The semi-simplicial topos $U^+_{\bullet,\text{ét}}$ is formed as follows: for each integer $n \geqslant 0$ there is the étale topos $U_{n,\text{ét}}$ and for each injective map $\delta\colon [n] = \{0,\dots,n\} \to [m] = \{0,\dots,m\}$ there is a morphism of topoi $\delta\colon U_{m,\text{ét}} \to U_{n,\text{ét}}$. A sheaf $F_\bullet$ on $U^+_{\bullet,\text{ét}}$ is a sheaf F_n on each $U_{n,\text{ét}}$ together with transition maps $\delta^{-1}F_n \to F_m$ for each injective map $\delta\colon [n] \to [m]$ that are compatible with composition; the sheaf $F_\bullet$ is *cartesian* if the transition maps are always isomorphisms.

The topos $U^+_{\bullet,\text{ét}}$ is naturally ringed by the flat sheaf $\mathcal{O}^+_{U_{\bullet,\text{ét}}}$. Here flat means that the transition maps $\delta^{-1}\mathcal{O}_{U_n,\text{ét}} \to \mathcal{O}_{U_m,\text{ét}}$ are flat. Let $\mathsf{Mod}(U^+_{\bullet,\text{ét}})$ denote the associated category of modules and $\mathsf{Mod}_{\mathrm{cart}}(U^+_{\bullet,\text{ét}})$ the subcategory of cartesian sheaves. Here cartesian means that the transition maps $\delta^*F_n = \delta^{-1}F_n \otimes_{\delta^{-1}\mathcal{O}_{U_{n,\text{ét}}}} \mathcal{O}_{U_{m,\text{ét}}} \to F_m$ are isomorphisms.

An $\mathcal{O}^+_{U_{\bullet,\text{ét}}}$-module is *quasi-coherent* if it is cartesian and its restriction to each $U_{n,\text{ét}}$ is quasi-coherent. Let $\mathsf{QCoh}(U^+_{\bullet,\text{ét}})$ denote the category of quasi-coherent sheaves. Let $\mathsf{D}(U^+_{\bullet,\text{ét}})$ be the unbounded derived category of $\mathsf{Mod}(U^+_{\bullet,\text{ét}})$, let $\mathsf{D}_{\mathrm{cart}}(U^+_{\bullet,\text{ét}})$ denote the subcategory whose objects have cartesian cohomology sheaves, and let $\mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ denote the subcategory with quasi-coherent cohomology sheaves. The semi-simplicial topos $U^+_{\bullet,\text{lis-ét}}$ and its various module categories are defined similarly.

Thus, there are natural morphisms of ringed topoi:

$$X_{\text{lis-ét}} \xleftarrow{p^+_{\bullet,\text{lis-ét}}} U^+_{\bullet,\text{lis-ét}} \xrightarrow{\mathrm{res}_{U_\bullet}} U^+_{\bullet,\text{ét}}. \tag{1.1}$$

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One way of phrasing smooth descent of quasi-coherent sheaves is that these morphisms of topoi induce equivalences of abelian categories:

$$\mathsf{QCoh}(X) \xleftarrow{(p^+_{\bullet,\text{lis-ét}})_*} \mathsf{QCoh}(U^+_{\bullet,\text{lis-ét}}) \xrightarrow{(\mathrm{res}_{U_\bullet})_*} \mathsf{QCoh}(U^+_{\bullet,\text{ét}}).$$

In [LO08, Example 2.2.5], it is shown that this can be improved to *unbounded cohomological descent*, that is, these morphisms of topoi induce equivalences of triangulated categories:

$$\mathsf{D}_{\mathrm{qc}}(X) \xleftarrow{\mathsf{R}(p^+_{\bullet,\text{lis-ét}})_*} \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{lis-ét}}) \xrightarrow{\mathsf{R}(\mathrm{res}_{U_\bullet})_*} \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}). \tag{1.2}$$

The morphisms $p^+_{\bullet,\text{lis-ét}}$ and $\mathrm{res}_{U_\bullet}$ have left-exact inverse image functors.

1.2 Operations in unbounded categories of modules

We now record for future reference some useful formulas [KS06, §§ 18.4 and 18.6]. If $\mathcal{M}$ and $\mathcal{N} \in \mathsf{D}(X)$, then there are

$$\begin{aligned} \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} &\in \mathsf{D}(X) \quad \text{(the derived tensor product)},\\ \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) &\in \mathsf{D}(X) \quad \text{(the derived sheaf Hom functor)},\\ \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) &\in \mathsf{D}(\mathsf{Ab}) \quad \text{(the derived global Hom functor)}. \end{aligned}$$

If in addition $\mathcal{P} \in \mathsf{D}(X)$, then we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})) \tag{1.3}$$

as well as a functorial quasi-isomorphism

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})). \tag{1.4}$$

Letting $\mathsf{R}\Gamma(X,-) = \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{O}_X,-)$, there is also a natural quasi-isomorphism

$$\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \simeq \mathsf{R}\Gamma\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}). \tag{1.5}$$

If $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathsf{D}_{\mathrm{qc}}(X)$, then $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$. Indeed, using homotopy colimits and triangles we reduce to the case where $\mathcal{M}$ and $\mathcal{N}$ are quasi-coherent sheaves. Then it follows from the definition of quasi-coherence [LM00, Définition 13.2.2]. Since the category $\mathsf{D}_{\mathrm{qc}}(X)$ is well generated [HNR14, Theorem B.1] and the functor $- \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{M} \colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X)$ preserves small coproducts, it admits a right adjoint [Nee01b, Theorem 8.4.4]

$$\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{M},-) \colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X).$$

In fact, if

$$\mathcal{Q}_X \colon \mathsf{D}(X) \to \mathsf{D}_{\mathrm{qc}}(X)$$

is the right adjoint to the inclusion $\mathsf{D}_{\mathrm{qc}}(X) \subseteq \mathsf{D}(X)$, which exists for the same reasons as above, then

$$\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{M},-) \simeq \mathcal{Q}_X(\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},-)).$$

Note that while the formation of $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},-)$ is smooth local on X, this is not true in general for $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{M},-)$. It is true, however, if $\mathcal{M}$ is perfect (Lemma 4.3(ii)).

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1.3 Direct and inverse images

For a morphism of algebraic stacks $f\colon X \to Y$, the induced morphism of ringed topoi $f_{\text{lis-ét}}\colon X_{\text{lis-ét}} \to Y_{\text{lis-ét}}$ does not necessarily have a left-exact inverse image functor [Beh03, 5.3.12]. Thus, the construction of the correct derived functors of $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is somewhat subtle. There are currently two approaches to constructing these functors. The first, due to Olsson [Ols07] and Laszlo and Olsson [LO08], uses cohomological descent. The other approach appears in the Stacks Project [Sta, Tag 07BD]. In this article, we will employ the approach of Olsson and Laszlo–Olsson, which we now briefly recall.

Let $f\colon X \to Y$ be a morphism of algebraic stacks. Let $q\colon V \to Y$ be a smooth surjection from an algebraic space. Let $U \to X \times_Y V$ be another smooth surjection from an algebraic space. Let $\tilde{f}\colon U \to V$ be the resulting morphism of algebraic spaces and let $p\colon U \to X$ be the resulting smooth covering. By (1.1), there is an induced 2-commutative diagram of ringed topoi as follows.

$$\begin{array}{ccccc} X & \xleftarrow{p^+_{\bullet,\text{lis-ét}}} & U^+_{\bullet,\text{lis-ét}} & \xrightarrow{\text{res}_{U_\bullet}} & U^+_{\bullet,\text{ét}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}^+_{\bullet,\text{lis-ét}}} & & \downarrow{\scriptstyle \tilde{f}^+_{\bullet,\text{ét}}} \\ Y & \xleftarrow[q^+_{\bullet,\text{lis-ét}}]{} & V^+_{\bullet,\text{lis-ét}} & \xrightarrow[\text{res}_{V_\bullet}]{} & V^+_{\bullet,\text{ét}} \end{array} \tag{1.6}$$

The 2-commutativity of the diagram above induces natural transformations:

$$\mathsf{R}(f_{\text{lis-ét}})_* \Rightarrow \mathsf{R}(q^+_{\bullet,\text{lis-ét}})_*\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^* \tag{1.7}$$

$$\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_* \Rightarrow \mathsf{R}(\text{res}_{V_\bullet})_*\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*\mathsf{L}(\text{res}_{U_\bullet})^*, \tag{1.8}$$

which are natural isomorphisms for those complexes with quasi-coherent cohomology that are sent to complexes with quasi-coherent cohomology by $\mathsf{R}(\tilde{f}^+_{\bullet,\text{lis-ét}})_*$ or $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*$.

Remark 1.1. Note, however, that if $f\colon X \to Y$ is not representable, then $\mathsf{R}(f_{\text{lis-ét}})_*$ does not, in general, send $\mathsf{D}_{\text{qc}}(X)$ to $\mathsf{D}_{\text{qc}}(Y)$; even if f is proper and étale and X and Y are smooth Deligne–Mumford stacks [Sta, 07DC]. The problem is that quasi-compact and quasi-separated morphisms of algebraic stacks can have unbounded cohomological dimension, which is in contrast to the cases of schemes and algebraic spaces [Sta, 073G]. In the next section we will clarify this with the concept of a *concentrated* morphism.

Another crucial observation here is that the morphism of topoi $\tilde{f}^+_{\bullet,\text{ét}}$ has a left-exact inverse image functor. The general theory now gives rise to an unbounded derived functor $\mathsf{L}(\tilde{f}^+_{\bullet,\text{ét}})^*\colon \mathsf{D}(V^+_{\bullet,\text{ét}}) \to \mathsf{D}(U^+_{\bullet,\text{ét}})$, which is left adjoint to $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\colon \mathsf{D}(U^+_{\bullet,\text{ét}}) \to \mathsf{D}(V^+_{\bullet,\text{ét}})$. The functor $\mathsf{L}(\tilde{f}^+_{\bullet,\text{ét}})^*$ is easily verified to preserve small coproducts and complexes with quasi-coherent cohomology. Using the equivalences of (1.2), we may now define a functor $\mathsf{L}f^*_{\text{qc}}\colon \mathsf{D}_{\text{qc}}(Y) \to \mathsf{D}_{\text{qc}}(X)$ such that $\mathcal{H}^0(\mathsf{L}f^*_{\text{qc}}\mathcal{M}[0]) \cong f^*\mathcal{M}$ whenever $\mathcal{M} \in \mathsf{QCoh}(Y)$. It is readily verified that the functor $\mathsf{L}f^*_{\text{qc}}$ is unique up to canonical natural isomorphism, that is, it does not depend on the choice of charts U and V. If $f\colon X \to Y$ is flat, then for all integers q and all $\mathcal{M} \in \mathsf{D}_{\text{qc}}(Y)$ there is a natural isomorphism

$$f^*\mathcal{H}^q(\mathcal{M}) \cong \mathcal{H}^q(\mathsf{L}f^*_{\text{qc}}\mathcal{M}). \tag{1.9}$$

Since the category $\mathsf{D}_{\text{qc}}(Y)$ is well generated [HNR14, Theorem B.1] and the functor $\mathsf{L}f^*_{\text{qc}}$ preserves small coproducts, it admits a right adjoint [Nee01b, Theorem 8.4.4]

$$\mathsf{R}(f_{\text{qc}})_*\colon \mathsf{D}_{\text{qc}}(X) \to \mathsf{D}_{\text{qc}}(Y).$$

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The functor above is closely related to the functors we have already seen. Indeed, since $\mathsf{L}f_{\mathrm{qc}}^{*}\mathcal{O}_Y[0] \simeq \mathcal{O}_X[0]$, it follows that if $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, then

$$\mathsf{R}\Gamma(Y, \mathsf{R}(f_{\mathrm{qc}})_{*}\mathcal{M}) \simeq \mathsf{R}\Gamma(X, \mathcal{M}). \tag{1.10}$$

We now describe $\mathsf{R}(f_{\mathrm{qc}})_{*}$ locally. Let

$$\mathcal{Q}_{V_{\bullet,\text{ét}}^{+}} : \mathsf{D}(V_{\bullet,\text{ét}}^{+}) \to \mathsf{D}_{\mathrm{qc}}(V_{\bullet,\text{ét}}^{+})$$

be a right adjoint to the natural inclusion functor $\mathsf{D}_{\mathrm{qc}}(V_{\bullet,\text{ét}}^{+}) \to \mathsf{D}(V_{\bullet,\text{ét}}^{+})$, which exists by [Nee01b, Theorem 8.4.4]. A straightforward calculation, utilizing the equivalences (1.2), induces a natural isomorphism of functors:

$$\mathsf{R}(f_{\mathrm{qc}})_{*} \simeq \mathsf{R}(q_{\bullet,\text{lis-ét}}^{+})_{*}\mathsf{L}(\mathrm{res}_{V_\bullet})^{*}\mathcal{Q}_{V_{\bullet,\text{ét}}^{+}}\mathsf{R}(\tilde{f}_{\bullet,\text{ét}}^{+})_{*}\mathsf{R}(\mathrm{res}_{U_\bullet})_{*}\mathsf{L}(p_{\bullet,\text{lis-ét}}^{+})^{*}. \tag{1.11}$$

The following lemma clarifies the situation somewhat.

LEMMA 1.2. *If $f\colon X \to Y$ is a morphism of algebraic stacks that is quasi-compact and quasi-separated, then the following hold.*

(i) *The restriction of $\mathsf{R}(f_{\text{lis-ét}})_{*}$ to $\mathsf{D}_{\mathrm{qc}}^{+}(X)$ factors through $\mathsf{D}_{\mathrm{qc}}^{+}(Y)$.*

(ii) *The restrictions of $\mathsf{R}(f_{\text{lis-ét}})_{*}$ and $\mathsf{R}(f_{\mathrm{qc}})_{*}$ to $\mathsf{D}_{\mathrm{qc}}^{+}(X)$ are isomorphic.*

(iii) *For each integer d, the restriction of the functor $\mathsf{R}(f_{\mathrm{qc}})_{*}$ to $\mathsf{D}_{\mathrm{qc}}^{[d,\infty)}(X)$ preserves direct limits (in particular, small coproducts).*

(iv) *Consider the following 2-cartesian diagram of algebraic stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If g is flat, then the base-change transformation

$$\mathsf{L}g_{\mathrm{qc}}^{*}\mathsf{R}(f_{\mathrm{qc}})_{*} \Rightarrow \mathsf{R}(f'_{\mathrm{qc}})_{*}\mathsf{L}(g')_{\mathrm{qc}}^{*}$$

is an isomorphism upon restriction to $\mathsf{D}_{\mathrm{qc}}^{+}(X)$.

Proof. Claim (i) is [Ols07, Lemma 6.20]. Claim (ii) follows from cohomological descent (1.2), claim (i) and equations (1.7), (1.8), and (1.11).

For (iii), by (ii), we may replace $\mathsf{R}(f_{\mathrm{qc}})_{*}$ by $\mathsf{R}(f_{\text{lis-ét}})_{*}$. The hypercohomology spectral sequence

$$\mathsf{R}^{r}(f_{\text{lis-ét}})_{*}\mathcal{H}^{s}(\mathcal{M}) \Rightarrow \mathsf{R}^{r+s}(f_{\text{lis-ét}})_{*}\mathcal{M} \tag{1.12}$$

now applies and it is thus sufficient to prove that the higher pushforwards

$$\mathsf{R}^{r}(f_{\text{lis-ét}})_{*}\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$$

preserve direct limits for every integer $r \geqslant 0$. This is local on Y for the smooth topology, so we may assume that Y is an affine scheme. Thus, it suffices to prove that the cohomology functors $H^{r}(X_{\text{lis-ét}}, -)\colon \mathsf{QCoh}(X) \to \mathsf{Ab}$ preserve direct limits for every integer $r \geqslant 0$. Since X is quasi-compact and quasi-separated, this is well known (e.g., [Sta, 0739]).

The base-change transformation of (iv) exists by functoriality of the adjoints. Applying (ii), we may replace $(f_{\mathrm{qc}})_{*}$ and $(f'_{\mathrm{qc}})_{*}$ by $(f_{\text{lis-ét}})_{*}$ and $(f'_{\text{lis-ét}})_{*}$, respectively. The statement is now local on Y and Y' for the smooth topology, so we may assume that both Y and Y' are affine schemes. Small modifications to the argument of [Sta, 073K] complete the proof. □

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In § 2 we describe a class of morphisms for which the conclusions of Lemma 1.2 remain valid in the unbounded derived category.

1.4 Comparison with definitions in derived algebraic geometry

We will now compare the derived category of quasi-coherent sheaves $\mathsf{D}_{\mathrm{qc}}(X)$ that we are using in this paper with derived categories of quasi-coherent sheaves as defined in derived algebraic geometry. This is not used in the remainder of the paper (but see Example 9.4).

First recall that if $\mathcal{A}$ is an abelian category, then the derived category $\mathsf{D}(\mathcal{A})$ is the homotopy category of a natural stable ∞-category $\mathcal{D}(\mathcal{A})$ [Lur16a, Definition 1.3.5.8]. If in addition $\mathcal{C} \subseteq \mathcal{A}$ is a weak Serre subcategory, then we can consider the full ∞-subcategory $\mathcal{D}_{\mathcal{C}}(\mathcal{A})$ of objects with cohomology in $\mathcal{C}$ and this has homotopy category $\mathsf{D}_{\mathcal{C}}(\mathcal{A})$. We thus define $\mathcal{D}_{\mathrm{qc}}(X) := \mathcal{D}_{\mathsf{QCoh}(X)}(\mathsf{Mod}(X))$, which has homotopy category $\mathsf{D}_{\mathrm{qc}}(X)$. If X is Deligne–Mumford, we also define $\mathcal{D}_{\mathrm{qc}}(X_{\text{ét}}) := \mathcal{D}_{\mathsf{QCoh}(X_{\text{ét}})}(\mathsf{Mod}(X_{\text{ét}}))$, which is equivalent to $\mathcal{D}_{\mathrm{qc}}(X)$ [LM00, Proposition 12.10.1].

Proposition 1.3 (cf. [DG13, Remark 1.2.3]). *Let X be an algebraic stack. Then*

$$\mathcal{D}_{\mathrm{qc}}(X) = \varprojlim_{\operatorname{Spec} A \to X} \mathcal{D}(\mathsf{Mod}(A)),$$

where the limit is taken in the ∞-category of ∞-categories and is over (i) the category of all affine schemes over X, (ii) the full subcategory of those smooth over X, or (iii) the subcategory of those smooth over X with only smooth X-morphisms $\operatorname{Spec} A \to \operatorname{Spec} B$.

We denote the limit of the right-hand side going over all morphisms by $\mathcal{QC}\mathsf{oh}(X)$. This is a stable ∞-category which is left- and right-complete, and has a t-structure with heart the abelian category $\mathsf{QCoh}(X)$ [Lur16b, Corollary 9.1.3.2 and Remark 9.1.3.3]. The ∞-category $\mathcal{QC}\mathsf{oh}(X)$ also makes sense for any contravariant functor X from affine schemes to groupoids and can be adapted to variants in derived algebraic geometry. This is how derived categories of quasi-coherent sheaves usually are defined in derived algebraic geometry; cf. [BFN10, § 3.1], [DG13, § 1.2], [Lur16b, § 6.2.2], and [GR17, I.3, § 1.1].

The category $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete [HNR14, Theorem B.1]. We do not use this fact in the proof and so we obtain an independent proof of the left-completeness of both $\mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X)$.

Proof of Proposition 1.3. The limit restricted to smooth morphisms or to smooth morphisms with smooth maps between them is also equivalent to $\mathcal{QC}\mathsf{oh}(X)$ [GR17, I.3, § 1.4.2]. Moreover, if $U \to X$ is a smooth presentation and $U^+_\bullet$ is the corresponding semi-simplicial algebraic space, then restricting the limit to the diagram $U^+_\bullet$ gives the same limit (use that X is a stack and that $\Delta^+ \subseteq \Delta$ is right cofinal; cf. [Lur16b, Proposition 6.2.3.1 and proof of Proposition 9.1.3.1] or [GR17, I.3, Corollary 1.3.11]). We may also instead take the limit over the site $U^+_{\bullet,\text{ét}}$ since $U^+_\bullet \colon \Delta^+ \to U^+_{\bullet,\text{ét}}$ is right cofinal.

We have a sequence of maps between ∞-categories

$$\begin{aligned}\mathcal{D}_{\mathrm{qc}}(X) \xrightarrow{\alpha} \mathcal{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}) \xrightarrow{\beta} \varprojlim_{V \in U^+_{\bullet,\text{ét}}} \mathcal{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}/V) \\ \xrightarrow{\gamma} \varprojlim_{V \in U^+_{\bullet,\text{ét}}} \mathcal{D}_{\mathrm{qc}}(V) \xleftarrow{\delta} \mathcal{QC}\mathsf{oh}(X).\end{aligned}$$

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That α is an equivalence follows by unbounded cohomological descent (1.2). That β is an equivalence follows from Lemma 1.6 applied to the semi-simplicial étale site $U^+_{\bullet,\text{ét}}$ since having quasi-coherent cohomology can be verified locally. The map γ comes from the morphism of topoi $\epsilon\colon U^+_{\bullet,\text{ét}}/V \to V_{\text{ét}}$. Since ϵ_* (restriction) is exact and ϵ^* is exact and fully faithful with essential image the cartesian modules, we obtain equivalences between cartesian modules and between derived categories of modules with cartesian cohomology sheaves; cf. [LM00, Proposition 12.10.1]. This shows that γ is an equivalence. We saw that δ was an equivalence above. □

Remark 1.4. The maps β and γ are also equivalences if we replace quasi-coherent cohomology with cartesian cohomology.

Remark 1.5. If in addition X has affine diagonal, or is noetherian and affine-pointed, then the natural functor $\mathsf{D}^+(\mathsf{QCoh}(X)) \to \mathsf{D}^+_{\text{qc}}(X)$ is an equivalence [HNR14, Appendix C]. It follows that $\mathcal{D}^+(\mathsf{QCoh}(X)) \to \mathcal{QC}\mathsf{oh}^+(X)$ is an equivalence, which can also be proven directly; cf. [DG13, Remark 1.2.10] and [GR17, I.3, Proposition 2.4.3]. Note that $\mathsf{D}(\mathsf{QCoh}(X))$ and $\mathcal{D}(\mathsf{QCoh}(X))$ are not always left-complete, e.g., when $X = B\mathbb{G}_a$ in positive characteristic [Nee11]. They are left-complete and hence coincide with $\mathsf{D}_{\text{qc}}(X)$ and $\mathcal{D}_{\text{qc}}(X)$, respectively, when $\mathsf{D}_{\text{qc}}(X)$ is compactly generated [HNR14, Theorem 1.2].

Lemma 1.6. *Let $(\mathcal{T}, \mathcal{O})$ be a ringed topos and let $(\mathcal{T}/U, \mathcal{O}|_U)$ denote the localized topos for any $U \in \mathcal{T}$. Then the assignment $U \mapsto \mathcal{D}(\mathsf{Mod}(\mathcal{O}|_U))$ is a sheaf, that is, there is a limit-preserving functor*

$$\begin{aligned} \mathcal{T}^\circ &\longrightarrow \mathsf{Cat}_\infty \\ U &\longmapsto \mathcal{D}(\mathsf{Mod}(\mathcal{O}|_U)). \end{aligned}$$

Proof. We identify a ringed topos with a $\mathbb{E}_\infty$-ringed ∞-topos by taking nerves. Such an ∞-topos is a 1-topos, that is, discrete, and $\mathcal{O}$ is discrete. We let $\mathcal{M}\mathsf{od}_{\mathcal{O}}$ denote the ∞-category of $\mathcal{O}$-module spectra. Its heart is the category of usual modules $\mathsf{Mod}(\mathcal{O})$ [Lur16b, Definition 2.1.0.1 and Remark 2.1.2.1]. By the universal property of derived categories, there is a functor $\mathcal{D}^+(\mathsf{Mod}(\mathcal{O}|_U)) \to \mathcal{M}\mathsf{od}_{\mathcal{O}|_U}$. This functor extends to a fully faithful functor $\mathcal{D}(\mathsf{Mod}(\mathcal{O}|_U)) \to \mathcal{M}\mathsf{od}_{\mathcal{O}|_U}$ whose essential image is the full subcategory of hypercomplete objects [Lur16b, Corollary 2.1.2.3].

By [Lur16b, Remark 2.1.0.5], there is a limit-preserving functor $U \mapsto \mathcal{M}\mathsf{od}_{\mathcal{O}|_U}$. Since being hypercomplete is a local property [Lur09, Remark 6.5.2.22], we obtain a limit-preserving functor $U \mapsto \mathcal{D}(\mathsf{Mod}(\mathcal{O}|_U))$. □

Remark 1.7 (Deligne–Mumford stacks). If X is a Deligne–Mumford stack, we can associate a spectral Deligne–Mumford stack $\mathfrak{X}$ to X. To $\mathfrak{X}$, one associates a stable ∞-category of quasi-coherent sheaves $\mathcal{QC}\mathsf{oh}(\mathfrak{X})$, a subcategory of $\mathcal{M}\mathsf{od}(\mathfrak{X}) = \mathcal{M}\mathsf{od}(\mathcal{O}_{X_{\text{ét}}})$. It is equivalent to $\mathcal{QC}\mathsf{oh}(X)$ as defined above [Lur16b, Proposition 6.2.4.1] and hence to $\mathcal{D}_{\text{qc}}(X)$. This can also be seen directly as follows [Lur16b, Corollary 2.2.6.2]. The ∞-category $\mathcal{D}(\mathsf{Mod}(\mathcal{O}_{X_{\text{ét}}}))$ can be identified with the full subcategory of hypercomplete objects of $\mathcal{M}\mathsf{od}(\mathfrak{X})$ [Lur16b, Corollary 2.1.2.3] and $\mathcal{QC}\mathsf{oh}(\mathfrak{X})$ can be identified with the full subcategory of hypercomplete objects with quasi-coherent homotopy sheaves [Lur16b, Proposition 2.2.6.1]. That is, $\mathcal{D}_{\text{qc}}(X_{\text{ét}}) = \mathcal{QC}\mathsf{oh}(\mathfrak{X})$.

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2. Concentrated morphisms of algebraic stacks

A morphism of schemes $f\colon X \to Y$ is concentrated if it is quasi-compact and quasi-separated [Lip09, § 3.9]. Concentrated morphisms of schemes are natural to consider when working with unbounded derived categories of quasi-coherent sheaves. Indeed, if f is concentrated, then:

(i) $\mathsf{R}(f_{\mathrm{qc}})_*$ coincides with the restriction of $\mathsf{R}(f_{\mathrm{Zar}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)$;
(ii) $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts; and
(iii) $\mathsf{R}(f_{\mathrm{qc}})_*$ is compatible with flat base change on Y.

Here, as before, $\mathsf{R}(f_{\mathrm{qc}})_*$ denotes the right adjoint to the unbounded derived functor $\mathsf{L}f_{\mathrm{qc}}^*\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$. In this section we isolate a class of morphisms of algebraic stacks, which we will also call concentrated, that enjoy the same properties.

Definition 2.1. Let $n \geqslant 0$ be an integer. A quasi-compact and quasi-separated morphism of algebraic stacks $f\colon X \to Y$ has *cohomological dimension* $\leqslant n$ if for all $i > n$ and all $\mathcal{M} \in \mathsf{QCoh}(X)$ we have that $\mathsf{R}^i(f_{\text{lis-ét}})_*\mathcal{M} = 0$ (by Lemma 1.2(ii), this is equivalent to $\mathsf{R}^i(f_{\mathrm{qc}})_*\mathcal{M} = 0$).

The next result is inspired by [Alp13, Proposition 3.9], where similar results are proven in the context of cohomologically affine morphisms. Note, however, that cohomologically affine morphisms are not quite the same as morphisms of cohomological dimension $\leqslant 0$ [Alp13, Remark 3.5].

Lemma 2.2. *Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated. Let $n \geqslant 0$ be an integer.*

(i) *Let $\alpha\colon f \Rightarrow f'$ be a 2-morphism. If f has cohomological dimension $\leqslant n$, then so has f'.*
(ii) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks that is faithfully flat. If $f_Z\colon X \times_Y Z \to Z$ has cohomological dimension $\leqslant n$, then so has f.*
(iii) *If f is affine, then it has cohomological dimension $\leqslant 0$.*
(iv) *Let $h\colon W \to X$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated and let $m \geqslant 0$ be an integer. If f (respectively h) has cohomological dimension $\leqslant n$ (respectively $\leqslant m$), then the composition $f \circ h\colon W \to Y$ has cohomological dimension $\leqslant m + n$.*
(v) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks that is quasi-affine. If f has cohomological dimension $\leqslant n$, then so has the 1-morphism $f_Z\colon X \times_Y Z \to Z$.*
(vi) *Let $g\colon Z \to Y$ be a 1-morphism of algebraic stacks. If f has cohomological dimension $\leqslant n$ and Y has quasi-affine diagonal, then the 1-morphism $f_Z\colon X \times_Y Z \to Z$ has cohomological dimension $\leqslant n$.*

Proof. The claim (i) is trivial. To address the claim (ii), we note that higher pushforwards commute with flat base change (Lemma 1.2(iv)). As faithfully flat morphisms are conservative, the morphism f has cohomological dimension $\leqslant n$. The claim (iii) follows trivially from (ii). The claim (iv) follows from the Leray spectral sequence.

We now address the claim (v). Denote the pullback of g by f as $g_X\colon Z_X \to X$ and throughout we fix $M \in \mathsf{QCoh}(Z_X)$. We first assume that the morphism g is a quasi-compact open immersion. In this situation the adjunction $(g_X)_{\mathrm{qc}}^*((g_X)_{\text{lis-ét}})_*M \to M$ is an isomorphism. For $i \geqslant 0$, we deduce that there are isomorphisms in $\mathsf{QCoh}(Z)$:

$$\mathsf{R}^i((f_Z)_{\text{lis-ét}})_*((g_X)_{\mathrm{qc}}^*((g_X)_{\text{lis-ét}})_*M) \to \mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M.$$

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Since higher pushforwards commute with flat base change, we deduce that for all $i \geqslant 0$ there are isomorphisms:

$$g^*\mathsf{R}^i(f_{\text{lis-ét}})_*(((g_X)_{\text{lis-ét}})_*M) \to \mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M.$$

Since $((g_X)_{\text{lis-ét}})_*M \in \mathsf{QCoh}(X)$, it follows that $f_Z\colon X_Z \to Z$ has cohomological dimension $\leqslant n$. Next assume that the morphism g is affine. Then the morphism g_X is also affine and so, by (iii), both morphisms have cohomological dimension $\leqslant 0$. By (iv), we conclude that the composition $f \circ g_X\colon Z_X \to Y$ has cohomological dimension $\leqslant n$. But we have a 2-isomorphism $f \circ g_X \Rightarrow g \circ f_Z$ and so, by (i), the morphism $g \circ f_Z$ has cohomological dimension $\leqslant n$. By the Leray spectral sequence, however, we see that there is an isomorphism for all $i \geqslant 0$:

$$(g_{\text{lis-ét}})_*\mathsf{R}^i((f_Z)_{\text{lis-ét}})_*M \to \mathsf{R}^i((g \circ f_Z)_{\text{lis-ét}})_*M.$$

Since the morphism g is affine, the functor g_* is faithful; thus, we conclude that the morphism f_Z has cohomological dimension $\leqslant n$. In general, a quasi-affine morphism $g\colon Z \to Y$ factors as $Z \xrightarrow{j} \overline{Z} \xrightarrow{\overline{g}} Y$, where the morphism j is a quasi-compact open immersion and the morphism $\overline{g}$ is affine. Combining the above completes the proof of (v).

To prove the claim (vi), we observe that by (ii) the statement is smooth local on Z; thus, we are free to assume that Z is an affine scheme. Since the diagonal of the stack Y is quasi-affine, the morphism $g\colon Z \to Y$ is quasi-affine. An application of (v) now gives the claim. □

We wish to point out that Lemma 2.2(vi) is false if Y does not have affine stabilizers [HR15, Remark 1.6].

Definition 2.3. A quasi-compact and quasi-separated morphism $f\colon X \to Y$ of algebraic stacks has *finite cohomological dimension* if there exists an integer $n \geqslant 0$ such that the morphism f has cohomological dimension $\leqslant n$.

Morphisms of quasi-compact and quasi-separated algebraic spaces have finite cohomological dimension [Sta, 073G]. Drinfeld and Gaitsgory [DG13, Theorem 1.4.2 and § 2] have shown that a morphism of quasi-compact and quasi-separated algebraic stacks $f\colon X \to Y$ has finite cohomological dimension if Y is a $\mathbb{Q}$-stack and f has affine stabilizers and finitely presented inertia. This result is refined and generalized in [HR15]: the condition on inertia is not required and in positive characteristic f has finite cohomological dimension exactly when f has linearly reductive stabilizers.

Definition 2.4. A morphism of algebraic stacks $f\colon X \to Y$ is *concentrated* if it is quasi-compact, quasi-separated, and, for every quasi-compact and quasi-separated algebraic stack Z and every morphism $g\colon Z \to Y$, the pulled back morphism $f_Z\colon X_Z \to Z$ has finite cohomological dimension.

By the result of Drinfeld and Gaitsgory, a quasi-compact and quasi-separated morphism of algebraic stacks $f\colon X \to Y$ is concentrated if Y is a $\mathbb{Q}$-stack and f has affine stabilizers.

The next result is immediate from Lemma 2.2.

Lemma 2.5. *Let $f\colon X \to Y$ be a 1-morphism of algebraic stacks that is quasi-compact and quasi-separated.*

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(i) *If f is concentrated, then it remains so after base change.*
(ii) *Let $g\colon Z \to Y$ be a 1-morphism that is faithfully flat. If $f_Z\colon X \times_Y Z \to Z$ is concentrated, then so is f.*
(iii) *If f is representable, then it is concentrated.*
(iv) *Let $h\colon Y \to W$ be a 1-morphism that is concentrated. Then the composition $h \circ f\colon X \to W$ is concentrated if and only if f is concentrated.*
(v) *Assume that Y is quasi-compact with quasi-affine diagonal. Then f is concentrated if and only if it has finite cohomological dimension.*

The main result of this section is the following theorem that refines Lemma 1.2.

Theorem 2.6. *Let $f\colon X \to Y$ be a concentrated 1-morphism of algebraic stacks.*

(i) *If Y is quasi-compact and quasi-separated, then there is an integer n such that the natural morphism*
$$\tau^{\geqslant j}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \to \tau^{\geqslant j}\mathsf{R}(f_{\mathrm{qc}})_*(\tau^{\geqslant j-n}\mathcal{M})$$
is a quasi-isomorphism for every integer j and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$.
(ii) *The restriction of $\mathsf{R}(f_{\text{lis-ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)$ coincides with $\mathsf{R}(f_{\mathrm{qc}})_*$.*
(iii) *The functor $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts.*
(iv) *If $g\colon Y' \to Y$ is a flat morphism of algebraic stacks, then the 2-cartesian square*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
induces a natural quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$:
$$\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \simeq \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.$$

Proof. For (ii), choose a diagram as in (1.6). By the natural transformations (1.7), (1.8), and (1.11), it is sufficient to prove that the restriction of $\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ factors through $\mathsf{D}_{\mathrm{qc}}(V^+_{\bullet,\text{ét}})$. This can be verified smooth-locally on Y, so we may assume that Y is quasi-compact and quasi-separated and f has cohomological dimension $\leqslant n$ for some integer n. In particular, $\mathsf{R}^i(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{M} = 0$ for every $i > n$ and $\mathcal{M} \in \mathsf{QCoh}(U^+_{\bullet,\text{ét}})$. By [LO08, Lemma 2.1.10], for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$ and integer j, the natural morphism
$$\tau^{\geqslant j}\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*\mathcal{M} \to \tau^{\geqslant j}\mathsf{R}(\tilde{f}^+_{\bullet,\text{ét}})_*(\tau^{\geqslant j-n}\mathcal{M}) \tag{2.1}$$
is an isomorphism. By Lemma 1.2(ii), the result follows. Note that the equation (2.1) now proves (i). Finally, the claims (iii) and (iv) follow from (i) and the corresponding results for the bounded below category in Lemma 1.2. □

Corollary 2.7. *If $f\colon X \to Y$ is an affine morphism of algebraic stacks, then there is a natural equivalence of triangulated categories*
$$\overline{f}^*\colon \mathsf{D}_{\mathrm{qc}}(Y_{\text{lis-ét}}, f_*\mathcal{O}_X) \to \mathsf{D}_{\mathrm{qc}}(X).$$

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Proof. Pick a diagram as in (1.6). Since f is an affine morphism, we may assume that $U_i = X \times_Y V_i$. The morphism of ringed topoi $\tilde{f}^+_{\bullet,\text{ét}} \colon U^+_{\bullet,\text{ét}} \to V^+_{\bullet,\text{ét}}$ factors as

$$(U^+_{\bullet,\text{ét}}, \mathcal{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{g} (V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_* \mathcal{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{k} (V^+_{\bullet,\text{ét}}, \mathcal{O}_{V^+_{\bullet,\text{ét}}}).$$

We claim that $g^{-1}(\tilde{f}^+_{\bullet,\text{ét}})_* \mathcal{O}_{U^+_{\bullet,\text{ét}}} \to \mathcal{O}_{U^+_{\bullet,\text{ét}}}$ is flat. It is sufficient to verify this upon restriction to each $(U_i)_{\text{ét}}$ and then work étale-locally; thus, we may assume that f is a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. Hence, it suffices to prove that the induced morphism of ringed sites $\overline{f} \colon ((\operatorname{Spec} B)_{\text{ét}}, \mathcal{O}_{\operatorname{Spec} B_{\text{ét}}}) \to ((\operatorname{Spec} A)_{\text{ét}}, (f_{\text{ét}})_* \mathcal{O}_{\operatorname{Spec} B})$ is flat. This can be verified at geometric points, so let $\mathfrak{p}$ be a prime ideal of B. We must prove that if $\mathfrak{q} = f(\mathfrak{p})$, then the induced ring homomorphism $B \otimes_A A^{\text{sh}}_{\mathfrak{q}} \to B^{\text{sh}}_{\mathfrak{p}}$ is flat, where sh denotes the strict henselization at the relevant prime ideal. Since $\operatorname{Spec}(B \otimes_A A^{\text{sh}}_{\mathfrak{q}})$ has flat diagonal over $\operatorname{Spec} B$ and $\operatorname{Spec} B^{\text{sh}}_{\mathfrak{p}}$ is a flat $\operatorname{Spec} B$-scheme, the assertion is clear and the claim is proved.

It follows that $g^* \colon \mathsf{Mod}(V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_* \mathcal{O}_{U^+_{\bullet,\text{ét}}}) \to \mathsf{Mod}(U^+_{\bullet,\text{ét}}, \mathcal{O}_{U^+_{\bullet,\text{ét}}})$ is exact. Moreover, since f is affine, it has cohomological dimension $\leqslant 0$ (Lemma 2.2(iii)) and is concentrated. In particular, the restriction of $\mathsf{R}g_*$ to $\mathsf{D}_{\text{qc}}(U^+_{\bullet,\text{ét}}, \mathcal{O}_{U^+_{\bullet,\text{ét}}})$ factors through $\mathsf{D}_{\text{qc}}(V^+_{\bullet,\text{ét}}, (\tilde{f}^+_{\bullet,\text{ét}})_* \mathcal{O}_{U^+_{\bullet,\text{ét}}})$ and is exact. It remains to prove that the restrictions of the adjunctions $\mathrm{Id} \Rightarrow \mathsf{R}g_* \mathsf{L}g^*$ and $\mathsf{L}g^* \mathsf{R}g_* \Rightarrow \mathrm{Id}$ to complexes with quasi-coherent cohomology sheaves are isomorphisms. By the exactness, it is sufficient to prove that the restrictions of the underived adjunctions $\mathrm{Id} \Rightarrow g_* g^*$ and $g^* g_* \Rightarrow \mathrm{Id}$ to quasi-coherent modules are isomorphisms. Unwinding the definitions, this is just the assertion that $\mathsf{QCoh}(X) \cong \mathsf{QCoh}(Y_{\text{lis-ét}}, f_* \mathcal{O}_X)$. By smooth descent, we may reduce to the situation where X and Y are affine schemes. The result now follows from [EGA, § II.1.4]. □

COROLLARY 2.8. *Let $f \colon X \to Y$ be a quasi-affine morphism of algebraic stacks and let $M \in \mathsf{D}_{\text{qc}}(X)$. If $\mathsf{R}(f_{\text{qc}})_* M \simeq 0$, then $M \simeq 0$, that is, $\mathsf{R}(f_{\text{qc}})_*$ is conservative.*

Proof. We may factor f as $X \xrightarrow{j} X' \xrightarrow{f'} Y$, where j is a quasi-compact open immersion and f' is an affine morphism. By Corollary 2.7, the result is true for f'. Hence, we are reduced to the situation where f is a quasi-compact open immersion. In this case, however, $\Delta_{X/Y} \colon X \to X \times_Y X$ is an isomorphism. By Theorem 2.6(iv), $\mathsf{L}f^*_{\text{qc}} \mathsf{R}(f_{\text{qc}})_* M \simeq M$ and the result follows. □

3. Triangulated categories

In this section we will recall some results on triangulated categories that may not be familiar to everyone. For excellent and comprehensive treatments of these topics, see [Nee92b] and [Tho97, § 2]. In particular, we will recall *thick* and *localizing* triangulated subcategories. This leads to the concept of *compact* objects and Thomason's localization theorem.

Throughout this section let $\mathcal{S}$ be a triangulated category with shift operator Σ.

A functor $F \colon \mathcal{S} \to \mathcal{S}'$ between triangulated categories is *triangulated* if F sends triangles to triangles and is compatible with shifts. We say that a full subcategory $\mathcal{R}$ of $\mathcal{S}$ is *triangulated* if the category $\mathcal{R}$ is triangulated and the inclusion functor $\mathcal{R} \to \mathcal{S}$ is triangulated. A subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *thick* (also known as épaisse or saturated) if it is full, triangulated, and every $\mathcal{S}$-direct summand of every $r \in \mathcal{R}$ belongs to $\mathcal{R}$.

Example 3.1. For a triangulated functor $F \colon \mathcal{S} \to \mathcal{S}'$, we denote by $\ker F$ the full triangulated subcategory consisting of those $x \in \mathcal{S}$ such that $F(x) \simeq 0$. The subcategory $\ker F \subseteq \mathcal{S}$ is thick.

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Example 3.2. Given triangulated subcategories $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{S}$ such that $\mathcal{R}_1$ is a thick subcategory of $\mathcal{S}$, then $\mathcal{R}_1$ is a thick subcategory $\mathcal{R}_2$.

Kernels of triangulated functors produce essentially all thick subcategories [Tho97, § 1.3]. Indeed, for every thick subcategory $\mathcal{R} \subseteq \mathcal{S}$, there is a *quotient* functor $Q\colon \mathcal{S} \to \mathcal{S}/\mathcal{R}$ such that $\mathcal{R} \cong \ker Q$, Q is essentially surjective, and satisfies a universal property [Nee01b, Theorem 2.1.8].

For a class $R \subseteq \mathcal{S}$, the *thick closure* of R is the smallest thick subcategory $\mathcal{R} \subseteq \mathcal{S}$ containing R. A subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *dense* if it is full, triangulated, and its thick closure coincides with $\mathcal{S}$.

If the triangulated category $\mathcal{S}$ is essentially small, then there is a notion of $\mathsf{K}_0(\mathcal{S})$: it is the free abelian group on the *set* of isomorphism classes of objects in $\mathcal{S}$ modulo the relation that given an $\mathcal{S}$-triangle $s_1 \to s_2 \to s_3$, then $[s_2] = [s_1] + [s_3]$. It is easy to see that for s, $t \in \mathcal{S}$, then $[s \oplus t] = [s] + [t]$ and $[s] = -[\Sigma s]$. Also, for every $\sigma \in \mathsf{K}_0(\mathcal{S})$, there exists $s \in \mathcal{S}$ such that $\sigma = [s]$. Given a triangulated functor $F\colon \mathcal{S} \to \mathcal{S}'$ between essentially small triangulated categories, there is an induced group homomorphism $\mathsf{K}_0(F)\colon \mathsf{K}_0(\mathcal{S}) \to \mathsf{K}_0(\mathcal{S}')$. The following is a nice result of Neeman [Nee92b, Corollary 0.10] (also see [TT90, 5.2.2] and [Tho97, Lemma 2.2]).

Lemma 3.3. *Let $\mathcal{S}$ be an essentially small triangulated category and let $\mathcal{R}$ be a dense subcategory. If $s \in \mathcal{S}$, then $s \in \mathcal{R}$ if and only if s belongs to the image of $\mathsf{K}_0(\mathcal{R})$ in $\mathsf{K}_0(\mathcal{S})$. In particular, if $s \in \mathcal{S}$, then $s \oplus \Sigma s \in \mathcal{R}$.*

A pair of triangulated functors $\mathcal{R} \to \mathcal{S} \xrightarrow{F} \mathcal{T}$ is *left-exact* (respectively *almost-exact*, respectively *exact*) if $\mathcal{R}$ is a thick subcategory of $\mathcal{S}$ via the functor $\mathcal{R} \to \mathcal{S}$ and the functor $F\colon \mathcal{S} \to \mathcal{T}$ factors through the quotient $\overline{F}\colon \mathcal{S}/\mathcal{R} \to \mathcal{T}$ and this functor is fully faithful (respectively fully faithful and dense, respectively an equivalence). The following (well-known) lemma will be useful.

Lemma 3.4. *Let $F\colon \mathcal{S} \to \mathcal{T}$ be a triangulated functor. If F has a right adjoint $G\colon \mathcal{T} \to \mathcal{S}$ such that the adjunction $\epsilon\colon FG \to \mathrm{Id}_{\mathcal{T}}$ is an isomorphism, then the sequence $\ker F \to \mathcal{S} \xrightarrow{F} \mathcal{T}$ is exact.*

Proof. We will show that F satisfies the universal property of the quotient. Let $P\colon \mathcal{S} \to \mathcal{P}$ be a triangulated functor such that $\ker F \subseteq \ker P$. We must prove that there are a functor $P'\colon \mathcal{T} \to \mathcal{P}$ and an isomorphism $\alpha\colon P \simeq P'F$ unique up to unique isomorphism. For the uniqueness, let P'_1, P'_2 be two such functors with isomorphisms $\alpha_i\colon P \simeq P'_iF$. Then the isomorphism $\alpha_2\alpha_1^{-1}\colon P'_1F \simeq P'_2F$ induces a unique isomorphism $P'_1 \simeq P'_1FG \simeq P'_2FG \simeq P'_2$ compatible with the α_i. For the existence, set $P' = PG$ and let $\alpha = P_*\eta$, where $\eta\colon \mathrm{Id}_{\mathcal{S}} \to GF$ is the unit of the adjunction. Now $F_*\eta$ is an isomorphism since ϵ is an isomorphism, so $\alpha = P_*\eta$ is an isomorphism since $\ker F \subseteq \ker P$. □

A triangulated category is said to be *closed under small coproducts* if it admits small categorical coproducts and small coproducts of triangles remain triangles. If the triangulated category $\mathcal{S}$ is closed under small coproducts, then we say that a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is *localizing* if it is a full triangulated subcategory, closed under small coproducts, and the functor $\mathcal{R} \to \mathcal{S}$ preserves small coproducts. For a class $R \subseteq \mathcal{S}$, where $\mathcal{S}$ is closed under small coproducts, there is a smallest subcategory $\mathcal{R} \subseteq \mathcal{S}$ that is localizing and contains R. We refer to $\mathcal{R}$ as the *localizing envelope* of R.

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Example 3.5. If a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is localizing, then it is thick. Indeed, given $r \in \mathcal{R}$ and $r \simeq r' \oplus r''$ in $\mathcal{S}$, the Eilenberg swindle produces an $\mathcal{S}$-isomorphism

$$r'' \oplus r \oplus r \oplus \cdots \simeq r \oplus r \oplus r \oplus \cdots .$$

Since $\mathcal{R}$ is localizing, $r'' \oplus r \oplus r \oplus \cdots \in \mathcal{R}$. The cone of the natural morphism $r \oplus r \oplus \cdots \to r'' \oplus r \oplus r \oplus \cdots$ is r''; thus, $r'' \in \mathcal{R}$.

A result of Neeman [Nee92b, Proposition 1.9] says that if a subcategory $\mathcal{R} \subseteq \mathcal{S}$ is localizing, then the quotient $Q\colon \mathcal{S} \to \mathcal{S}/\mathcal{R}$ preserves small coproducts. In particular, since the quotient is essentially surjective, the category $\mathcal{S}/\mathcal{R}$ is closed under small coproducts.

Example 3.6. If $F\colon \mathcal{S} \to \mathcal{S}'$ is a triangulated functor that preserves small coproducts, then the subcategory $\ker F$ is localizing.

An object $s \in \mathcal{S}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{S}}(s, -)$ preserves small coproducts. Denote by $\mathcal{S}^c$ the full subcategory of compact objects of $\mathcal{S}$.

Example 3.7. Let A be a ring. A complex of A-modules is compact in $\mathsf{D}(A)$ if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules [Sta, 07LT]. That is, the compact objects of $\mathsf{D}(A)$ are the *perfect* complexes of A-modules.

Example 3.8. Let $F\colon \mathcal{S} \to \mathcal{S}'$ be a triangulated functor that admits a right adjoint $G\colon \mathcal{S}' \to \mathcal{S}$. If G preserves small coproducts, then F sends $\mathcal{S}^c$ to $\mathcal{S}'^c$ [Nee96, Theorem 5.1 '$\Rightarrow$'].

Example 3.9. If $f\colon X' \to X$ is a concentrated morphism of algebraic stacks, then $\mathsf{L}f^*_{\mathrm{qc}}$ sends $\mathsf{D}_{\mathrm{qc}}(X)^c$ to $\mathsf{D}_{\mathrm{qc}}(X')^c$. This follows by combining Example 3.8 with Theorem 2.6(iii).

A class $S \subseteq \mathcal{S}$ is *generating* if given $x \in \mathcal{S}$ such that $\mathrm{Hom}_{\mathcal{S}}(\Sigma^n s, x) = 0$ for all $s \in S$ and $n \in \mathbb{Z}$, then $x \simeq 0$. The triangulated category $\mathcal{S}$ is *compactly generated* if it admits a *set* of generators consisting of compact objects.

Example 3.10. Let A be a ring. Denote the unbounded derived category of A-modules by $\mathsf{D}(A)$. Then the set $\{A\}$ compactly generates $\mathsf{D}(A)$. Hence, $\mathsf{D}(A)$ is compactly generated.

Example 3.11. This is a refinement of Example 3.8. Let $F\colon \mathcal{S} \to \mathcal{S}'$ be a triangulated functor that admits a right adjoint $G\colon \mathcal{S}' \to \mathcal{S}$ that preserves small coproducts. In addition, assume that G is conservative (i.e., $G(x) \simeq 0$ implies that $x \simeq 0$). If $\mathcal{S}$ is compactly generated by a class S, then $\mathcal{S}'$ is compactly generated by the class $\{F(s) : s \in S\}$.

We now recall Thomason's localization theorem, which was proved in this generality by Neeman [Nee92b, Nee96, Theorem 2.1] (also see [TT90, 5.1]).

Theorem 3.12 (Thomason's localization). *Consider an exact sequence of triangulated categories $\mathcal{R} \to \mathcal{S} \xrightarrow{F} \mathcal{T}$ that are closed under small coproducts. If the triangulated category $\mathcal{S}$ is compactly generated and $\mathcal{R}$ is the localizing envelope of a subset $R \subseteq \mathcal{S}^c$, then there is an induced sequence $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$ which is almost-exact. In particular, $\mathcal{R}^c = \mathcal{S}^c \cap \mathcal{R}$ and $\mathcal{R}^c$ is the thick closure of R.*

Combining Theorem 3.12 with the elementary Lemma 3.3 produces something very surprising, which was observed by Neeman [Nee92b, Corollary 0.9].

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Corollary 3.13. *In Theorem 3.12, assume that the category $\mathcal{T}^c$ is essentially small. Then, for every $t \in \mathcal{T}^c$, there exist an $s \in \mathcal{S}^c$ and an isomorphism $t \oplus \Sigma t \simeq F(s)$.*

Another useful corollary is the following [Nee96, Theorem 2.1.2].

Corollary 3.14. *In Theorem 3.12, suppose that R is a generating set for $\mathcal{S}$; then $\mathcal{R} = \mathcal{S}$.*

4. Perfect complexes, projection formulas, and finite duality

We now use the results of the previous section to prove some useful results for derived categories of algebraic stacks. We begin with the notion of a perfect complex on an algebraic stack.

4.1 Perfect complexes

We recall some notions from [SGA6, Exposé II] (also see [Sta, Tag 08FK]). If A is a ring, then a complex of A-modules P is *strictly perfect* if it is a bounded complex of finitely generated and projective A-modules. More generally, if A is a sheaf of rings on a site E, then a complex $P = (P^k)$ of A-modules is *strictly perfect* if it is a bounded complex and each term P^k is a direct summand of a finite free A-module. A complex $P \in \mathsf{D}(A)$ is *perfect* if it is locally strictly perfect.

Let X be an algebraic stack. Then a complex $\mathcal{P}$ on X is *perfect* if it is a perfect object of $\mathsf{D}(X)$. Note that all perfect complexes on X belong to $\mathsf{D}_{\mathrm{qc}}(X)$. The following lemma provides a useful criterion for perfection.

Lemma 4.1. *Let X be an algebraic stack and let $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$. The following conditions are equivalent:*

(i) *$\mathcal{P}$ is perfect; and*
(ii) *for every $x \in |X|$, there exists a flat morphism $f\colon \operatorname{Spec} A \to X$ with image containing x such that $\mathsf{R}\Gamma(\operatorname{Spec} A, \mathsf{L}f_{\mathrm{qc}}^*\mathcal{P})$ is a strictly perfect complex of A-modules.*

Moreover, every perfect complex on an affine scheme is strictly perfect.

Proof. It is sufficient to prove that if $A \to B$ is a faithfully flat ring homomorphism and $M \in \mathsf{D}(A)$, then M is a strictly perfect complex of A-modules if and only if $M \otimes_A B$ is a strictly perfect complex of B-modules. This is [Sta, Tag 068T]. □

Example 4.2. Let X be an algebraic stack. Then every flat $\mathcal{O}_X$-module of finite presentation defines a perfect complex in $\mathsf{D}_{\mathrm{qc}}(X)$. In particular, if $f\colon X \to Y$ is a morphism of algebraic stacks that is finite, flat, and of finite presentation, then $f_*\mathcal{O}_X$ is perfect in $\mathsf{D}_{\mathrm{qc}}(Y)$.

We recall the following well-known definition (e.g., [Bra14, Definition 4.7.1] or [BFN10, Definition 3.3]). Let X be an algebraic stack. An object $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ is *dualizable* if there exists $\mathcal{P}^\vee \in \mathsf{D}_{\mathrm{qc}}(X)$ together with morphisms $e\colon \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \to \mathcal{O}_X$, $c\colon \mathcal{O}_X \to \mathcal{P} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}^\vee$ such that the two induced maps

$$\mathcal{O}_X \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \xrightarrow{c\otimes \mathrm{Id}_{\mathcal{P}}} \mathcal{P} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \xrightarrow{\mathrm{Id}_{\mathcal{P}}\otimes e} \mathcal{P},$$

$$\mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\mathrm{Id}_{\mathcal{P}^\vee}\otimes c} \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}^\vee \xrightarrow{e\otimes \mathrm{Id}_{\mathcal{P}^\vee}} \mathcal{P}^\vee$$

are isomorphisms. It is standard that $\mathcal{P}$ dualizable implies that $\mathcal{P}^\vee \simeq \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X)$ and the e and c maps are simply those arising from the adjunction between $- \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}$ and $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{P}, -)$.

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The following lemma is straightforward but crucial.

LEMMA 4.3. *Let X be an algebraic stack and let $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ be a perfect complex.*

(i) *The double duality morphism $\mathcal{P} \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X), \mathcal{O}_X)$ is a quasi-isomorphism.*

(ii) *The restriction of the functor $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, -)\colon \mathsf{D}(X) \to \mathsf{D}(X)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}_{\mathrm{qc}}(X)$ and preserves small coproducts in $\mathsf{D}_{\mathrm{qc}}(X)$. Moreover, if $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, then there is a natural quasi-isomorphism*

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{M} \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}).$$

In particular, $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{P}, -) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, -) \simeq \mathcal{P}^{\vee} \otimes -$ on $\mathsf{D}_{\mathrm{qc}}(X)$.

(iii) *If $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ is dualizable, then it is perfect.*

In particular, for every algebraic stack X, the notions of perfect and dualizable objects in $\mathsf{D}_{\mathrm{qc}}(X)$ coincide.

Proof. To prove (i), we note that the existence of the double duality morphism follows from $\mathsf{R}\mathcal{H}om$-$\otimes$ adjunction. That it is a quasi-isomorphism can be verified smooth-locally on X, so we may assume that $X = \operatorname{Spec} A$ is an affine scheme. To prove (ii), we may argue similarly to reduce to the affine setting. Now the collection of all $\mathcal{P}$ that satisfy the conclusions of (i) and (ii) is closed under finite coproducts, direct summands, shifts, and the taking of cones, that is, it is a thick subcategory of $\mathsf{D}_{\mathrm{qc}}(X) \simeq \mathsf{D}(A)$. Since $\mathcal{O}_X$ satisfies the conclusions of (i) and (ii) and X is affine, (i) and (ii) follow.

For (iii), since dualizable implies locally dualizable, it follows that we may assume that X is affine. In this case, the result is classical. One may also argue as follows: $\mathcal{Q}$ being dualizable implies that there is an isomorphism of functors $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{Q}, -) \simeq \mathcal{Q}^{\vee} \otimes^{\mathsf{L}}_{\mathcal{O}_X} -$ from $\mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}_{\mathrm{qc}}(X)$. In particular, the functor $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_X}(\mathcal{Q}, -)\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X)$ preserves small coproducts. Taking global sections, we see that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q}, -)$ preserves small coproducts. Hence, $\mathcal{Q}$ is compact in $\mathsf{D}_{\mathrm{qc}}(X)$. But X is affine, so $\mathcal{Q}$ is perfect. □

4.2 Compact complexes

We now move on to a description of the compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$ and their relationship to perfect complexes.

LEMMA 4.4. *Let X be a quasi-separated algebraic stack.*

(i) *If $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)^c$, then $\mathcal{Q}$ is perfect.*

(ii) *If $\mathcal{P}$ is perfect on X and $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)^c$, then $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$.*

(iii) *If X is concentrated, then $\mathcal{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$; in particular, if $\mathcal{P}$ is perfect on X, then $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$.*

Proof. We first prove (i). Consider a smooth morphism $p\colon \operatorname{Spec} A \to X$. It follows from the quasi-separatedness of X that p is quasi-compact, quasi-separated, and representable. In particular, p is concentrated (Lemma 2.5(iii)) and so $\mathsf{L}p^*_{\mathrm{qc}}\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(\operatorname{Spec} A)^c$ by Example 3.9. The claim now follows from Example 3.7 and Lemma 4.1.

To prove (ii), we note that if $\mathcal{M} \in \mathsf{D}(X)$, then (1.3) implies that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M})).$$

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By Lemma 4.3(ii), the restriction of $\mathsf{R}\mathcal{H}om(\mathcal{P}, -)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ preserves small coproducts. Since $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)^c$, it follows that the restriction of $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}, -)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ preserves small coproducts. Hence, $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)^c$.

For (iii), we note that if $\mathcal{M} \in \mathsf{D}(X)$, then by definition $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) = \mathsf{R}\Gamma(X_{\text{lis-ét}}, \mathcal{M})$. Since X is concentrated, Theorem 2.6(iii) implies that the restriction of $\mathsf{R}\Gamma(X_{\text{lis-ét}}, -)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ preserves small coproducts. Thus, $\mathcal{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$. The latter claim follows from the former and (ii). □

In the following lemma we provide criteria for a perfect complex on an algebraic stack to be compact. We have not seen this characterization in the literature before.

Lemma 4.5. *Let X be a quasi-compact and quasi-separated algebraic stack and let $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ be a perfect complex. The following conditions are equivalent:*

(i) *$\mathcal{P}$ is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$;*
(ii) *there exists an integer $r \geqslant 0$ such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{N}[i]) = 0$ for all $\mathcal{N} \in \mathsf{QCoh}(X)$ and $i > r$; and*
(iii) *there exists an integer $r \geqslant 0$ such that the natural map*

$$\tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \to \tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{P}, \tau^{\geqslant j-r}\mathcal{M})$$

is a quasi-isomorphism for all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and integers j.

Proof. Assume that (ii) does not hold. Then there are an infinite sequence of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}_1, \mathcal{M}_2, \dots$ and a strictly increasing sequence of integers $d_1 < d_2 < \cdots$ such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i]) \neq 0$ for every i. Since $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete [HNR14, Theorem B.1], there is a quasi-isomorphism in $\mathsf{D}_{\mathrm{qc}}(X)$:

$$\bigoplus_{i=1}^{\infty} \mathcal{M}_i[d_i] \simeq \prod_{i=1}^{\infty} \mathcal{M}_i[d_i].$$

This implies that the natural morphism

$$\bigoplus_{i=1}^{\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i]) \to \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{P}, \bigoplus_{i=1}^{\infty} \mathcal{M}_i[d_i]\right) \simeq \prod_{i=1}^{\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}_i[d_i])$$

is not an isomorphism. In particular, $\mathcal{P}$ is not compact. Thus, by the contrapositive, we have proved the implication (i)⇒(ii).

For (ii)⇒(iii), first choose a diagram as in (1.6) with $V = Y = \operatorname{Spec}\mathbb{Z}$. Let $\mathcal{P}^+_{\bullet,\text{ét}} = \mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathcal{P}$ and $\mathcal{M}^+_{\bullet,\text{ét}} = \mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathcal{M}$. Note that

$$\mathsf{R}(\mathrm{res}_{U_\bullet})_*\mathsf{L}(p^+_{\bullet,\text{lis-ét}})^*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_{U^+_{\bullet,\text{ét}}}}(\mathcal{P}^+_{\bullet,\text{ét}}, \mathcal{M}^+_{\bullet,\text{ét}})$$

has quasi-coherent cohomology (Lemma 4.3) and that $\mathsf{RHom}_{\mathcal{O}_{U^+_{\bullet,\text{ét}}}}(\mathcal{P}^+_{\bullet,\text{ét}}, \mathcal{N}[i]) = 0$ for all $i > r$ and $\mathcal{N} \in \mathsf{QCoh}(U^+_{\bullet,\text{ét}})$. It is enough to prove that

$$\tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_{U^+}}(\mathcal{P}^+_{\bullet,\text{ét}}, \mathcal{M}) \to \tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_{U^+}}(\mathcal{P}^+_{\bullet,\text{ét}}, \tau^{\geqslant j-r}\mathcal{M})$$

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for every integer j and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(U^+_{\bullet,\text{ét}})$. This follows as in the proof of [LO08, Lemma 2.1.10] with ϵ_* replaced by $\mathrm{Hom}_{\mathcal{O}_{U^+_{\bullet,\text{ét}}}}(\mathcal{P}^+_{\bullet,\text{ét}}, -)$.

Finally, for (iii)⇒(i), we note that because $\mathcal{P}$ is perfect, it is dualizable. Thus, for all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and integers j, there are natural quasi-isomorphisms:

$$\begin{aligned}\tau^{\geqslant j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) &\simeq \tau^{\geqslant j}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \tau^{\geqslant j-r}\mathcal{M}) \\ &\simeq \tau^{\geqslant j}\mathsf{R}\Gamma(X_{\text{lis-ét}}, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \tau^{\geqslant j-r}\mathcal{M}).\end{aligned}$$

Also, since $\mathcal{P}$ is perfect, the restriction of the functor $\mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} (-)$ to $\mathsf{D}^{\geqslant j-r}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}^{\geqslant j-r'}_{\mathrm{qc}}(X)$ for some fixed integer r'. The result now follows from Lemma 1.2(iii). □

For future reference, we summarize the situation in the following remark.

Remark 4.6. Let X be a quasi-compact and quasi-separated algebraic stack. Then the following are equivalent:

(i) every perfect object of $\mathsf{D}_{\mathrm{qc}}(X)$ is compact;
(ii) the structure sheaf $\mathcal{O}_X$ is compact;
(iii) X has finite cohomological dimension; and
(iv) the derived global section functor $\mathsf{R}\Gamma\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{Ab})$ commutes with small coproducts.

The equivalence of the first two conditions follows from Lemma 4.4. The structure sheaf is compact if and only if X has finite cohomological dimension (Lemma 4.5). The last condition is equivalent to the definition of $\mathcal{O}_X$ being compact.

4.3 Supports and generation

Let X be a scheme and let $M \in \mathsf{QCoh}(X)$. The *support* of M is the subset

$$\mathrm{supp}(M) = \{x \in |X| : M_x \neq 0\}.$$

More generally, if X is an algebraic stack and $M \in \mathsf{QCoh}(X)$, then we define the *support* of M as follows: let $p\colon U \to X$ be a smooth surjection from a scheme U; then $\mathrm{supp}(M) = p(\mathrm{supp}(p^*M))$. It is easily verified that this is well defined (i.e., independent of the cover p).

Definition 4.7. Let X be an algebraic stack and let $E \in \mathsf{D}_{\mathrm{qc}}(X)$. Define the *cohomological support* of E to be the subset

$$\mathrm{supph}(E) = \bigcup_{n\in\mathbb{Z}} \mathrm{supp}(\mathcal{H}^n(E)) \subseteq |X|.$$

Recall that if X is a quasi-separated algebraic stack, then, for every $x \in |X|$, there is a quasi-affine monomorphism $i_x\colon \mathcal{G}_x \hookrightarrow X$ with image x, where $\mathcal{G}_x$ is a gerbe over a field $\kappa(x)$, the residue field at x [Ryd11, Theorem B.2]. We refer to this data as the *residual gerbe at* x. We now have the following lemma.

Lemma 4.8. *Let X be a quasi-compact and quasi-separated algebraic stack. Let E be a perfect complex on X.*

(i) *Then $x \in \mathrm{supph}(E)$ if and only if the complex $\mathsf{L}(i_x)^*_{\mathrm{qc}}E$ is not acyclic in $\mathsf{D}_{\mathrm{qc}}(\mathcal{G}_x)$, where $i_x\colon \mathcal{G}_x \to X$ is the residual gerbe.*

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(ii) *If* $f\colon X' \to X$ *is a quasi-compact and quasi-separated morphism, then* $\operatorname{supph}(\mathsf{L}f^*_{\mathrm{qc}}E) = f^{-1}\operatorname{supph}(E)$.

(iii) $\operatorname{supph}(E)$ *is closed with quasi-compact complement.*

Proof. For (i), $x \in \operatorname{supph}(E)$ if and only if there exist a smooth and surjective morphism $p\colon U \to X$, where U is a scheme, and $u \in U$ such that $p(u) = x$ and $(\mathsf{L}p^*_{\mathrm{qc}}E)_u$ is not acyclic in $\mathsf{D}_{\mathrm{qc}}(\mathcal{O}_{U,u})$. By [Tho97, Lemma 3.3(a)], this is equivalent to $(\mathsf{L}p^*_{\mathrm{qc}}E) \otimes \kappa(u)$ not being acyclic in $\mathsf{D}_{\mathrm{qc}}(\kappa(u))$. Since $\operatorname{Spec}\kappa(u) \to \mathcal{G}_x$ is faithfully flat, the result follows.

For (ii), we use (i) and argue as in [Tho97, Lemma 3.3(b)], with the additional observation that for every $x' \in |X'|$, the induced morphism on residual gerbes $\mathcal{G}_{x'} \to \mathcal{G}_{f(x')}$ is faithfully flat.

For (iii), by (ii), the conclusion may be verified smooth-locally. In particular, we may assume that X is an affine scheme, and the result follows from [Tho97, Lemma 3.3(c)]. □

If X is an algebraic stack and $Z \subseteq |X|$, then we define

$$\mathsf{D}_{\mathrm{qc},|Z|}(X) = \{\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X) : \operatorname{supph}(\mathcal{M}) \subseteq Z\}.$$

If $j\colon U \hookrightarrow X$ is a flat monomorphism, e.g., an open immersion, then

$$\mathsf{D}_{\mathrm{qc},|X\setminus U|}(X) = \{\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X) : j^*\mathcal{M} \simeq 0\}.$$

Lemma 4.9. *Let* X *be an algebraic stack and let* $\mathcal{P} \in \mathsf{D}_{\mathrm{qc}}(X)$ *be a perfect complex with support* $Z = \operatorname{supph}(\mathcal{P})$. *If* $\mathcal{M} \in \mathsf{D}_{\mathrm{qc},|Z|}(X)$, *then* $\mathcal{M} \simeq 0$ *if and only if* $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \simeq 0$.

Proof. The question is local on X for the smooth topology, so we may assume that X is an affine scheme. By [BN93, Proposition 6.1], there is a perfect generating complex $K \in \mathsf{D}_{\mathrm{qc},|Z|}(X)^c$ (a Koszul complex). As $\mathcal{P}$ is perfect and the support of $\mathcal{P}$ is $|Z|$, it follows from [Nee92a, Lemma A.3] that K is in the thick closure of $\mathcal{P}$, so $\mathcal{P}$ is also a generator of $\mathsf{D}_{\mathrm{qc},|Z|}(X)$. Thus, $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \simeq 0$ if and only if $\mathcal{M} \simeq 0$. □

Lemma 4.10. *Let* X *be a quasi-compact and quasi-separated algebraic stack and let* $j\colon U \to X$ *be a quasi-compact open immersion with complement* $|Z|$.

(i) *If* $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ *is generated by a set whose elements have compact image in* $\mathsf{D}_{\mathrm{qc}}(X)$, *then there exists a compact object* $\mathcal{Q}$ *of* $\mathsf{D}_{\mathrm{qc}}(X)$ *with support* $|Z|$.

(ii) *If* $\mathsf{D}_{\mathrm{qc}}(X)$ *is generated by a set of compact objects* $\{\mathcal{Q}_b\}_{b\in B}$ *and there exists a perfect complex* $\mathcal{P}$ *on* X *with support* $|Z|$, *then* $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ *is generated by the set* $\{\mathcal{Q}_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}\}_{b\in B}$ *(whose elements have compact image in* $\mathsf{D}_{\mathrm{qc}}(X)$*).*

Proof. For (i), let $\{\mathcal{Q}_\lambda\}_{\lambda\in\Lambda}$ be a set of generators for $\mathsf{D}_{\mathrm{qc},|Z|}(X)$ whose elements have compact image in $\mathsf{D}_{\mathrm{qc}}(X)$. Let $z \in |Z|$ be a point and choose a representative, that is, a field k and a 1-morphism of algebraic stacks $\overline{z}\colon \operatorname{Spec}k \to X$ with image z. Since the diagonal of X is quasi-compact and quasi-separated, it follows that $\overline{z}$ is quasi-compact and quasi-separated. By Lemma 1.2(iv), it follows that $\mathsf{L}j^*_{\mathrm{qc}}\mathsf{R}(\overline{z}_{\mathrm{qc}})_*\mathcal{O}_{\operatorname{Spec}k} \simeq 0$ and so there exist a $\lambda \in \Lambda$, an integer n, and a non-zero morphism $\mathcal{Q}_\lambda[n] \to \mathsf{R}(\overline{z}_{\mathrm{qc}})_*\mathcal{O}_{\operatorname{Spec}k}$. By adjunction, $\mathsf{L}\overline{z}^*_{\mathrm{qc}}\mathcal{Q}_\lambda \neq 0$ and we deduce that $\bigcup_{\lambda\in\Lambda}\operatorname{supph}(\mathcal{Q}_\lambda) = |Z|$. It suffices to show that there is a finite subset $\Lambda' \subseteq \Lambda$ such that $\bigcup_{\lambda\in\Lambda'}\operatorname{supph}(\mathcal{Q}_\lambda) = |Z|$. This is obvious if X is noetherian or, more generally, if Z has a finite number of irreducible components.

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In complete generality, we note that $|Z|$ is constructible (by hypothesis) and that $\operatorname{supph}(\mathcal{Q}_\lambda)$ is constructible for every λ (by Lemma 4.8(iii)). Indeed, both subsets are closed with quasi-compact complement. We conclude that $|Z| = \bigcup_{\lambda\in\Lambda} \operatorname{supph}(\mathcal{Q}_\lambda)$ has a finite subcovering since the constructible topology is quasi-compact.

For (ii), first note that the complex $\mathcal{Q}_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}$, which belongs to $\mathsf{D}_{\mathrm{qc},|Z|}(X)$, is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ (Lemma 4.4(ii)). Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qc},|Z|}(X)$ and suppose that $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}_b \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{P}, \mathcal{M}) \simeq 0$. By adjunction (1.3), $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}_b, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M})) \simeq 0$. Since the set $\{\mathcal{Q}_b\}_{b\in B}$ is generating, it follows that $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{M}) \simeq 0$. Thus, $\mathcal{M} \simeq 0$ by Lemma 4.9. □

4.4 Projection formula

A typical application of Corollary 3.14 is given by the following proposition. The given argument is a variant of [Nee96, Proposition 5.3], though we have not seen this proposition in the literature before.

Proposition 4.11 (Strong projection formula). *Let A be a ring and let $\pi\colon X \to \operatorname{Spec} A$ be a morphism of algebraic stacks. Let $\mathcal{Q}$ be a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ and let $\mathcal{G} \in \mathsf{D}_{\mathrm{qc}}(X)$. Then, for every $I \in \mathsf{D}(A)$, there is a natural quasi-isomorphism*

$$\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \otimes^{\mathsf{L}}_A I \simeq \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Proof. First we describe the morphism: by adjunction, there is a natural morphism

$$\mathsf{L}\pi^*_{\mathrm{qc}}\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G})$$

and so by (1.3) there is a natural morphism

$$(\mathsf{L}\pi^*_{\mathrm{qc}}\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G})) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I \to \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I.$$

By (1.3) again, there is a natural morphism

$$\mathsf{L}\pi^*_{\mathrm{qc}}[\mathsf{R}(\pi_{\mathrm{qc}})_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \otimes^{\mathsf{L}}_A I] \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

By adjunction and (1.5) and (1.10), we deduce the existence of the required natural morphism

$$\phi_I\colon \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \otimes^{\mathsf{L}}_A I \to \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Let $\mathcal{K} \subseteq \mathsf{D}(A)$ be the full subcategory with objects those I such that ϕ_I is a quasi-isomorphism. It remains to show that $\mathcal{K} = \mathsf{D}(A)$. Clearly, $\mathcal{K}$ is a triangulated subcategory that contains $A[k]$ for every integer k. Moreover, since $\mathcal{Q}$ is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$, $\mathcal{K}$ is closed under small coproducts. The result now follows from Corollary 3.14. □

A straightforward implication is the usual projection formula for concentrated morphisms of algebraic stacks.

Corollary 4.12 (Projection formula). *Let $f\colon X \to Y$ be a concentrated 1-morphism of algebraic stacks. The natural map*

$$(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{N} \to \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{N})$$

is a quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(Y)$.

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Proof. By adjunction, there is a natural morphism

$$\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{N} \to \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{N}.$$

By adjunction again, we deduce the existence of a natural morphism

$$\psi_{\mathcal{N}} \colon (\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{N} \to \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{N}).$$

It remains to show that $\psi_{\mathcal{N}}$ is a quasi-isomorphism for every $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(Y)$. Note that the verification of this is smooth local on Y, so, by Theorem 2.6(iv), we may reduce to the situation where Y is an affine scheme. By Lemma 4.4(iii), $\mathcal{O}_X$ is compact and the result now follows from Proposition 4.11. □

4.5 Tor-independent base change

Let $f \colon X \to Y$ and $g \colon Y' \to Y$ be morphisms of algebraic stacks. We say that f and g are *tor-independent* if for every smooth morphism $\operatorname{Spec} A \to Y$ and every pair of smooth morphisms $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$ and $\operatorname{Spec} A' \to Y' \times_Y \operatorname{Spec} A$, we have that $\operatorname{Tor}_i^A(B, A') = 0$ for all $i > 0$. Equivalently, $\mathcal{T}or_i^{Y,f,g}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0$ for every integer $i > 0$ (see [Hal17, Appendix C] for details). Note that if g is flat, then it is tor-independent of every f. The projection formula of Corollary 4.12 is powerful enough to prove a very general tor-independent base-change result which extends Theorem 2.6(iv).

Corollary 4.13. *Fix the following* 2*-cartesian square of algebraic stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If f and g are tor-independent and f is concentrated, then there is a natural quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$:

$$\mathsf{L}g_{\mathrm{qc}}^*\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \simeq \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.$$

Proof. By Theorem 2.6(iv), the result can be verified smooth-locally on Y and Y'. Thus, we may assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. In particular, g and g' are affine. Since g is affine, $\mathsf{R}(g_{\mathrm{qc}})_*$ is conservative (Corollary 2.8). Hence, it is sufficient to verify that the morphism in question is a quasi-isomorphism after application of the functor $\mathsf{R}(g_{\mathrm{qc}})_*$. By the projection formula applied to g and then f, there are natural quasi-isomorphisms

$$\begin{aligned} \mathsf{R}(g_{\mathrm{qc}})_*\mathsf{L}g_{\mathrm{qc}}^*\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} &\simeq \mathsf{R}(f_{\mathrm{qc}})_*\mathcal{M} \otimes_{\mathcal{O}_Y} \mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'} \\ &\simeq \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'}). \end{aligned}$$

Note, however, that because f and g are tor-independent and g is affine, the natural map

$$\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'} \to \mathsf{R}(g'_{\mathrm{qc}})_*\mathcal{O}_{X'}$$

is a quasi-isomorphism. Indeed, this may be verified smooth-locally on X, so we may assume that $X = \operatorname{Spec} C$. The morphism in question corresponds to the map $C \otimes^{\mathsf{L}}_A A' \to (C \otimes_A A')[0]$ in $\mathsf{D}(C)$, which is a quasi-isomorphism because f and g are tor-independent. With the projection formula and functoriality, we now obtain the following natural sequence of quasi-isomorphisms:

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$$\begin{aligned}\mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}(g_{\mathrm{qc}})_*\mathcal{O}_{Y'}) &\simeq \mathsf{R}(f_{\mathrm{qc}})_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}(g'_{\mathrm{qc}})_*\mathcal{O}_{X'})\\ &\simeq \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{R}(g'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}\\ &\simeq \mathsf{R}(g_{\mathrm{qc}})_*\mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{M}.\end{aligned}$$

The result follows. □

Note that in the setting of derived algebraic geometry, tor-independence is not necessary to obtain a base-change result [BFN10, Proposition 3.10].

4.6 Finite duality

Using the projection formula, we can also establish finite duality.

THEOREM 4.14. *Let $f\colon X \to Y$ be a concentrated morphism of algebraic stacks.*

(i) $\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y)$ *admits a right adjoint* $f^\times$.

(ii) *For every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$, there is a natural quasi-isomorphism*

$$\mathsf{R}(f_{\mathrm{qc}})_*f^\times(\mathcal{M}) \simeq \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X, \mathcal{M}).$$

(iii) *If f is affine, then for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$ there is a natural quasi-isomorphism*

$$f^\times(\mathcal{M}) \simeq \overline{f}^*\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}),$$

where $\overline{f}^*$ is the functor from Corollary 2.7.

(iv) *If f is affine and $f_*\mathcal{O}_X$ is perfect, then for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$ there is a natural quasi-isomorphism*

$$f^\times(\mathcal{O}_Y) \otimes^{\mathsf{L}}_{\mathcal{O}_X} f^*(\mathcal{M}) \simeq f^\times(\mathcal{M}).$$

In particular, $f^\times$ preserves small coproducts. Moreover, $f^\times$ is compatible with tor-independent base change on Y. If in addition f is surjective, then $f^\times$ is conservative.

Proof. By Theorem 2.6(iii), the functor $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts. Since $\mathsf{D}_{\mathrm{qc}}(X)$ is well generated [HNR14, Theorem B.1], the existence of $f^\times$ follows from [Nee01b, Proposition 1.20]. Now fix $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(Y)$; then there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{N}, \mathsf{R}(f_{\mathrm{qc}})_*f^\times(\mathcal{M})) &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{N}, f^\times(\mathcal{M}))\\ &\cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathsf{L}f^*_{\mathrm{qc}}\mathcal{N}, \mathcal{M})\\ &\cong \mathrm{Hom}_{\mathcal{O}_Y}((\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{N}, \mathcal{M})\\ &\cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{N}, \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X, \mathcal{M})).\end{aligned}$$

The penultimate isomorphism follows from the projection formula (Corollary 4.12). By the Yoneda lemma, this proves (ii).

We now address (iii). Let $\tilde{f}^\times\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$ be the functor

$$\tilde{f}^\times(\mathcal{M}) = \overline{f}^*\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}),$$

where $\overline{f}^*$ comes from the equivalence of Corollary 2.7. We claim that there is a natural transformation of functors $\tilde{f}^\times \Rightarrow f^\times$. To see this, let $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$; then there are natural morphisms

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$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}, \tilde{f}^\times(\mathcal{M})) &\to \operatorname{Hom}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}, \mathsf{R}(f_{\mathrm{qc}})_*\tilde{f}^\times(\mathcal{M}))\\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}, \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}))\\ &\to \operatorname{Hom}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}, \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{M}))\\ &\cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{N}, \mathcal{M})\\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}, f^\times(\mathcal{M})).\end{aligned}$$

By the Yoneda lemma, we have the claim. Since f is affine, to prove that the natural transformation $\tilde{f}^\times \Rightarrow f^\times$ is an isomorphism, it is sufficient to prove that it is after application of $\mathsf{R}(f_{\mathrm{qc}})_*$ (Lemma 2.8). This follows from the definition of $\tilde{f}^\times$ and (ii).

We now treat (iv). By adjunction and the projection formula (Corollary 4.12) we obtain a natural morphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$:

$$f^\times(\mathcal{O}_Y) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(\mathcal{M}) \to f^\times(\mathcal{M}).$$

Since f is affine, it is sufficient to prove that this morphism is an isomorphism after application of $\mathsf{R}(f_{\mathrm{qc}})_*$ (Lemma 2.8). By (ii) and the projection formula (Corollary 4.12), we see that it is sufficient to prove that the induced morphism

$$\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{M} \to \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M})$$

is a quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y)$. But $f_*\mathcal{O}_X$ is perfect, so Lemma 4.3(ii) now gives the claim.

For the compatibility of $f^\times$ with base change, we consider the following tor-independent 2-cartesian diagram of algebraic stacks.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Adjointness and tor-independent base change (Corollary 4.13) provide a natural transformation $\mathsf{L}g'^*_{\mathrm{qc}}f^\times \to (f')^\times \mathsf{L}g^*_{\mathrm{qc}}$ of functors that we must show is an isomorphism. Tor-independent base change also implies that there is a quasi-isomorphism: $\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X \simeq \mathsf{R}(f'_{\mathrm{qc}})_*\mathsf{L}g'^*_{\mathrm{qc}}\mathcal{O}_X$. Since $\mathsf{R}(f_{\mathrm{qc}})_*\mathcal{O}_X$ is perfect, it follows that $\mathsf{R}(f'_{\mathrm{qc}})_*\mathcal{O}_{X'}$ is perfect. By the formula just determined for $f^\times$, we thus see that it is sufficient to prove that $\mathsf{L}g'^*_{\mathrm{qc}}f^\times(\mathcal{O}_Y) \to (f')^\times \mathsf{L}g^*_{\mathrm{qc}}(\mathcal{O}_Y)$ is a quasi-isomorphism. Since f (and so f') is affine, it is sufficient to verify this after application of $\mathsf{R}(f'_{\mathrm{qc}})_*$. This observation, together with (ii) and tor-independent base change, shows that it is sufficient to prove that the morphism

$$\mathsf{L}g^*_{\mathrm{qc}}\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_{Y'}}(\mathsf{L}g^*_{\mathrm{qc}}(f_*\mathcal{O}_X), \mathcal{O}_{Y'})$$

is a quasi-isomorphism. But $f_*\mathcal{O}_X$ and $\mathsf{L}g^*_{\mathrm{qc}}(f_*\mathcal{O}_X)$ are both perfect and so dualizable (Lemma 4.3). In particular, the derived pullback of the dual of $f_*\mathcal{O}_X$ coincides with the dual of $\mathsf{L}g^*_{\mathrm{qc}}(f_*\mathcal{O}_X)$. It follows that the asserted map is a quasi-isomorphism and the claim follows.

Finally, we address the conservativity. For this, it is sufficient to observe that if f is surjective, then $\operatorname{supph}(f_*\mathcal{O}_X) = |Y|$. But $\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}) \simeq 0$ if and only if $\mathcal{M} \simeq 0$ (Lemma 4.9). □

Corollary 4.15. *If $f\colon X \to Y$ is a finite and faithfully flat morphism of finite presentation between algebraic stacks, then the functor*

$$f^\times(\mathcal{M}) = \overline{f}^*\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}), \quad \text{where } \mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(Y),$$

is right adjoint to $\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y)$. Moreover, $f^\times$ is compatible with arbitrary base change on Y, $f^\times(\mathcal{O}_Y) \otimes^{\mathsf{L}}_{\mathcal{O}_X} f^*(-) \simeq f^\times(-)$, preserves small coproducts, and is conservative.

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4.7 Coherent functors

Combining the strong projection formula of Proposition 4.11 with the characterization of compact objects in Lemma 4.5, we can prove most of Theorem D.

Corollary 4.16. *Let A be a noetherian ring and let $\pi\colon X \to \operatorname{Spec} A$ be a morphism of finite type between noetherian algebraic stacks. Suppose that:*

(i) *for every $i \geqslant 0$ and $\mathcal{M} \in \mathsf{Coh}(X)$, the cohomology $H^i(X_{\text{lis-ét}}, \mathcal{M})$ is a finitely generated A-module (e.g., π proper); and*
(ii) *$\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated.*

Then, for every $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, the functor

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))\colon \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

is coherent.

Proof. We begin by observing that the coherent functors $\mathsf{Mod}(A) \to \mathsf{Mod}(A)$ constitute a full abelian subcategory of the category of A-linear functors, which is closed under products (where everything is computed 'pointwise') [Hal14, Example 4.9]. Let $\mathcal{T} \subseteq \mathsf{D}_{\mathrm{qc}}(X)$ denote the full subcategory with objects those $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(X)$ where the functor $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))$ is coherent for every $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. In particular, $\mathcal{T}$ is closed under small coproducts, shifts, and triangles. By Corollary 3.14, it is enough to prove that $\mathcal{T}$ contains the compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$. If $\mathcal{Q} \in \mathsf{D}_{\mathrm{qc}}(X)$ is compact, then the strong projection formula (Proposition 4.11) implies that there is a natural quasi-isomorphism

$$\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \otimes^{\mathsf{L}}_A I \simeq \mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}} I).$$

Since $\mathcal{Q}$ is compact, it is perfect (Lemma 4.4(i)) and so $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. The assumption on preservation of coherence implies that $\mathsf{R}(\pi_{\mathrm{qc}})_*$ sends $\mathsf{D}^+_{\mathsf{Coh}}(X)$ to $\mathsf{D}^+_{\mathsf{Coh}}(A)$. In particular, $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \simeq \mathsf{R}(\pi_{\mathrm{qc}})_* \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \in \mathsf{D}^+_{\mathsf{Coh}}(A)$. By Lemma 4.5, we also have $\mathsf{RHom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G}) \in \mathsf{D}^b(A)$. Thus, the functor $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}\pi^*_{\mathrm{qc}}(-))$ is coherent [Hal14, Example 4.13] and we deduce the result. □

5. Presheaves of triangulated categories

Throughout this section we fix a small category $\mathcal{D}$ that admits all finite limits. Let **TCat** denote the 2-category of triangulated categories. A *$\mathcal{D}$-presheaf of triangulated categories* is a 2-functor $\mathcal{T}\colon \mathcal{D}^\circ \to \mathbf{TCat}$.

Given a morphism $f\colon U \to V$ in $\mathcal{D}$, there is an induced pullback functor $f^*_{\mathcal{T}}\colon \mathcal{T}(V) \to \mathcal{T}(U)$. When there is no cause for confusion, we will suppress the subscript $\mathcal{T}$ from $f^*_{\mathcal{T}}$. For any such f (not necessarily a monomorphism), we let

$$\mathcal{T}_{V \setminus U}(V) = \ker(f^*\colon \mathcal{T}(V) \to \mathcal{T}(U)).$$

We say that $\mathcal{T}$ *has adjoints* if for every morphism $f\colon U \to V$ in $\mathcal{D}$, the pullback functor $f^*\colon \mathcal{T}(V) \to \mathcal{T}(U)$ admits a right adjoint $f_*\colon \mathcal{T}(U) \to \mathcal{T}(V)$.

Definition 5.1. Suppose that $\mathcal{T}$ is a $\mathcal{D}$-presheaf of triangulated categories with adjoints. Let $f\colon U \to V$ be a morphism in $\mathcal{D}$ and let $N \in \mathcal{T}(V)$. We denote by $\eta^f_N\colon N \to f_* f^* N$ the

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unit of the adjunction. A morphism $g\colon W \to V$ in $\mathcal{D}$ is $\mathcal{T}$-*preflat* if for every cartesian square in $\mathcal{D}$

$$\begin{array}{ccc} U_W & \xrightarrow{g_U} & U \\ {\scriptstyle f_W}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{g} & V \end{array}$$

the natural transformation $g^* f_* \to (f_W)_*(g_U)^*$ is an isomorphism. A morphism $g\colon W \to V$ in $\mathcal{D}$ is $\mathcal{T}$-*flat* if for every morphism $V' \to V$, the pullback $g'\colon W' \to V'$ of g is $\mathcal{T}$-preflat.

Note that because monomorphisms are stable under base change, $\mathcal{T}$-flat monomorphisms are stable under base change.

Example 5.2. Let Y be an algebraic stack that is quasi-compact and quasi-separated. Let $\mathbf{Rep}^{\mathrm{fp}}/Y$ denote the category of 1-morphisms $X \to Y$ that are representable and of finite presentation. The category $\mathbf{Rep}^{\mathrm{fp}}/Y$ is small. We have a $\mathbf{Rep}^{\mathrm{fp}}/Y$-presheaf of triangulated categories $\mathsf{D}_{\mathrm{qc}}\colon (\mathbf{Rep}^{\mathrm{fp}}/Y)^\circ \to \mathbf{TCat}$ that sends $X \to Y$ to $\mathsf{D}_{\mathrm{qc}}(X)$ and a 1-morphism $f\colon X' \to X$ in $\mathbf{Rep}^{\mathrm{fp}}/Y$ to $\mathsf{L}f_{\mathrm{qc}}^*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X')$. The functor $\mathsf{L}f_{\mathrm{qc}}^*$ admits a right adjoint $\mathsf{R}(f_{\mathrm{qc}})_*$, so D_{qc} is a presheaf with adjoints.

By Theorem 2.6(iv), if f is a flat morphism, then it is D_{qc}-flat. Conversely, if $f\colon X' \to X$ is D_{qc}-flat, then f is flat. Indeed, this is local on the source and target of f, so it is sufficient to show that if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is D_{qc}-preflat, then B is a flat A-algebra. For this, we note that if I is an ideal of A, then corresponding to $i\colon \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ we see that there is a quasi-isomorphism $(A/I) \otimes_A^{\mathsf{L}} B \simeq (B/IB)[0]$. That is, for all $n > 0$ and ideals I of A, we have that $\operatorname{Tor}_A^n(B, A/I) = 0$; hence, B is flat over A. It follows that the D_{qc}-flat monomorphisms are the quasi-compact open immersions [EGA, IV.17.9.1].

Example 5.3. Our notion of $\mathcal{T}$-flatness is not always optimal. In particular, it is weaker than expected in the derived setting. If $\mathcal{T}$ is a presheaf of triangulated categories *with t-structures*, then a better definition is that f is $\mathcal{T}$-flat if f^* is t-exact.

To illustrate this, suppose that $\mathcal{D} = \mathsf{SCR}^\circ$ is the ∞-category of affine derived schemes, that is, the opposite category to the ∞-category of simplicial commutative rings. Further, let $\mathcal{T} = \mathsf{Mod}(-)$ be the functor that takes a simplicial commutative ring A to the stable ∞-category $\mathsf{Mod}(A)$ of (not necessarily connective) A-modules. Then every morphism in $\mathcal{D}$ is $\mathcal{T}$-flat whereas $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat exactly when the pullback $B \overset{\mathsf{L}}{\otimes}_A -\colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$ is t-exact. Nevertheless, just as in the non-derived case, the finitely presented $\mathcal{T}$-flat monomorphisms are exactly the quasi-compact open immersions since every monomorphism of derived schemes is formally étale [TV08, 2.2.2.5(2)].

Lemma 5.4. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories with adjoints. Fix the following commutative diagram in $\mathcal{D}$.*

$$\begin{array}{ccc} W_V & \xrightarrow{j_W} & W \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & X \end{array}$$

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(i) *If j is a $\mathcal{T}$-preflat monomorphism in $\mathcal{D}$, then the adjunction $j^*j_* \to \mathrm{Id}_{\mathcal{T}(V)}$ is an isomorphism.*
(ii) *$f^*\colon \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}(W)$ factors through $\mathcal{T}_{W\setminus W_V}(W)$.*
(iii) *If the diagram is cartesian and j is $\mathcal{T}$-preflat, then the functor $f_*\colon \mathcal{T}_{W\setminus W_V}(W) \to \mathcal{T}(X)$ factors through $\mathcal{T}_{X\setminus V}(X)$ and is right adjoint to $f^*\colon \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}_{W\setminus W_V}(W)$.*

Proof. If $j\colon V \to X$ is a monomorphism, then the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Id}_V} & V \\ {\scriptstyle \mathrm{Id}_V}\downarrow & & \downarrow{\scriptstyle j} \\ V & \xrightarrow{j} & X \end{array}$$

is cartesian, whence $j^*j_* \simeq (\mathrm{Id}_V)_*(\mathrm{Id}_V)^* \simeq \mathrm{Id}_{\mathcal{T}(V)}$.

By functoriality, (ii) is trivial. For (iii), given $M \in \mathcal{T}_{W\setminus W_V}(W)$, then $j^*f_*M \simeq (f_V)_*(j_W)^*M \simeq 0$ and hence $f_*M \in \mathcal{T}_{X\setminus V}(X)$. □

DEFINITION 5.5. Fix a $\mathcal{D}$-presheaf of triangulated categories $\mathcal{T}$ with adjoints. A *Mayer–Vietoris $\mathcal{T}$-square* is a cartesian diagram in $\mathcal{D}$:

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

satisfying the following three conditions:

(i) j is a $\mathcal{T}$-flat monomorphism;
(ii) the natural transformation $f^*j_* \to j'_*f_U^*$ is an isomorphism; and
(iii) the induced functor $f^*\colon \mathcal{T}_{X\setminus U}(X) \to \mathcal{T}_{X'\setminus U'}(X')$ is an equivalence of categories.

Condition (ii) for a Mayer–Vietoris $\mathcal{T}$-square is satisfied if f is a $\mathcal{T}$-(pre)flat morphism. By tor-independent base change (Corollary 4.13), if $\mathcal{T} = \mathsf{D}_{\mathrm{qc}}$, then condition (ii) is satisfied for every f. In [HR16], we will consider applications of these Mayer–Vietoris triangles to a result of Moret-Bailly [Mor96]. For this intended application, it is essential that we permit f to be non-flat.

Example 5.6. We continue with Example 5.2. Let $f\colon X' \to X$ be a representable, quasi-compact, and quasi-separated étale neighborhood of a closed subset $|Z| \subseteq |X|$ with quasi-compact complement $|U|$. Let $j\colon U \hookrightarrow X$ be the resulting quasi-compact open immersion. Then the cartesian square

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

is a Mayer–Vietoris D_{qc}-square. To see this, it remains to prove that the functor $\mathsf{L}f_{\mathrm{qc}}^*$ induces the desired equivalence. Now the exact functor $f^*\colon \mathsf{Mod}(X) \to \mathsf{Mod}(X')$ admits an exact left adjoint $f_!\colon \mathsf{Mod}(X') \to \mathsf{Mod}(X)$ [Sta, 03DI]. Explicitly, for $M \in \mathsf{Mod}(X')$, we have that $f_!M$ is the sheafification of the presheaf

$$(V \to X) \longmapsto \bigoplus_{\phi \in \mathrm{Hom}_X(V,X')} M(V \xrightarrow{\phi} X').$$

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Note that the natural map $M \to f^* f_! M$ is an isomorphism for all $M \in \mathsf{Mod}(X')$ such that $j'^* M = 0$. Also, if $N \in \mathsf{Mod}(X)$ and $j^* N = 0$, then the natural map $f_! f^* N \to N$ is an isomorphism. The exactness of the adjoint pair $(f_!, f^*)$ now gives an adjoint pair on the level of derived categories $(f_!, f^*)\colon \mathsf{D}(X) \leftrightarrows \mathsf{D}(X')$ and that the relations just given also hold on the derived category. Next, we observe that the restriction of f^* to $\mathsf{D}_{\mathrm{qc}}(X)$ coincides with $\mathsf{L}f^*_{\mathrm{qc}}$. Thus, it remains to prove that if $M \in \mathsf{D}_{\mathrm{qc}}(X')$ and $\mathsf{L}(j')^*_{\mathrm{qc}} M \simeq 0$, then $f_! M \in \mathsf{D}_{\mathrm{qc}}(X)$. The exactness of $f_!$ and $\mathsf{L}(j')^*_{\mathrm{qc}}$ shows that is sufficient to prove this result when M is a quasi-coherent sheaf such that $(j')^* M = 0$. Note that $(U \xrightarrow{j} X, X' \xrightarrow{f} X)$ is an étale cover of X, $j^* f_! M = (f_U)_! (j')^* M = 0$, and $f^* f_! M \cong M$. We deduce that étale-locally $f_! M$ is quasi-coherent. By descent, $f_! M$ is quasi-coherent and the result is proved. For a different proof in a more general context, see [HR16, Proposition 4.2].

Example 5.7 (Étale cohomology). Let Y be an algebraic stack that is quasi-compact and quasi-separated. Let Λ be a noetherian ring such that Λ is torsion and $|\Lambda|$ is invertible on Y. We have the derived category $\mathsf{D}^+_{\mathrm{cart}}(Y, \Lambda)$ of bounded below lisse-étale Λ-modules on Y with cartesian cohomology [LM00, 12.10]. More generally, we have a $\mathbf{Rep}^{\mathrm{fp}}/Y$-presheaf of triangulated categories $\mathsf{D}^+_{\mathrm{cart}}(-, \Lambda)\colon (\mathbf{Rep}^{\mathrm{fp}}/Y)^\circ \to \mathbf{TCat}$ that sends $X \to Y$ to $\mathsf{D}^+_{\mathrm{cart}}(X, \Lambda)$ and a 1-morphism $f\colon X' \to X$ in $\mathbf{Rep}^{\mathrm{fp}}/Y$ to $f^*\colon \mathsf{D}^+_{\mathrm{cart}}(X, \Lambda) \to \mathsf{D}^+_{\mathrm{cart}}(X', \Lambda)$ [Ols07, 9.16]. By smooth base change, f_* takes cartesian sheaves to cartesian sheaves, so the functor f^* admits a right adjoint $\mathsf{R}f_*$ [Ols07, Proposition 9.9], i.e., $\mathsf{D}^+_{\mathrm{cart}}(-, \Lambda)$ is a presheaf with adjoints. Moreover, étale and smooth morphisms are $\mathsf{D}^+_{\mathrm{cart}}(-, \Lambda)$-flat.

If $f\colon X' \to X$ is an étale neighborhood of $|Z| \subseteq |X|$ as in the previous example, then the resulting square is a Mayer–Vietoris $\mathsf{D}^+_{\mathrm{cart}}(-, \Lambda)$-square. Indeed, if $i\colon Z \to X$ is a closed immersion for some scheme structure on Z, then i^* and i_* induce an equivalence of categories $\mathsf{D}^+_{\mathrm{cart}, |X \setminus U|}(X, \Lambda) \cong \mathsf{D}^+_{\mathrm{cart}}(Z, \Lambda)$.

We also have a subpresheaf $\mathsf{D}^+_{\mathrm{c}}(-, \Lambda)$, where $\mathsf{D}^+_{\mathrm{c}}(X, \Lambda) \subseteq \mathsf{D}^+_{\mathrm{cart}}(X, \Lambda)$ consists of the objects with constructible cohomology sheaves [LM00, 18.6]. If Y is quasi-excellent of finite dimension, $\Lambda = \mathbb{Z}/N\mathbb{Z}$, and N is invertible on Y, then $\mathsf{D}^+_{\mathrm{c}}(-, \Lambda)$ has adjoints (Deligne–Gabber's finiteness theorem) and an étale neighborhood is a Mayer–Vietoris $\mathsf{D}^+_{\mathrm{c}}(-, \Lambda)$-square.

These results are also valid for unbounded derived categories [LZ12, 6.3.3–6.3.4].

Example 5.8. Consider the following 2-commutative diagram of triangulated functors

$$\begin{array}{ccc} \mathcal{R}' & \xleftarrow{j'^*} & \mathcal{S}' \\ {\scriptstyle r^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathcal{R} & \xleftarrow{j^*} & \mathcal{S} \end{array}$$

and assume that they all have right adjoints, which we will denote as f_*, j_*, r_*, and j'_*, respectively. Let $\mathcal{K} = \ker(j^*)$ and $\mathcal{K}' = \ker(j'^*)$. Let $\mathcal{D}$ be the category consisting of the following objects and arrows.

$$\begin{array}{ccc} R' & \xrightarrow{j'} & S' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{j} & S \end{array}$$

There is a $\mathcal{D}$-presheaf of triangulated categories $\mathcal{T}$ with adjoints such that $\mathcal{T}(S) = \mathcal{S}$ etc. The square above is a Mayer–Vietoris $\mathcal{T}$-square if and only if the following conditions are satisfied:

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(i) the natural transformations $j^*j_* \to \mathrm{Id}$ and $j'^*j'_* \to \mathrm{Id}$ are isomorphisms;
(ii) the natural transformation $j^*f_* \to r_*j'^*$ is an isomorphism;
(iii) the natural transformation $f^*j_* \to j'_*r^*$ is an isomorphism; and
(iv) the induced functor $\mathcal{K} \to \mathcal{K}'$ is an equivalence of categories.

Condition (i) implies that $\mathcal{R} = \mathcal{S}/\mathcal{K}$, $\mathcal{R}' = \mathcal{S}'/\mathcal{K}'$ and $\mathcal{K}$, $\mathcal{K}'$ are Bousfield subcategories of $\mathcal{S}$, $\mathcal{S}'$ respectively (Lemma 3.4 and [Nee01b, ch. 9]).

Mayer–Vietoris $\mathcal{T}$-squares give rise to many nice properties. In particular, we obtain a familiar distinguished triangle.

LEMMA 5.9. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories with adjoints. Consider the following Mayer–Vietoris $\mathcal{T}$-square in $\mathcal{D}$.*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

(i) *If $N \in \mathcal{T}(X)$, then there is a unique map d that makes the triangle*

$$N \xrightarrow{\begin{pmatrix} \eta^j_N \\ \eta^f_N \end{pmatrix}} j_*j^*N \oplus f_*f^*N \xrightarrow{\begin{pmatrix} \eta^f_{j_*j^*N} & -f_*f^*\eta^j_N \end{pmatrix}} f_*f^*j_*j^*N \xrightarrow{d} N[1]$$

distinguished. Moreover, this d is functorial in N.

(ii) *Let $M \in \mathcal{T}_{X'\setminus U'}(X')$ and let $N \in \mathcal{T}(X)$. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, N) \cong \mathrm{Hom}_{\mathcal{T}(X')}(M, f^*N).$$

(iii) *Given $N_U \in \mathcal{T}(U)$, $N' \in \mathcal{T}(X')$, and an isomorphism $\delta\colon j'^*N' \to f_U^*N_U$, define N by a distinguished triangle in $\mathcal{T}(X)$:*

$$N \longrightarrow j_*N_U \oplus f_*N' \xrightarrow{\begin{pmatrix} \eta^f_{j_*N_U} & -\alpha \end{pmatrix}} f_*f^*j_*N_U \longrightarrow N[1],$$

where $\alpha\colon f_*N' \to f_*f^*j_*N_U$ is the composition:

$$f_*N' \xrightarrow{f_*\eta^{j'}_{N'}} f_*j'_*j'^*N' \xrightarrow{f_*j'_*\delta} f_*j'_*f_U^*N_U \cong f_*f^*j_*N_U.$$

Then the induced maps $j^*N \to N_U$ and $f^*N \to N'$ are isomorphisms.

(iv) *If $N \in \mathcal{T}(X)$ satisfies $j^*N \in \mathcal{T}(U)^c$, $f^*N \in \mathcal{T}(X')^c$, and $f_U^*j^*N \in \mathcal{T}(U')^c$, then $N \in \mathcal{T}(X)^c$.*

Proof. An equivalent formulation of (i) is that

$$\begin{array}{ccc} N & \xrightarrow{\eta^j_N} & j_*j^*N \\ {\scriptstyle \eta^f_N}\downarrow & & \downarrow{\scriptstyle \eta^f_{j_*j^*N}} \\ f_*f^*N & \xrightarrow{f_*f^*\eta^j_N} & f_*f^*j_*j^*N \end{array}$$

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is a homotopy cartesian square [Nee01b, Definition 1.4.1] whose differential d is unique and is functorial in N. To see that the square is cartesian, first choose C such that we have a distinguished triangle

$$C \xrightarrow{l} N \xrightarrow{\eta^j_N} j_*j^*N \xrightarrow{m} C[1].$$

Since j is a $\mathcal{T}$-flat monomorphism, $j^*\eta^j_N$ is an isomorphism (Lemma 5.4(i)). It follows that $j^*C \cong 0$, so $C \in \mathcal{T}_{X\setminus U}(X)$, and η^f_C is an isomorphism. We thus obtain the following morphism of distinguished triangles

$$\begin{array}{ccccccc} C & \xrightarrow{l} & N & \xrightarrow{\eta^j_N} & j_*j^*N & \xrightarrow{m} & C[1] \\ \| & & \downarrow{\scriptstyle \eta^f_N} & & \downarrow{\scriptstyle u} & & \| \\ C & \xrightarrow{\eta^f_N \circ l} & f_*f^*N & \xrightarrow{f_*f^*\eta^j_N} & f_*f^*j_*j^*N & \xrightarrow{(\eta^f_{C[1]})^{-1}\circ f_*f^*m} & C[1] \end{array}$$

for some morphism u. We can certainly let $u = \eta^f_{j_*j^*N}$ and we will soon see that this is actually the only possible u. On the other hand, we can choose u such that the middle square is a homotopy cartesian square by the octahedral axiom [Nee01b, Lemma 1.4.3]. After applying j_*j^* to the middle square and adjoining to it the natural square relating u and j_*j^*u, we obtain the following commutative diagram.

$$\begin{array}{ccccc} j_*j^*N & \xrightarrow{j_*j^*\eta^j_N} & j_*j^*j_*j^*N & \xleftarrow{\eta^j_{j_*j^*N}} & j_*j^*N \\ \downarrow{\scriptstyle j_*j^*\eta^f_N} & & \downarrow{\scriptstyle j_*j^*u} & & \downarrow{\scriptstyle u} \\ j_*j^*f_*f^*N & \xrightarrow{j_*j^*f_*f^*\eta^j_N} & j_*j^*f_*f^*j_*j^*N & \xleftarrow{\eta^j_{f_*f^*j_*j^*N}} & f_*f^*j_*j^*N \end{array}$$

Since j is a $\mathcal{T}$-flat monomorphism, the horizontal maps are all isomorphisms and it follows that $u = \eta^f_{j_*j^*N}$. Moreover, it is readily verified that the induced differential

$$d := l[1] \circ (\eta^f_C)^{-1}[1] \circ f_*f^*m \colon f_*f^*j_*j^*N \to N[1]$$

is independent of the choice of the triangle $C \xrightarrow{l} N \xrightarrow{\eta^j_N} j_*j^*N \xrightarrow{m} C[1]$. The functoriality of the Mayer–Vietoris triangle now follows from the construction. Finally, to show that d is unique, if $d' \colon f_*f^*j_*j^*N \to N[1]$ is another morphism that makes a distinguished triangle, then there is an induced morphism of distinguished triangles as follows.

$$\begin{array}{ccccccc} N & \longrightarrow & j_*j^*N \oplus f_*f^*N & \longrightarrow & f_*f^*j_*j^*N & \xrightarrow{d} & N[1] \\ \| & & \| & & \downarrow{\scriptstyle \theta} & & \| \\ N & \longrightarrow & j_*j^*N \oplus f_*f^*N & \longrightarrow & f_*f^*j_*j^*N & \xrightarrow{d'} & N[1] \end{array}$$

It remains to show that θ is the identity morphism. Splitting up the sum in the middle square, we obtain the following commutative diagram.

$$\begin{array}{ccc} f_*f^*N & \xrightarrow{f_*f^*\eta^j_N} & f_*f^*j_*j^*N \\ \| & & \downarrow{\scriptstyle \theta} \\ f_*f^*N & \xrightarrow{f_*f^*\eta^j_N} & f_*f^*j_*j^*N \end{array}$$

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Applying j_*j^* to this diagram, we may append another square on the right to obtain the commutative diagram

$$\begin{array}{ccccc} j_*j^*f_*f^*N & \xrightarrow{j_*j^*f_*f^*\eta^j_N} & j_*j^*f_*f^*j_*j^*N & \xleftarrow{\eta^j_{f_*f^*j_*j^*N}} & f_*f^*j_*j^*N \\ \| & & \downarrow{\scriptstyle j_*j^*\theta} & & \downarrow{\scriptstyle \theta} \\ j_*j^*f_*f^*N & \xrightarrow{j_*j^*f_*f^*\eta^j_N} & j_*j^*f_*f^*j_*j^*N & \xleftarrow{\eta^j_{f_*f^*j_*j^*N}} & f_*f^*j_*j^*N \end{array}$$

where the horizontal arrows are isomorphisms. It follows that θ is the identity.

To obtain the isomorphism in (ii), first note that by the definition of a Mayer–Vietoris square, the co-unit $f^*f_*M \to M$ is an isomorphism since $M \in \mathcal{T}_{X'\setminus U'}(X')$. This gives us a natural isomorphism

$$\mathrm{Hom}_{\mathcal{T}(X')}(M, f^*N) \cong \mathrm{Hom}_{\mathcal{T}(X')}(f^*f_*M, f^*N) \cong \mathrm{Hom}_{\mathcal{T}(X)}(f_*M, f_*f^*N).$$

Now apply the homological functor $\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, -)$ to the triangle $N \to j_*j^*N \oplus f_*f^*N \to f_*f^*j_*j^*N$ from (i). Since $j^*f_*M \simeq 0$, we obtain an isomorphism $\mathrm{Hom}_{\mathcal{T}(X)}(f_*M, N) \cong \mathrm{Hom}_{\mathcal{T}(X)}(f_*M, f_*f^*N)$ and the result follows.

For (iii), the natural maps $v_j\colon j^*N \to N_U$ and $v_f\colon f^*N \to N'$ are obtained by adjunction from the maps $v_j^\vee\colon N \to j_*N_U$ and $v_f^\vee\colon N \to f_*N'$ in the defining triangle of N. The defining triangle exhibits N as a homotopy pullback. We may thus find a morphism of distinguished triangles as follows [Nee01b, Lemma 1.4.4] (octahedral axiom).

$$\begin{array}{ccccccc} C & \longrightarrow & N & \xrightarrow{v_j^\vee} & j_*N_U & \longrightarrow & C[1] \\ \| & & \downarrow{\scriptstyle v_f^\vee} & & \downarrow{\scriptstyle \eta^f_{j_*N_U}} & & \| \\ C & \longrightarrow & f_*N' & \xrightarrow{\alpha} & f_*f^*j_*N_U & \longrightarrow & C[1] \end{array}$$

Since j is a $\mathcal{T}$-flat monomorphism, we have that $j^*\alpha$ is an isomorphism, so $j^*C \cong 0$. It follows that $j^*v_j^\vee\colon j^*N \to j^*j_*N_U$ is an isomorphism and hence that $v_j\colon j^*N \to N_U$ is an isomorphism (Lemma 5.4(i)).

Now, by (i), we have the following morphism of distinguished triangles.

$$\begin{array}{ccccccc} N & \longrightarrow & j_*j^*N \oplus f_*f^*N & \longrightarrow & f_*f^*j_*j^*N & \xrightarrow{d} & N[1] \\ \| & & \downarrow{\scriptstyle j_*v_j \oplus f_*v_f} & & \downarrow{\scriptstyle \theta} & & \| \\ N & \longrightarrow & j_*N_U \oplus f_*N' & \longrightarrow & f_*f^*j_*N_U & \longrightarrow & N[1] \end{array}$$

As before, it follows that $\theta = f_*f^*j_*v_j$ by considering the application of j_*j^* to the middle square. Since v_j is an isomorphism, it follows that f_*v_f is an isomorphism. Now, if we let W be a cone of v_f, then $f_*W \simeq 0$. We will be done if we can show that j'^*v_f is an isomorphism. Indeed, it would then follow that $j'^*W \simeq 0$ and so $W \simeq f^*f_*W \simeq f^*0 \simeq 0$. To this end, since the following diagram commutes:

$$\begin{array}{ccc} j'^*f^*N & \xrightarrow{j'^*v_f} & j'^*N' \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \delta} \\ f_U^*j^*N & \xrightarrow{f_U^*v_j} & f_U^*N_U \end{array}$$

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and all appearing morphisms except j'^*v_f are known to be isomorphisms, it follows that j'^*v_f is an isomorphism.

For (iv), let $h \in \{j, f, j \circ f_U\}$. Since h^* admits a right adjoint, it commutes with small coproducts. Thus, if $\{Q_\lambda\}$ is a set of objects of $\mathcal{T}(X)$, then

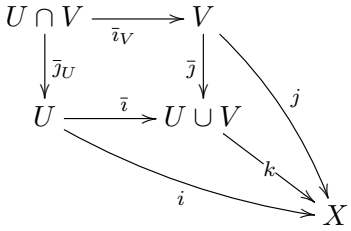
$$\begin{aligned}\bigoplus_\lambda \operatorname{Hom}(N, h_*h^*Q_\lambda) &\cong \bigoplus_\lambda \operatorname{Hom}(h^*N, h^*Q_\lambda)\\ &\cong \operatorname{Hom}\left(h^*N, \bigoplus_\lambda h^*Q_\lambda\right)\\ &\cong \operatorname{Hom}\left(h^*N, h^*\left(\bigoplus_\lambda Q_\lambda\right)\right)\\ &\cong \operatorname{Hom}\left(N, h_*h^*\left(\bigoplus_\lambda Q_\lambda\right)\right).\end{aligned}$$

The result now follows by consideration of the Mayer–Vietoris triangles associated to Q_λ and $\bigoplus_\lambda Q_\lambda$, together with the long exact sequence given by the homological functor $\operatorname{Hom}(N, -)$. □

In the following definition, we axiomatize the required properties of open immersions of algebraic stacks.

Definition 5.10. Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories with adjoints. Let $\mathcal{L}$ be a collection of morphisms in $\mathcal{D}$. We say that $\mathcal{L}$ *supports* $\mathcal{T}$ if it satisfies the following five conditions:

(i) if $j\colon U \to X$ belongs to $\mathcal{L}$, then j is a $\mathcal{T}$-flat monomorphism;
(ii) if $j\colon U \to V$ is an isomorphism, then j belongs to $\mathcal{L}$;
(iii) if $j\colon U \to V$ and $k\colon V \to X$ belong to $\mathcal{L}$, then $k \circ j$ belongs to $\mathcal{L}$;
(iv) if $j\colon U \to X$ belongs to $\mathcal{L}$ and $f\colon X' \to X$ is a morphism in $\mathcal{D}$, then the induced morphism $j'\colon U \times_X X' \to X'$ belongs to $\mathcal{L}$; and
(v) if $i\colon U \to X$ and $j\colon V \to X$ belong to $\mathcal{L}$, then there exists a commutative diagram



where k belongs to $\mathcal{L}$, and the square is a Mayer–Vietoris $\mathcal{T}$-square.

Note that $U \cap V = U \times_X V$.

Example 5.11. We continue Example 5.6. Let $\mathcal{I}$ be the collection of morphisms in $\mathbf{Rep}^{\mathrm{fp}}/Y$ that are open immersions. By Example 5.6 and standard arguments, $\mathcal{I}$ supports D_{qc}.

We now have a straightforward lemma.

Lemma 5.12. *Let $\mathcal{T}$ be a $\mathcal{D}$-presheaf of triangulated categories with adjoints. Let $\mathcal{L}$ be a collection of morphisms in $\mathcal{D}$ that supports $\mathcal{T}$. If $i\colon U \to X$ and $j\colon V \to X$ belong to $\mathcal{L}$, then:*

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(i) $\mathcal{T}_{X\setminus U\cup V}(X) = \mathcal{T}_{X\setminus U}(X) \cap \mathcal{T}_{X\setminus V}(X)$; *and*
(ii) *there is an exact sequence of triangulated categories:*

$$\mathcal{T}_{X\setminus U\cup V}(X) \to \mathcal{T}_{X\setminus V}(X) \xrightarrow{i^*} \mathcal{T}_{U\setminus U\cap V}(U).$$

Proof. (i) Certainly, we have $\mathcal{T}_{X\setminus U\cup V}(X) \subseteq (\ker i^*) \cap (\ker j^*)$. For the other inclusion, let $M \in (\ker i^*) \cap (\ker j^*)$. If $k\colon U \cup V \to X$ denotes the morphism induced by Definition 5.10(v), then $k^*M \in \mathcal{T}(U \cup V)$. Now let $\overline{k}\colon U \cap V \to U \cup V$ denote the induced morphism. There is a triangle in $\mathcal{T}(U \cup V)$:

$$k^*M \to \bar{\imath}_*\bar{\imath}^*k^*M \oplus \bar{\jmath}_*\bar{\jmath}^*k^*M \to \overline{k}_*\overline{k}^*k^*M.$$

Functoriality induces isomorphisms $\bar{\imath}^*k^* \simeq i^*$, $\bar{\jmath}^*k^* \simeq j^*$, and $\overline{k}^*k^* \simeq \bar{\jmath}_U^*i^*$. By hypothesis, i^*M and j^*M vanish, so the triangle gives $k^*M \simeq 0$. Hence, $M \in \mathcal{T}_{X\setminus U\cup V}(X)$.

(ii) Lemma 5.4(ii) shows that the functor $i^*\colon \mathcal{T}_{X\setminus V}(X) \to \mathcal{T}(U)$ factors through $\mathcal{T}_{U\setminus U\cap V}(U)$ and the kernel is $(\ker i^*) \cap (\ker j^*) = \mathcal{T}_{X\setminus U\cup V}(X)$ by (i). Also, Lemma 5.4(iii) shows that $i_*\colon \mathcal{T}_{U\setminus U\cap V}(U) \to \mathcal{T}_{X\setminus V}(X)$ is a right adjoint to i^* and Lemma 5.4(i) shows that the natural transformation $i^*i_* \to \mathrm{Id}$ is an isomorphism. It follows that the sequence is exact by Lemma 3.4. □

6. Descent of compact generation

For this section we fix a small category $\mathcal{D}$ that admits all finite limits. We also fix a collection $\mathcal{L}$ of morphisms in $\mathcal{D}$.

Definition 6.1. An *admissible* $(\mathcal{L}, \mathcal{D})$*-presheaf* of triangulated categories is a $\mathcal{D}$-presheaf $\mathcal{T}$ of triangulated categories with adjoints (§ 5) satisfying:

(i) for all $X \in \mathcal{D}$, the triangulated category $\mathcal{T}(X)$ is closed under small coproducts;
(ii) for all $(f\colon X \to Y) \in \mathcal{D}$, the pushforward $f_*\colon \mathcal{T}(X) \to \mathcal{T}(Y)$ preserves small coproducts; and
(iii) $\mathcal{L}$ supports $\mathcal{T}$ (Definition 5.10).

Example 6.2. We continue Example 5.11: D_{qc} is an admissible $(\mathcal{I}, \mathbf{Rep}^{\mathrm{fp}}/Y)$-presheaf of triangulated categories. Indeed, the only non-trivial condition is that f_* preserves small coproducts, which follows from Theorem 2.6(iii) since representable morphisms are concentrated.

Definition 6.3. Let β be a cardinal and let $X \in \mathcal{D}$. We say that an admissible $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories $\mathcal{T}$ is *compactly generated with* $\mathcal{L}$*-supports by* β *objects at* X if for every $j\colon V \to X$ in $\mathcal{L}$ the triangulated category $\mathcal{T}_{X\setminus V}(X)$ is generated by a set of cardinality $\leqslant\beta$ whose elements have compact image in $\mathcal{T}(X)$.

In practice, $\mathcal{D}$ will often contain an initial object $\emptyset$ and, for every $X \in \mathcal{D}$, it will be the case that $(\emptyset \to X) \in \mathcal{L}$ and $\mathcal{T}(\emptyset) \simeq 0$. Hence, in this situation, $\mathcal{T}(X) = \mathcal{T}_{X\setminus\emptyset}(X)$ is also compactly generated by a set of cardinality $\leqslant\beta$. Also, observe that if β is a finite cardinal, then $\mathcal{T}$ can always be compactly generated with supports by *one* object at X. In this section we will give conditions on $\mathcal{T}$ that guarantee that the condition of compact generation with $\mathcal{L}$-supports by β objects descends along certain morphisms and diagrams in $\mathcal{D}$.

Our first result is of an elementary nature and is similar to the arguments of Toën [Toë12, Lemma 4.11]. First we require a definition.

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Definition 6.4. Let $\mathcal{T}$ be an admissible $(\mathcal{L},\mathcal{D})$-presheaf of triangulated categories. A morphism $f\colon X'\to X$ in $\mathcal{D}$ is $\mathcal{T}$-*quasi-perfect with respect to* $\mathcal{L}$ if the following three conditions are satisfied:

(i) f is $\mathcal{T}$-flat (Definition 5.1);
(ii) if $P\in\mathcal{T}(X')^c$, then $f_*P\in\mathcal{T}(X)^c$;
(iii) f_* admits a right adjoint $f^\times$ such that for every $j\colon V\to X$ in $\mathcal{L}$, the restriction of $f^\times$ to $\mathcal{T}_{X\setminus V}(X)$ factors through $\mathcal{T}_{X'\setminus V\times_X X'}(X')$.

By Example 3.8, a potentially easy way to verify condition (ii) above is for $f^\times$ to preserve small coproducts. To verify condition (iii) above, it is sufficient to prove the following: for every $j\colon V\to X$ in $\mathcal{L}$, if $j'\colon V'\to X'$ is the pullback of j along f and $f_V\colon V'\to V$ is the projection to V, then f_* and $(f_V)_*$ both admit right adjoints and the natural transformation $j'^*f^\times\to f_V^\times j^*$ is an isomorphism of functors.

Example 6.5. We continue with Example 6.2. If $q\colon W'\to W$ is a finite and faithfully flat morphism of finite presentation, then q is D_{qc}-quasi-perfect with respect to $\mathcal{I}$ (Corollary 4.15). In [HR15, Appendix A], we prove that if $q\colon W'\to W$ is a proper, smooth, and locally schematic morphism of noetherian algebraic stacks, then q is D_{qc}-quasi-perfect with respect to $\mathcal{I}$.

We now have the first important result of this section.

Proposition 6.6. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L},\mathcal{D})$-presheaf of triangulated categories. Let $f\colon X'\to X$ be a morphism in $\mathcal{D}$ that is $\mathcal{T}$-quasi-perfect with respect to $\mathcal{L}$. If the functor $f^\times$ (which exists because f is $\mathcal{T}$-quasi-perfect) is conservative and $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X', then $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X. In fact, if $j\colon V\to X$ belongs to $\mathcal{L}$, let $V'=X'\times_X V$ and let $\mathcal{B}'\subseteq\mathcal{T}(X')^c\cap\mathcal{T}_{X'\setminus V'}(X')$ be a subset of cardinality $\leqslant\beta$ generating $\mathcal{T}_{X'\setminus V'}(X')$; then $f_*\mathcal{B}'\subseteq\mathcal{T}(X)^c\cap\mathcal{T}_{X\setminus V}(X)$ and $f_*\mathcal{B}'$ generates $\mathcal{T}_{X\setminus V}(X)$.*

Proof. It suffices to prove the latter assertion. Set $\mathcal{B}=f_*\mathcal{B}'=\{f_*P : P\in\mathcal{B}'\}$. Then $\mathcal{B}$ has cardinality $\leqslant\beta$, and $\mathcal{B}\subseteq\mathcal{T}_{X\setminus V}(X)$ by Lemma 5.4(iii). Moreover, $\mathcal{B}\subseteq\mathcal{T}(X)^c$, since f is $\mathcal{T}$-quasi-perfect with respect to $\mathcal{L}$. It remains to show that $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$. Let $N\in\mathcal{T}_{X\setminus V}(X)$ satisfy $\operatorname{Hom}_{\mathcal{T}(X)}(f_*P[n],N)=0$ for all $P\in\mathcal{B}'$ and all $n\in\mathbb{Z}$. Since f is $\mathcal{T}$-quasi-perfect with respect to $\mathcal{L}$, it follows that $f^\times N\in\mathcal{T}_{X'\setminus V'}(X')$. As $\mathcal{B}'$ is generating for $\mathcal{T}_{X'\setminus V'}(X')$, we may conclude that $f^\times N\simeq 0$. By assumption, $f^\times$ is conservative. Thus, $N\simeq 0$ and $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$. □

Our next descent result is deeper, relying on Thomason's localization theorem 3.12. First, however, we require a lemma.

Lemma 6.7. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L},\mathcal{D})$-presheaf of triangulated categories. Suppose that $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at $X\in\mathcal{D}$. Let $W\to V$ and $V\to X$ belong to $\mathcal{L}$. Then the following hold:*

(i) *$\mathcal{T}_{X\setminus W}(X)$ is closed under small coproducts and the subcategory $\mathcal{T}_{X\setminus V}(X)\subseteq\mathcal{T}_{X\setminus W}(X)$ is localizing;*
(ii) *$\mathcal{T}_{X\setminus V}(X)$ is the localizing envelope of a set of compact objects of $\mathcal{T}_{X\setminus W}(X)$; and*
(iii) *$\mathcal{T}_{X\setminus V}(X)^c=\mathcal{T}_{X\setminus W}(X)^c\cap\mathcal{T}_{X\setminus V}(X)$.*

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Proof. First, observe that $\mathcal{T}(X)$ is closed under small coproducts. Also, if $f\colon X' \to X$ is a morphism in $\mathcal{D}$, then f^* admits a right adjoint, so f^* preserves small coproducts. Hence, we see that $\mathcal{T}_{X\setminus X'}(X)$ is a localizing subcategory of $\mathcal{T}(X)$. In particular, $\mathcal{T}_{X\setminus X'}(X)$ is closed under small coproducts. The claim (i) is now immediate.

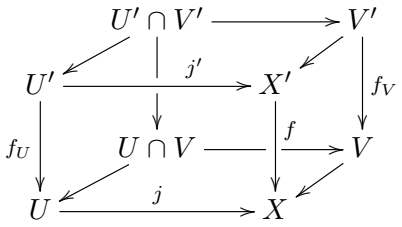
By hypothesis, $\mathcal{T}_{X\setminus V}(X)$ is generated by a subset R that has compact image in $\mathcal{T}(X)$ and hence also in $\mathcal{T}_{X\setminus W}(X)$. Let $\mathcal{R} \subseteq \mathcal{T}_{X\setminus W}(X)$ denote the localizing envelope of R; then $\mathcal{R} \subseteq \mathcal{T}_{X\setminus V}(X)$. Applying Corollary 3.14 to $R \subseteq \mathcal{T}_{X\setminus V}(X)$, we find that $\mathcal{R} = \mathcal{T}_{X\setminus V}(X)$, proving (ii). The claim (iii) is now an immediate consequence of (ii) and Thomason's theorem 3.12. □

PROPOSITION 6.8. *Let β be a cardinal. Let $\mathcal{T}$ be an admissible $(\mathcal{L},\mathcal{D})$-presheaf of triangulated categories. Consider a Mayer–Vietoris $\mathcal{T}$-square (Definition 5.5)*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

with $j \in \mathcal{L}$. If $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at U and X', then $\mathcal{T}$ is compactly generated with $\mathcal{L}$-supports by β objects at X.

Proof. Let $V \to X$ belong to $\mathcal{L}$. Form the following cartesian cube.



By Lemma 5.12, we have an exact sequence

$$\mathcal{T}_{X'\setminus U'\cup V'}(X') \to \mathcal{T}_{X'\setminus V'}(X') \to \mathcal{T}_{U'\setminus U'\cap V'}(U'). \tag{6.1}$$

The category $\mathcal{T}_{X'\setminus V'}(X')$ is compactly generated and, by Lemma 6.7(i), it is also closed under small coproducts. By Lemma 6.7(ii), the subcategory $\mathcal{T}_{X'\setminus U'\cup V'}(X') \subseteq \mathcal{T}_{X'\setminus V'}(X')$ is the localizing envelope of a set of compact objects of $\mathcal{T}_{X'\setminus V'}(X')$.

Now let $P \in \mathcal{T}(U)^c \cap \mathcal{T}_{U\setminus U\cap V}(U)$. Then $f_U^*P \in \mathcal{T}(U')^c$ since $(f_U)_*$ preserves coproducts (Example 3.8) and $f_U^*P \in \mathcal{T}_{U'\setminus U'\cap V'}(U')$ (Lemma 5.4(ii)). Thus, $f_U^*P \in \mathcal{T}_{U'\setminus U'\cap V'}(U')^c$ by Lemma 6.7(iii). We now apply Thomason's localization theorem, in the form of Corollary 3.13, to the exact sequence (6.1). This gives us $P' \in \mathcal{T}_{X'\setminus V'}(X')^c = \mathcal{T}(X')^c \cap \mathcal{T}_{X'\setminus V'}(X')$ and an isomorphism $j'^*P' \simeq f_U^*(P \oplus P[1])$. As in Lemma 5.9(iii), form the following triangle in $\mathcal{T}(X)$:

$$\tilde{P} \to j_*(P \oplus P[1]) \oplus f_*P' \to f_*f^*j_*(P \oplus P[1]).$$

By Lemma 5.9((iii),(iv)), we have that $j^*\tilde{P} \simeq P \oplus P[1]$ and $f^*\tilde{P} \simeq P'$ and that $\tilde{P} \in \mathcal{T}(X)^c$. Since $j_*j^*\tilde{P}$, $f_*f^*\tilde{P}$ and $f_*f^*j_*j^*\tilde{P} \in \mathcal{T}_{X\setminus V}(X)$, it follows that $\tilde{P} \in \mathcal{T}_{X\setminus V}(X)$.

Now let $Q \in \mathcal{T}_{X'\setminus U'\cup V'}(X')$ and note that $f_*Q \in \mathcal{T}_{X\setminus U\cup V}(X)$ (Lemma 5.12(i)). Moreover, $f^*f_*Q \to Q$ is an isomorphism, because $f^* : \mathcal{T}_{X\setminus U}(X) \to \mathcal{T}_{X'\setminus U'}(X')$ is an equivalence of categories. We also have that $j^*f_*Q \simeq 0$ and $j'^*f^*f_*Q \simeq 0$. Thus, if in addition $Q \in \mathcal{T}(X')^c$, then $f_*Q \in \mathcal{T}(X)^c$ by Lemma 5.9(iv).

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By hypothesis, there is a subset $\mathcal{B}_0 \subseteq \mathcal{T}(U)^c \cap \mathcal{T}_{U\setminus U\cap V}(U)$ (respectively $\mathcal{B}_1 \subseteq \mathcal{T}(X')^c \cap \mathcal{T}_{X'\setminus U'\cup V'}(X')$) of cardinality $\leqslant\beta$ generating $\mathcal{T}_{U\setminus U\cap V}(U)$ (respectively $\mathcal{T}_{X'\setminus U'\cup V'}(X')$). Define

$$\mathcal{B} = \{\tilde{P} : P \in \mathcal{B}_0\} \cup \{f_*Q : Q \in \mathcal{B}_1\}.$$

If β is infinite, then the cardinality of $\mathcal{B}$ is $\leqslant\beta$ and, if β is finite, then the same is true of $\mathcal{B}$. By the above considerations, $\mathcal{B} \subseteq \mathcal{T}(X)^c \cap \mathcal{T}_{X\setminus V}(X)$ and it remains to show that $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$.

Let $M \in \mathcal{T}_{X\setminus V}(X)$, so that $f^*M \in \mathcal{T}_{X'\setminus V'}(X')$ and $j^*M \in \mathcal{T}_{U\setminus U\cap V}(U)$. Suppose that $\operatorname{Hom}_{\mathcal{T}(X)}(f_*Q[n], M) = 0$ for all $Q \in \mathcal{B}_1$ and all $n \in \mathbb{Z}$. By Lemma 5.9(ii), we see that $\operatorname{Hom}_{\mathcal{T}(X')}(Q[n], f^*M) = 0$ for all $Q \in \mathcal{B}_1$ and all $n \in \mathbb{Z}$. Let K be a cone of $f^*M \to j'_*j'^*f^*M$. Note that

$$\operatorname{Hom}_{\mathcal{T}(X')}(Q[n], j'_*j'^*f^*M) = \operatorname{Hom}_{\mathcal{T}(X')}(j'^*Q[n], j'^*f^*M) = 0,$$

so $\operatorname{Hom}_{\mathcal{T}(X')}(Q[n], K) = 0$. Since $K \in \mathcal{T}_{X'\setminus U'\cup V'}(X')$ and $\mathcal{B}_1$ is generating, we see that $K \simeq 0$, so that $f_*f^*M \to f_*j'_*j'^*f^*M \simeq f_*f^*j_*j^*M$ is an isomorphism. From the Mayer–Vietoris triangle $M \to j_*j^*M \oplus f_*f^*M \to f_*f^*j_*j^*M$, we deduce that the natural map $M \to j_*j^*M$ is an isomorphism for all such M.

Now suppose that M also satisfies $\operatorname{Hom}_{\mathcal{T}(X)}(\tilde{P}[n], M) = 0$ for all $P \in \mathcal{B}_0$ and $n \in \mathbb{Z}$. Since the natural map $M \to j_*j^*M$ is an isomorphism, it follows that $\operatorname{Hom}_{\mathcal{T}(U)}(j^*\tilde{P}[n], j^*M) = 0$ for all $P \in \mathcal{B}_0$ and $n \in \mathbb{Z}$. By Lemma 5.9(iii), $j^*\tilde{P} \simeq P \oplus P[1]$ and so $\operatorname{Hom}_{\mathcal{T}(U)}(P[n], j^*M) = 0$ for all $P \in \mathcal{B}_0$ and all $n \in \mathbb{Z}$. By assumption, $\mathcal{B}_0$ generates $\mathcal{T}_{U\setminus U\cap V}(U)$ and thus $j^*M \simeq 0$. Since $M \simeq j_*j^*M \simeq 0$, we deduce that $\mathcal{B}$ generates $\mathcal{T}_{X\setminus V}(X)$. □

We are now in a position to prove the main technical result of the article.

Theorem 6.9. *Let X be a quasi-compact and quasi-separated algebraic stack. Let $\mathcal{D}$ be $\mathbf{Rep}^{\mathrm{fp}}/X$ or one of the full subcategories $\mathbf{Rep}^{\mathrm{fp,qff,sep}}/X$ or $\mathbf{Rep}^{\mathrm{fp,\acute{e}t,sep}}/X$. Let $\mathcal{I}$ denote the set of open immersions in $\mathcal{D}$. Let $\mathcal{T}$ be a presheaf of triangulated categories on $\mathcal{D}$. Assume that:*

(i) *$\mathcal{T}(W)$ is closed under small coproducts for all $W \in \mathcal{D}$;*

(ii) *for every morphism $f\colon W_1 \to W_2$ in $\mathcal{D}$, the pullback $f^*\colon \mathcal{T}(W_2) \to \mathcal{T}(W_1)$ admits a right adjoint f_* that preserves small coproducts;*

(iii) *for every cartesian square in $\mathcal{D}$*

$$\begin{array}{ccc} U_W & \xrightarrow{g_U} & U \\ {\scriptstyle f_W}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{g} & V \end{array}$$

such that g is flat, the natural transformation $g^*f_* \to (f_W)_*(g_U)^*$ is an isomorphism;

(iv) *for every open immersion $U \to W$ and étale neighborhood $f\colon W' \to W$ of $W\setminus U$, the pullback f^* induces an equivalence $\mathcal{T}_{W\setminus U}(W) \to \mathcal{T}_{W'\setminus U'}(W')$;*

(v) *for every finite faithfully flat morphism $W' \to W$ of finite presentation, the functor $f_*\colon \mathcal{T}(W') \to \mathcal{T}(W)$ admits a right adjoint $f^\times$ that preserves small coproducts, is conservative, and commutes with pullback along open immersions.*

Let $\mathcal{C} \subseteq \mathcal{D}$ be the collection of all objects W such that for every separated étale morphism $q\colon W' \to W$ in $\mathcal{D}$ and every open immersion $V' \to W'$ in $\mathcal{I}$, the triangulated category $\mathcal{T}_{W'\setminus V'}(W')$ is generated by a set of cardinality $\leqslant \beta$ whose elements have compact image in $\mathcal{T}(W')$.

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If $p\colon W \to X$ is a separated, quasi-finite, and faithfully flat morphism in $\mathcal{D}$ such that $W \in \mathcal{C}$, then $X \in \mathcal{C}$.

Proof. Condition (ii) says that $\mathcal{T}$ has adjoints. Condition (iii) says that flat morphisms are $\mathcal{T}$-flat. Conditions (iii) and (iv) imply that étale neighborhoods are Mayer–Vietoris squares. In particular, $\mathcal{I}$ supports $\mathcal{T}$. Combining this with conditions (i) and (ii), we conclude that $\mathcal{T}$ is an admissible $(\mathcal{I}, \mathcal{D})$-presheaf of triangulated categories.

By assumption, there exists an object $W \in \mathcal{C}$ with $W \to X$ separated, quasi-finite, and faithfully flat. We will apply [Ryd11, Theorem 6.1] to deduce that $X \in \mathcal{C}$. To do this, we need to verify the following three conditions for a flat morphism $q\colon W' \to W$ in $\mathcal{D}$:

(D1) if $W \in \mathcal{C}$ and q is étale and separated, then $W' \in \mathcal{C}$;

(D2) if $W' \in \mathcal{C}$ and q is finite and surjective, then $W \in \mathcal{C}$; and

(D3) if q is an étale neighborhood of $W \backslash U$, where $U \to W$ is an open immersion in $\mathcal{D}$, and U and W' belong to $\mathcal{C}$, then $W \in \mathcal{C}$.

Now (D1) tautologically follows from the definition of $\mathcal{C}$. For (D2), condition (v) implies that p is $\mathcal{T}$-quasi-perfect with respect to $\mathcal{I}$. By Proposition 6.6, we deduce that (D2) is satisfied. As noted previously, conditions (iii) and (iv) imply that étale neighborhoods are Mayer–Vietoris $\mathcal{T}$-squares. By Proposition 6.8, (D3) is satisfied. The result follows. □

7. Algebraic stacks with the β-resolution property

Let X be an algebraic stack. Recall that X is said to have the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module M is a quotient of a direct sum of locally free $\mathcal{O}_X$-modules of finite type. The resolution property is a subtle and difficult property, although it is always satisfied for quasi-projective schemes. It has been studied systematically by several authors, with notable contributions due to Thomason [Tho87], Totaro [Tot04], and Gross [Gro10, Gro17].

The following simple refinements of the resolution property will be useful for us. Let $\mathbf{VB}(X) \subseteq \mathsf{QCoh}(X)$ denote the subcategory of locally free $\mathcal{O}_X$-modules of finite type. Let β be a cardinal. We say that X has the *β-resolution property* if there exists a subset $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leqslant \beta$ with the property that every quasi-coherent $\mathcal{O}_X$-module M is a quotient of a direct sum of elements of $\mathcal{B}$. If, in addition, it can be arranged that $\mathcal{B}$ consists of vector bundles that are compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$, then we say that X has the *compact β-resolution property*.

If X is concentrated, then every locally free $\mathcal{O}_X$-module of finite type is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ (Lemma 4.4(iii)). In particular, if X also has the β-resolution property, then X has the compact β-resolution property. Since quasi-compact and quasi-separated schemes and algebraic spaces are concentrated, the compact β-resolution property and the β-resolution property coincide for schemes and algebraic spaces.

The following simple lemma will be important.

Lemma 7.1 [Gro17, Proposition 1.8(v)]. *Let $f\colon X \to Y$ be a quasi-affine morphism of algebraic stacks. Let β be a cardinal. If Y has the β-resolution property or the compact β-resolution property, then so does X.*

Proof. If $\mathcal{B} \subseteq \mathbf{VB}(Y)$ is a resolving set of cardinality β, then $f^*\mathcal{B} = \{f^*E : E \in \mathcal{B}\}$ is a resolving set of cardinality β. Indeed, f^* is right-exact and $f^*f_*M \to M$ is surjective for every $M \in \mathsf{QCoh}(X)$. In addition, if $\mathcal{B}$ consists of compact objects of $\mathsf{D}_{\mathrm{qc}}(Y)$, then $f^*\mathcal{B}$ consists of compact objects in $\mathsf{D}_{\mathrm{qc}}(X)$ (Example 3.9). □

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Remark 7.2. Similarly, there is the following partial converse: if $f\colon X \to Y$ is finite and faithfully flat of finite presentation and X has the β-resolution property or the compact β-resolution property, then so does Y. In this case, one takes $f_*\mathcal{B}$ as the resolving set and uses that f_* is right-exact and that $f_*f^!M \to M$ is surjective [Gro17, Proposition 1.13]. But $f^! = f^\times$ preserves coproducts (Corollary 4.15), so f_* preserves compact objects (Example 3.8).

Proposition 7.3. *Let X be an algebraic stack. If X is quasi-affine, then X has the* 1*-resolution property. If X is quasi-compact and quasi-separated with affine stabilizer groups, then the following are equivalent:*

(i) *X has the $\aleph_0$-resolution property;*
(ii) *X has the resolution property; and*
(iii) *$X = [V/\operatorname{GL}_{n,\mathbb{Z}}]$, where V is a quasi-affine scheme.*

When these conditions hold, X has affine diagonal.

Proof. The first statement follows from the fact that $\mathcal{O}_X \in \mathsf{QCoh}(X)$ is a generator if X is quasi-affine. Trivially, (i) $\implies$ (ii). That (ii) $\implies$ (iii) is Totaro's theorem [Tot04, Gro17]. Finally, to see that (iii) $\implies$ (i), it is enough to prove that $B\operatorname{GL}_{n,\mathbb{Z}}$ has the $\aleph_0$-resolution property since $X \to B\operatorname{GL}_{n,\mathbb{Z}}$ is quasi-affine. That $B\operatorname{GL}_{n,\mathbb{Z}}$ has the resolution property is a special case of [Tho87, Lemma 2.4]: every coherent sheaf on $B\operatorname{GL}_{n,\mathbb{Z}}$ is a quotient of a finite-dimensional subrepresentation of a finite number of copies of the regular representation. Since there is a countable number of vector bundles on $B\operatorname{GL}_{n,\mathbb{Z}}$, the $\aleph_0$-resolution property holds.

The last statement follows since $[V/\operatorname{GL}_{n,\mathbb{Z}}]$ has affine diagonal. □

Question 7.4. Every algebraic stack that admits a finite flat cover $V \to X$ with V quasi-affine has the compact 1-resolution property by Remark 7.2. Are all stacks with the (compact) 1-resolution property of this form? Is every algebraic space with the 1-resolution property quasi-affine?

Many quotient stacks have the resolution property.

Example 7.5. Let $S = \operatorname{Spec} R$ be a regular scheme of dimension at most 1 (e.g., $R = \mathbb{Z}$ or R is a field). Let $G \to S$ be a flat affine group scheme of finite type and let V be an algebraic space with an action of G. Then $X = [V/G]$ has the resolution property in the following cases [Tot04, Theorem 2.1].

(i) V is quasi-affine.
(ii) V is normal, noetherian, and has an ample family of line bundles (e.g., V is quasi-projective) and G is an extension of a finite flat group scheme by a smooth group scheme with connected fibers (this is automatic if R is a field).
(iii) V has an ample family of G-equivariant line bundles (e.g., V is quasi-projective and G is acting linearly).

For (i), use that BG_S has the resolution property [Tho87, Lemma 2.4] and Lemma 7.1. For (ii), use [Tho87, Lemmas 2.10 and 2.14] and for (iii) use [Tho87, Lemma 2.6]. Note that in (ii) it is crucial that V is normal to apply Sumihiro's theorem [Sum75, Theorem 1.6] and deduce that sufficiently high powers of the line bundles carry a G-action. In fact, this is false otherwise (cf. [Tot04, § 9]) and whether X has the resolution property in this case is not known in general. Alternatively, we could assume that V is a quasi-projective scheme with a linear action of G, as in [BFN10, Corollary 3.22].

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We conclude this section with a simple lemma.

LEMMA 7.6. *Let X be a quasi-compact and quasi-separated algebraic stack. The following conditions are equivalent.*

(i) *X has the compact β-resolution property.*
(ii) *X has the β-resolution property and there exists a vector bundle F such that $\operatorname{supp}(F) = X$ and F is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$.*
(iii) *There are a vector bundle F on X that is a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ and a subset $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leqslant \beta$ such that the set $\{F \otimes E\}_{E\in\mathcal{B}}$ generates $\mathsf{QCoh}(X)$.*
(iv) *There are a subset $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leqslant \beta$ that generates $\mathsf{QCoh}(X)$ and an integer $r \geqslant 0$ such that $\operatorname{Ext}^i_{\mathcal{O}_X}(E, N) = 0$ for all $i > r$, all $E \in \mathcal{B}$, and all $N \in \mathsf{QCoh}(X)$.*
(v) *There are a subset $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leqslant \beta$ that generates $\mathsf{QCoh}(X)$ and an integer $r \geqslant 0$ such that the natural map*

$$\tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_X}(E, \mathcal{M}) \to \tau^{\geqslant j}\mathsf{RHom}_{\mathcal{O}_X}(E, \tau^{\geqslant j-r}\mathcal{M})$$

is a quasi-isomorphism for all integers j, all $E \in \mathcal{B}$, and all $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$.

Proof. (i)⇒(ii): there is a surjection $F \twoheadrightarrow O_X$ with F a vector bundle on X that is compact in $\mathsf{D}_{\mathrm{qc}}(X)$. Note that $\operatorname{supp}(F) = X$.

(ii)⇒(iii): if $\mathcal{C} \subseteq \mathbf{VB}(X)$ is a resolving set of vector bundles, then so is the set $\{F \otimes F^\vee \otimes E\}_{E\in\mathcal{C}}$. Indeed, the evaluation map $F \otimes F^\vee \twoheadrightarrow \mathcal{O}_X$ is surjective. We take $\mathcal{B} = \{F^\vee \otimes E\}_{E\in\mathcal{C}}$.

(iii)⇒(iv): for all i, all $E \in \mathcal{B}$, and $N \in \mathsf{QCoh}(X)$, we have $\operatorname{Ext}^i_{\mathcal{O}_X}(F \otimes E, N) = \operatorname{Ext}^i_{\mathcal{O}_X}(F, E^\vee \otimes N)$. Now choose $r \geqslant 0$ as in Lemma 4.5 for F. An identical argument gives (iii)⇒(v). Also, (v)⇒(iv) is trivial.

(iv)⇒(i): immediate from Lemma 4.5. □

8. Crisp stacks

In this section we define β-crisp stacks and show that the compact β-resolution property implies β-crispness.

DEFINITION 8.1. Let β be a cardinal. Let X be an algebraic stack. We say that X satisfies the *β-Thomason condition* if:

(i) $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a set of cardinality $\leqslant \beta$; and
(ii) for every quasi-compact open immersion $j\colon U \hookrightarrow X$, there exists a perfect object P of $\mathsf{D}_{\mathrm{qc}}(X)$ with support $X\backslash U$.

We say that X is *β-crisp* if for every representable, étale, separated, and quasi-compact morphism $X' \to X$ the stack X' satisfies the β-Thomason condition.

By Lemmas 4.10 and 8.2, an equivalent definition for β-crispness is that for every representable étale morphism $W \to X$ that is quasi-compact and separated, and every quasi-compact open immersion $j\colon U \hookrightarrow W$, the triangulated category

$$\mathsf{D}_{\mathrm{qc},|W|\backslash|U|}(W) = \{M \in \mathsf{D}_{\mathrm{qc}}(W) : j^*M \simeq 0\}$$

is generated by a set of cardinality $\leqslant \beta$ consisting of compact objects of $\mathsf{D}_{\mathrm{qc}}(W)$.

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Lemma 8.2. *Let $f\colon X \to Y$ be a quasi-affine morphism of algebraic stacks. Let β be a cardinal. If $\mathsf{D}_{\mathrm{qc}}(Y)$ is compactly generated by β objects, then so is $\mathsf{D}_{\mathrm{qc}}(X)$. In fact, if $\mathcal{B} \subseteq \mathsf{D}_{\mathrm{qc}}(Y)^c$ is a subset that generates $\mathsf{D}_{\mathrm{qc}}(Y)$, then the set $\{\mathsf{L}f_{\mathrm{qc}}^* B : B \in \mathcal{B}\}$ compactly generates $\mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. Since f is concentrated, $\mathsf{R}(f_{\mathrm{qc}})_*$ preserves small coproducts. Since $\mathsf{R}(f_{\mathrm{qc}})_*$ is conservative (Corollary 2.8), the result follows from Example 3.11. □

Remark 8.3. We do not know if the analogue of Lemma 8.2 holds for the β-Thomason condition or even β-crispness.

The main result of this section is the following.

Proposition 8.4. *Let X be a quasi-compact algebraic stack with affine diagonal. Let β be a cardinal. If X has the compact β-resolution property, then it is β-crisp. In particular, concentrated stacks with the β-resolution property are β-crisp. In fact, $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by any resolving set of compact vector bundles.*

Proof. By Lemma 7.1, the compact β-resolution property is preserved under quasi-affine morphisms. By Zariski's main theorem [LM00, Théorèm 16.5], étale morphisms that are quasi-compact, separated, and representable are quasi-affine. Thus, it is enough to prove the following statement: if $j\colon V \hookrightarrow X$ is a quasi-compact open immersion with complement $|Z|$, then there exists a generating subset $\mathcal{B}_{|Z|} \subseteq \mathsf{D}_{\mathrm{qc},|Z|}(X)$, of cardinality $\leqslant \beta$, with compact image in $\mathsf{D}_{\mathrm{qc}}(X)$.

Choose a resolving set $\mathcal{B} \subseteq \mathbf{VB}(X)$ of cardinality $\leqslant \beta$ and an integer $r \geqslant 0$ as in Lemma 7.6(v). Let $M \in \mathsf{D}_{\mathrm{qc}}(X)$. We claim that if $n \in \mathbb{Z}$ is such that $\mathcal{H}^n(M) \neq 0$, then there exist an $E \in \mathcal{B}$ and a non-zero morphism $E[-n] \to M$ in $\mathsf{D}_{\mathrm{qc}}(X)$. We prove this claim by a small modification (which is likely well known, e.g., [DG13, Remark 1.2.10]) to the argument of Neeman [Nee96, Example 1.10].

Thus, for all $E \in \mathcal{B}$, all $n \in \mathbb{Z}$, and all $M \in \mathsf{D}_{\mathrm{qc}}(X)$, we have

$$\operatorname{Hom}_{\mathcal{O}_X}(E, M[n]) = \operatorname{Hom}_{\mathcal{O}_X}(E, \tau^{\geqslant -r} M[n]).$$

We may consequently assume that $M \in \mathsf{D}^+_{\mathrm{qc}}(X)$. By [Lur04, Theorem 3.8], the natural functor $\mathsf{D}^+(\mathsf{QCoh}(X)) \to \mathsf{D}^+_{\mathrm{qc}}(X)$ is an equivalence of triangulated categories. Hence, we are free to assume that M is a complex $(\cdots \to M^k \xrightarrow{d^k} M^{k+1} \to \cdots)$ of quasi-coherent $\mathcal{O}_X$-modules. By assumption, $\mathcal{H}^n(M) \neq 0$, so there exist $E \in \mathcal{B}$ and a morphism $E \to \ker(d^n)$ such that $E \to \ker(d^n) \to \mathcal{H}^n(M)$ is non-zero. The composition $E \to \ker(d^n) \to M^n$ thus induces a non-zero morphism $E \to M[n]$ in $\mathsf{D}_{\mathrm{qc}}(X)$ and we deduce the claim.

We now return to the proof of the proposition. The above considerations show that the set $\mathcal{B}$ compactly generates $\mathsf{D}_{\mathrm{qc}}(X)$. Now let $i\colon Z \hookrightarrow X$ be a closed immersion with support $|Z|$. Since X has the resolution property and $j\colon V \to X$ is quasi-compact, we may choose i such that the quasi-coherent ideal sheaf I defining Z in X is of finite type. It follows that there is a surjection $F \to I$, where F is a finite direct sum of objects of $\mathcal{B}$. Corresponding to the morphism $s\colon F \to \mathcal{O}_X$, we obtain a section $s^\vee \in \Gamma(X, F^\vee)$ with vanishing locus $|Z|$. If $K(s^\vee)$ is the resulting Koszul complex [FL85, IV.2], then $K(s^\vee)$ is a perfect complex on X with support $|Z|$. By Lemma 4.10(ii), we deduce the claim. □

We conclude this section with examples of algebraic stacks that are crisp.

Example 8.5. Let A be a ring. Then $\operatorname{Spec} A$ is 1-crisp. Indeed, $\operatorname{Spec} A$ has the 1-resolution property and is concentrated; thus, the result follows from Proposition 8.4.

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Example 8.6. Let X be a concentrated stack with affine stabilizers and the resolution property. Then X has the $\aleph_0$-resolution property and affine diagonal (Proposition 7.3) and hence is $\aleph_0$-crisp (Proposition 8.4).

Examples are stacks of the form $X = [V/G]$, where V and G are as in Example 7.5 and either $S = \operatorname{Spec}(\mathbb{Q})$ or G is linearly reductive (e.g., a torus). Indeed, under these assumptions on G, the classifying stack BG is concentrated, so X is concentrated since $X \to BG$ is representable. More generally, we can take any stack $X = [V/G]$ as in Example 7.5 with linearly reductive stabilizers. Such stacks are concentrated by [HR15, Theorem C].

Remark 8.7. Let X be a quasi-compact algebraic stack with affine diagonal and the resolution property. When X is concentrated, $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated (Proposition 8.4) and $\mathsf{D}_{\mathrm{qc}}(X)$ is an example of a unital algebraic stable homotopy category [HPS97, Definition 1.1.4]. Note that the localizing envelope of a set of compact generators is the whole category (Corollary 3.14).

The proof of Proposition 8.4 actually shows that even if X is not concentrated, then $\mathsf{D}(\mathsf{QCoh}(X))$ is perfectly generated. Note that since $\mathsf{D}(\mathsf{QCoh}(X))$ may not be compactly generated, Corollary 3.14 does not apply. Nonetheless, $\mathsf{D}(\mathsf{QCoh}(X))$ is well generated [Nee01a, Theorem 0.2] and there is a version of Corollary 3.14 for well-generated triangulated categories [Nee01b, Theorem 1.14]. This result and others are also discussed in [AJPV17]. Thus, $\mathsf{D}(\mathsf{QCoh}(X))$ is a non-algebraic stable homotopy category in the sense of [HPS97, Definition 1.1.4]. Note that this says nothing about perfect or compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$, because the functor $\mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ can fail to be fully faithful or essentially surjective (e.g., if $X = B\mathbb{G}_a$ in positive characteristic [HNR14]). Compact generation of $\mathsf{D}_{\mathrm{qc}}(X)$, however, is sufficient to prove that $\mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is an equivalence [HNR14].

9. Quasi-finite flat locality of β-crispness and applications

We are now in a position to prove Theorems A–C and address the applications mentioned in the Introduction.

Proof of Theorem C. Take $\mathcal{D} = \mathbf{Rep}^{\mathrm{fp}}/X$. By Examples 5.2, 5.6, 6.2, and 6.5, the $\mathcal{D}$-presheaf of triangulated categories D_{qc} satisfies conditions (i)–(v) of Theorem 6.9. The result now follows from Lemma 4.10 and Theorem 6.9. □

Proving Theorems A and B is now very simple.

Proof of Theorem A. By [Ryd11, Theorem 7.1], there exists a locally quasi-finite flat morphism $p\colon X' \to X$, where X' is a scheme. Since X is quasi-compact, we may further assume that X' is an affine scheme, and consequently the morphism p is also quasi-compact and separated. The result now follows by combining Example 8.5 with Theorem C. □

Proof of Theorem B. Since X is of s-global type, there exist an integer $N > 0$, a quasi-affine $\mathbb{Q}$-scheme V with an action of GL_N, together with an étale, representable, separated, and finitely presented morphism $p\colon [V/\mathrm{GL}_N] \to X$. Now the result follows from Theorem C and Example 8.6. □

We now recall some results of Sumihiro and Brion.

Proposition 9.1 (Sumihiro and Brion). *Let X be a variety over a field k. Let G be an affine algebraic k-group scheme acting on X. Assume that X is either:*

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(i) *geometrically normal; or*
(ii) *quasi-projective and:*
  (a) *geometrically seminormal; or*
  (b) $\operatorname{char} k = p > 0$*; or*
  (c) *the action is linearizable; or*
  (d) G^0 *is a torus.*

Then there exist a finite field extension k'/k, a quasi-projective variety W' over k' with a linear action of $G' = (G \otimes_k k')^0_{\mathrm{red}}$, and an étale G'-equivariant morphism $f\colon W' \to X_{k'}$.

Proof. Choose a finite field extension k'/k such that $X' = X \times_k k'$ is normal (respectively seminormal, respectively G' is smooth, respectively G' is a split torus). By construction, G' is then a smooth connected group scheme.

If X' is normal, then, by Sumihiro's theorem, we can choose f as a Zariski-open covering [Sum74, Lemma 8] (see [Sum75, Theorem 3.8] when k is not algebraically closed, or replace k with a finite field extension). If X' is seminormal and quasi-projective, then an étale f exists by Brion [Bri15, Theorem 4.7]. If $\operatorname{char} k > 0$, then an étale f exists by [Bri15, § 4.3]. If the action is already linearizable, then let f be the identity. If G' is a split torus, then an étale f, with W' affine, exists by [Bri15, Theorem 4.8]. □

In [AHR15, Theorem 2.6], it is proved that if G acts with linearly reductive stabilizers at closed points (e.g., if G is linearly reductive), then the result of Proposition 9.1 holds for any algebraic space of finite type (not necessarily normal, quasi-projective, or even separated). In particular, if G^0 is a torus, we may drop the requirement that X is quasi-projective.

The corollary of Theorem B is a special case of the following.

Corollary 9.2. *Let (X, G, k) and (W', G', k') be as in Proposition 9.1. Then:*

(i) *$[W'/G']$ has the resolution property;*
(ii) *the map $[W'/G'] \to [X'/G'] \to [X/G]$ is quasi-finite and faithfully flat; and*
(iii) *$[X/G]$ is of s-global type.*

If in addition, $\operatorname{char} k = 0$ or $(G \otimes_k \overline{k})^0_{\mathrm{red}}$ is a torus, then:

(iv) *$[W'/G']$ is concentrated and $\aleph_0$-crisp;*
(v) *$\mathsf{D}(\mathsf{QCoh}_G(X)) = \mathsf{D}_{\mathrm{qc}}([X/G])$ is compactly generated; and*
(vi) *for every G-invariant open subset $U \subseteq X$, there exists a compact perfect G-equivariant complex with support exactly $X \backslash U$.*

Proof. (i) is Example 7.5(iii), (ii) is by construction, and (iii) follows from (ii) and [Ryd15, Proposition 2.8(iii)]. Under the additional assumption on k and G, we have that BG' is concentrated [HR15, Theorem B] and hence so is $[W'/G']$ and (iv) follows from Proposition 8.4. It follows that $[X/G]$ is $\aleph_0$-crisp by Theorem C. We always have that $\mathsf{QCoh}_G(X) = \mathsf{QCoh}([X/G])$ and we have that $\mathsf{D}(\mathsf{QCoh}([X/G])) = \mathsf{D}_{\mathrm{qc}}([X/G])$ since $\mathsf{D}_{\mathrm{qc}}([X/G])$ is compactly generated [HNR14]. □

Example 9.3 (Brauer groups). Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Let $\alpha \in H^2(X, \mathbb{G}_m)$ be an element of the (bigger) cohomological Brauer group and let $\mathcal{X}$ denote the $\mathbb{G}_m$-gerbe corresponding to α. Since X has quasi-finite diagonal, there exists

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a quasi-finite flat presentation $p\colon X' \to X$ such that X' is affine and $p^*\alpha = 0$. Indeed, recall that X admits a quasi-finite flat presentation $q\colon U \to X$ by an affine scheme [Ryd15, Theorem 7.1] and then we may trivialize $q^*\alpha$ by a further surjective étale morphism of schemes. In particular, $\mathcal{X} \times_X X' = X' \times B\mathbb{G}_m$ has the resolution property and is cohomological affine. It follows that $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X X')$ and $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ have countable sets of compact generators by Proposition 8.4 and Theorem C.

Let $\mathcal{D} = \mathbf{Rep}^{\mathrm{fp}}/X$ and let $\mathcal{T} = \mathsf{D}(\mathsf{QCoh}^{\alpha}(-))$ be the presheaf of derived categories of α-twisted sheaves. Then the conditions of Theorem 6.9 are satisfied. Indeed, there is a canonical decomposition $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X T) = \mathsf{D}(\mathsf{QCoh}(\mathcal{X} \times_X T)) = \bigoplus_{m\in\mathbb{Z}} \mathsf{D}(\mathsf{QCoh}^{m\alpha}(T))$ which is respected by pullbacks.[1] Since $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X -)$ satisfies conditions (i)–(v) of Theorem 6.9, so does $\mathcal{T}$.

Thus, $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X')) = \mathsf{D}(\mathsf{QCoh}(X')) = \mathsf{D}_{\mathrm{qc}}(X')$ is compactly generated by one object with supports. It follows that $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X))$ is compactly generated by one object with supports (Theorem 6.9). The endomorphism algebra of this object, in a dg-enhancement of $\mathsf{D}(\mathsf{QCoh}^{\alpha}(X))$, is a derived Azumaya algebra [Toë12].

Alternatively, one could argue as follows. Take the degree-one part of the compact generators of $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$. This gives a countable generating set $\{P_i\}$ of compact objects in $\mathsf{D}^{\alpha}_{\mathrm{qc}}(X)$. For sufficiently large $n > 0$, the direct sum $P := \bigoplus_{i=1}^{n} P_i$ gives a compact object that locally generates $\mathsf{D}^{\alpha}_{\mathrm{qc}}(X)$. Indeed, since $\mathsf{D}^{\alpha}_{\mathrm{qc}}(X') = \mathsf{D}_{\mathrm{qc}}(X')$ and X' is affine, it is sufficient that $\operatorname{supph}(p^*P) = |X'|$ (Lemma 4.9). This compact local generator P is enough to produce a derived Azumaya algebra [Toë12, Proposition 4.6].

This latter argument also works for any quasi-compact and quasi-separated algebraic stack X such that $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ has a set of compact generators. In this case, take $p\colon X' \to X$ as a smooth presentation by an affine scheme such that $p^*\alpha = 0$. By [AHR15, Theorem 2.26], $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated for any algebraic stack X of finite type over a field, with affine diagonal, and linearly reductive stabilizers at closed points. We can thus conclude that for such X, every cohomological Brauer class comes from a derived Azumaya algebra.

Example 9.4 (Sheaves of linear categories on derived stacks). Let $(X, \mathcal{O}_X)$ be a derived (or spectral) Deligne–Mumford stack. The 0-truncation $(X, \pi_0\mathcal{O}_X)$ is an ordinary Deligne–Mumford stack with the same underlying topos X. In fact, even for a non-connective $\mathbb{E}_\infty$-algebra A, the category of étale A-algebras is equivalent to the category of étale $\pi_0 A$-algebras [Lur16a, Theorem 7.5.0.6].

Let $F \in \mathrm{QStk}(X)$ be a quasi-coherent stack on X [Lur11a, § 8], e.g., $F = \mathsf{QCoh}(X)$. For every object U in the small étale topos of X, this gives an $\mathcal{O}_X(U)$-linear ∞-category $F(U)$. Let hF be the presheaf of triangulated categories that assigns to each étale $U \to X$ the homotopy category of $F(U)$. Compact generation of $F(U)$ is a statement about its homotopy category [Lur16a, Remark 1.4.4.3]. Moreover, since the conditions (i)–(v) of Theorem 6.9 can all be verified étale-locally, it follows that Theorem 6.9 can be applied to hF to deduce compact generation of $F(X)$ from local compact generation of F.

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[1] Here we have tacitly used [HNR14] to identify $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X T) = \mathsf{D}(\mathsf{QCoh}(\mathcal{X} \times_X T))$, which holds since $\mathsf{D}_{\mathrm{qc}}(\mathcal{X} \times_X T)$ is compactly generated. In general, one could *define* $\mathsf{D}^{\alpha}_{\mathrm{qc}}(X)$ as the degree-one part of $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$.

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Appendix A. Generators from above

Let $w\colon W \to X$ be an étale, separated, finitely presented, and representable morphism of algebraic stacks. Define

$$\mathcal{P}'(w) = \{\mathsf{R}(w_{\mathrm{qc}})_*\mathsf{R}(u_{\mathrm{qc}})_*P : (u\colon U \to W) \in \mathbf{Rep}^{\mathrm{fp},\text{ét},\mathrm{sep}}/W \text{ and } P \in \mathsf{D}_{\mathrm{qc}}(U)^c\}$$

and let $\mathcal{P}(w) \subseteq \mathsf{D}_{\mathrm{qc}}(X)$ be the smallest thick subcategory of $\mathsf{D}_{\mathrm{qc}}(X)$ containing $\mathcal{P}'(w)$. The following lemma, requested by Neeman, is a natural generalization to algebraic stacks of a result that has found applications to Grothendieck duality for schemes [Nee14, Lemma 3.1].

Lemma A.1. *Let $w\colon W \to X$ be an étale, separated, finitely presented, representable, and surjective morphism of algebraic stacks. If X is quasi-compact and quasi-separated and $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated, then*

$$\mathsf{D}_{\mathrm{qc}}(X)^c \subseteq \mathcal{P}(w).$$

Proof. Let $\mathcal{C} \subseteq \mathbf{Rep}^{\mathrm{fp},\text{ét},\mathrm{sep}}/X$ be the subcategory with objects those V such that $\mathsf{D}_{\mathrm{qc}}(V)^c \subseteq \mathcal{P}(W \times_X V \to V)$. Note that if $W \times_X V \to V$ admits a section, then it is clear that $V \in \mathcal{C}$. In particular, it follows immediately that $W \in \mathcal{C}$. Now we will prove that $X \in \mathcal{C}$ using [Ryd11, Theorem 6.1]. To do this, we need to verify the following three conditions for a morphism $v\colon V' \to V$ in $\mathbf{Rep}^{\mathrm{fp},\text{ét},\mathrm{sep}}/X$:

(D1) if $V \in \mathcal{C}$, then $V' \in \mathcal{C}$;

(D2) if $V' \in \mathcal{C}$ and v is finite and surjective, then $V \in \mathcal{C}$; and

(D3) if v is an étale neighborhood of $V\backslash U$, where $j\colon U \to V$ is an open immersion in $\mathbf{Rep}^{\mathrm{fp},\text{ét},\mathrm{sep}}/X$, and U and V' belong to $\mathcal{C}$, then $V \in \mathcal{C}$.

For (D1), Lemma 8.2 and Theorem 3.12 imply that $\mathsf{D}_{\mathrm{qc}}(V')^c$ is the smallest thick subcategory of $\mathsf{D}_{\mathrm{qc}}(V')$ containing $\mathsf{L}v_{\mathrm{qc}}^*Q$, where $Q \in \mathsf{D}_{\mathrm{qc}}(V)^c$. A simple argument involving flat base change (Theorem 2.6(iv)) and the preservation of compact objects under concentrated morphisms (Example 3.9) now shows that $V' \in \mathcal{C}$. For (D2), Proposition 6.6 and Theorem 3.12 imply that $\mathsf{D}_{\mathrm{qc}}(V)^c$ is the smallest thick subcategory of $\mathsf{D}_{\mathrm{qc}}(V)$ containing the collection of complexes $\mathsf{R}(v_{\mathrm{qc}})_*Q$, where $Q \in \mathsf{D}_{\mathrm{qc}}(V')^c$. The property now follows from the trivial observation that

$$\{\mathsf{R}(v_{\mathrm{qc}})_*M : M \in \mathcal{P}(W \times_X V' \to V')\} \subseteq \mathcal{P}(W \times_X V \to V).$$

For (D3), we note that the Mayer–Vietoris triangle of Lemma 5.9(i) implies that if $P \in \mathsf{D}_{\mathrm{qc}}(V)^c$, then there is a distinguished triangle

$$P \to \mathsf{R}(j_{\mathrm{qc}})_*\mathsf{L}j_{\mathrm{qc}}^*P \oplus \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{L}f_{\mathrm{qc}}^*P \to \mathsf{R}(f_{\mathrm{qc}})_*\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}(j_{\mathrm{qc}})_*\mathsf{L}j_{\mathrm{qc}}^*P \to P[1].$$

Since $\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}(j_{\mathrm{qc}})_* \simeq \mathsf{R}(j'_{\mathrm{qc}})_*\mathsf{L}f'^*_{\mathrm{qc}}$, where $j'\colon U' = U \times_V V' \to V'$ and $f'\colon U' \to U$ are the projections, it follows immediately that $\mathsf{D}_{\mathrm{qc}}(V)^c$ is contained in the smallest thick subcategory of $\mathsf{D}_{\mathrm{qc}}(V)$ containing the objects:

- $\{\mathsf{R}(j_{\mathrm{qc}})_*M : M \in \mathcal{P}(W \times_X U \to U)\}$;
- $\{\mathsf{R}(f_{\mathrm{qc}})_*M : M \in \mathcal{P}(W \times_X V' \to V')\}$; and
- $\{\mathsf{R}((f \circ j')_{\mathrm{qc}})_*M : M \in \mathcal{P}(W \times_X U' \to U')\}$.

But all the objects above are contained in $\mathcal{P}(W \times_X V \to V)$ and the claim follows. Since $W \in \mathcal{C}$, we may now conclude that $X \in \mathcal{C}$. □

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