ON METABELIAN GROUPS

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1. Introduction

In this note we present some results on relationships between certain verbal subgroups of metabelian groups. To state these results explicitly we need some notation. As usual

$$x^{-1}y^{-1}xy = [x, y] \text{ and } [[x, y], z] = [x, y, z];$$

further $[x, 0y] = x$ and $[x, ky] = [x, (k-1)y, y]$ for all positive integers k. The s-th term $\gamma_s(G)$ of the lower central series of a group G is the subgroup of G generated by $[a_1, \cdots, a_s]$ for all $a_1, \cdots, a_s$ in G. A group G is metabelian if $[[a_1, a_2], [a_3, a_4]] = e$ (the identity element) for all a_1, a_2, a_3, a_4, in G, and has exponent k if $a^k = e$ for all a in G.

This investigation was motivated by the observation that, if a group G belongs to a proper subvariety of $\boldsymbol{A}_p\boldsymbol{A}_p$ (the variety of extensions of elementary abelian p-groups by elementary abelian p-groups), then for some positive integer s

$$[a_1, (p-1)a_2, (p-1)a_1, (p-1)a_3, \cdots, (p-1)a_s] = e$$

for all $a_1, \cdots, a_s$ in G. [2]

Let s be an integer greater than 1 and let $\boldsymbol{n} = (n_1, \cdots, n_s)$ be an ordered s-tuplet of positive integers. We write $\varepsilon_{\boldsymbol{n}}(G)$ for the subgroup of a group G generated by

$$[a_1, n_2a_2, (n_1-1)a_1, n_3a_3, \cdots, n_sa_s]$$

for all $a_1, \cdots, a_s$ in G. A group G is an $\varepsilon_{\boldsymbol{n}}$-group if $\varepsilon_{\boldsymbol{n}}(G)$ is the trivial subgroup.

THEOREM. *Let s be an integer greater than 1, let $\boldsymbol{n} = (n_1, \cdots, n_s)$ be an s-tuplet of positive integers, and let $m = \sum_{i=1}^{s} n_i$. In a metabelian $\varepsilon_{\boldsymbol{n}}$-group G*

(A) $\gamma_{m+1}(G)$ *has exponent dividing*

[1] Part of this work was done at the fifth Summer Research Institute of the Australian Mathematical Society.

[2] Now superseded by a complete description of the subvarieties of $\boldsymbol{A}_p\boldsymbol{A}_p$ (Kovács-Newman [2]) which, however, makes use of the results obtained here.

$$k = \frac{1}{n_1!} \prod_{i=1}^{s} \prod_{j=1}^{n_i} j!,$$

and (B) $\gamma_t(G)/\gamma_{t+1}(G)$ *has exponent dividing*

$$h = \frac{\text{g.c.d. } (n_1, n_2)}{n_1 n_2} \prod_{i=1}^{s} n_i!$$

for $t \in \{m, m+1, \cdots\}$ *except for* $s = 2, t = m$ *when the exponent divides*

$$\frac{n_1+n_2}{\text{g.c.d. } (n_1, n_2)} h.$$

This theorem is best possible in a number of senses:

1) $\gamma_{m-1}(G)/\gamma_m(G)$ need not have finite exponent — every nilpotent group of class $m-1$ is an ε_n-group;

2) if p is a prime divisor of h, then p may divide the exponent of $\gamma_t(G)/\gamma_{t+1}(G)$ for all t in $\{m, m+1, \cdots\}$ — the wreath product of a cyclic group of order p by a countably infinite elementary abelian p-group is then a metabelian ε_n-group;

3) if $s = 2$ and n_1+n_2 is a prime, then n_1+n_2 may divide the exponent of $\gamma_m(G)/\gamma_{m+1}(G)$ — Example 3.2;

4) in a non-metabelian ε_n-group $\gamma_m(G)/\gamma_{m+1}(G)$ need not have finite exponent — Example 3.1 exhibits a group of this kind which is both abelian-by-nilpotent-of-class-two and nilpotent-of-class-two-by-abelian.

The theorem includes as special cases all the related results we know: Theorem 1.10 of Gruenberg [1]; Weston [3]; and an unpublished result of Mrs. U. Heineken (verbally communicated by Dr. H. Heineken). Gruenberg's theorem gives a finite exponent for a term of the lower central series of a soluble Engel group — the term depending on the soluble length and the Engel condition. Our result improves the term of the lower central series involved for the metabelian case; using his techniques (embodied in his Lemma 4.4), we can improve his result also in the general soluble case —we will not, however, write these results down explicitly here as they are most unlikely to be best possible.

We thank the referee for making suggestions which have lead to improvements of some of our original results and to shortening of some of our original proofs.

2. Proof of theorem

The proof consists largely of commutator calculations. Let $a, a_1, \cdots, a_t$, b, c be elements of a metabelian group G and d an element of the (abelian) commutator subgroup $G' = \gamma_2(G)$ of G. We start from the following well-known or easily verifiable identities:

(1) $[a, b] = [b, a]^{-1}$

(2) $[a, bc] = [a, c][a, b][a, b, c]$

(3) $[a, b, c][b, c, a][c, a, b] = e$

(4) $[ad, b] = [a, b][d, b]$

(5) $[d, a_1, \cdots, a_t] = [d, a_{1\sigma}, \cdots, a_{t\sigma}]$ for every permutation σ of $\{1, \cdots, t\}$

(6) $[d^i, b] = [d, b]^i$ for every integer i.

Let r be a positive integer. A straight-forward induction on r using the above identities gives

(7) $[d, rab] = [d, ra][d, (r-1)a, b]^r \pi$

where π is a product of powers of commutators of the form $[d, ja, kb]$ where $j+k \geqq r$ and $k \geqq 2$.

To make the proof easier to follow we break it up using a number of lemmas which we prove later. Throughout these lemmas G is a metabelian group.

(2.1) Lemma. *Let n, t be positive integers. If $x \in G'$, $[x, b_1, \cdots, b_{n+1}] = e$ for all $b_1, \cdots, b_{n+1}$ in G, and $[x, na]^t = e$ for all a in G, then*

$$[x, a_1, \cdots, a_n]^{n!t} = e$$

for all $a_1, \cdots, a_n$ in G.

(2.2) Lemma. *Let n, t be positive integers. If $x \in G'$ and $[x, na]^t = e$ for all a in G, then $[x, (n-1)a, b]^{n!t} = e$ for all a, b in G.*

(2.3) Lemma. *Let n, r be positive integers. If $\gamma_{n+r+1}(G)$ is trivial and $[a, rb, (n-1)a] = e$ for all a, b in G, then $\gamma_{n+r}(G)$ has exponent dividing $(n-1)!\,(r-1)!\,(n+r)$.*

(2.4) Lemma. *Let n, r be positive integers. If $\gamma_{n+r+2}(G)$ is trivial and $[a, rb, (n-1)a, c] = e$ for all a, b, c in G, then $\gamma_{n+r+1}(G)$ has exponent dividing $(n-1)!(r-1)!$ g.c.d. (n, r)*

Proof of theorem.

(A) (induction on s)

$s = 2$: Since $[a_1, n_2a_2, (n_1-1)a_1] = e$ for all a_1, a_2 in G, replacing a_1, by $a[c, d]$, a_2 by b, and using (4) gives

$$[c, d, n_2b, (n_1-1)a] = e$$

for all a, b, c, d in G. The results follows after n_1+n_2-1 applications of Lemma 2.2.

$s > 2$: Let $\mathbf{n}' = (n_1, \cdots, n_{s-1})$. By the inductive hypothesis $x^{k'}$ belongs to $\varepsilon_{\mathbf{n}'}(G)$ for all x in $\gamma_{m+1-n_s}(G)$ where

$$k' = \frac{1}{n_1!}\prod_{i=1}^{s-1}\prod_{j=1}^{n_i} j!.$$

Hence, using (4), $[x, n_s a_s]^{k'} = e$ for all x in $\gamma_{m+1-n_s}(G)$ and all a_s in G. The result follows after n_s applications of Lemma 2.2.

(B) Consider first the case $t = m$. Without loss of generality $\gamma_{m+1}(G) = E$. The case $s = 2$ is the content of Lemma 2.3. If $s > 2$, then $s-2$ applications of Lemma 2.1 give

$$[a_1, n_2 a_2, (n_1-1)a_1, b_1, \cdots, b_{m-n_1-n_2}]^{n_3!\cdots n_s!} = e$$

for all $a_1, a_2, b_1, \cdots, b_{m-n_1-n_2}$ in G. Let $\mathbf{n}' = (n_1, n_2, 1)$. By Lemma 2.4 $x^{h'}$ belongs to $\varepsilon_{\mathbf{n}'}(G)\gamma_{n_1+n_2+2}(G)$ for all x in $\gamma_{n_1+n_2+1}(G)$ where

$$h' = (n_1-1)!\,(n_2-1)!\ \text{g.c.d.}\ (n_1, n_2).$$

The result follows from these last two statements.

Since the exponent of $\gamma_{t+1}(G)/\gamma_{t+2}(G)$ divides the exponent of $\gamma_t(G)/\gamma_{t+1}(G)$, it remains to show that $\gamma_{m+1}(G)/\gamma_{m+2}(G)$ has exponent dividing h when $s = 2$. This follows at once from Lemma 2.4 and the observation that if $\varepsilon_{(n_1, n_2)}(G)$ is trivial, then so is $\varepsilon_{(n_1, n_2, 1)}(G)$.

PROOF OF LEMMA 2.1. The result for arbitrary t follows at once from the case $t = 1$. Replacing a by $a_1 \cdots a_n$ and expanding using (2) gives

$$\prod_{i_1=1}^{n}\cdots\prod_{i_n=1}^{n}[x, a_{i_1}, \cdots, a_{i_n}] = e \text{ for all } a_1, \cdots, a_n \text{ in } G.$$

Since G is metabelian the left-hand side of this equation can be written in the form $\prod_1\prod_1'$ where $\prod_1$ is the product of commutators involving a_1 and $\prod_1'$ is the product of commutators not involving a_1. The relation $\prod_1\prod_1' = e$ holds for all a_1 in G and so in particular for $a_1 = e$. Thus $\prod_1' = e$ for all $a_2, \cdots, a_n$ in G. Hence $\prod_1 = e$ for all $a_1, \cdots, a_n$ in G. Repeating this argument for $a_2, \cdots, a_n$ in turn gives

$$\prod[x, a_{i_1}, \cdots, a_{i_n}] = e$$

where $(i_1, \cdots, i_n)$ runs over all permutations of $\{1, \cdots, n\}$. Hence, by (5), $[x, a_1, \cdots, a_n]^{n!} = e$ as required.

PROOF OF LEMMA 2.2. Again the result for arbitrary t follows from the case $t = 1$. For $i \in \{0, \cdots, n-1\}$ let f be the function defined by $f(i) = n!/(n-i)!$ and let S_i be the statement:

$$[d, (n-i)a, ib]^{f(i)}\pi_{i+1}(a, b) = e$$

for all a, b in G where $\pi_{i+1}(a, b)$ is a (possibly empty) product of powers of commutators of the form $[d, ja, kb]$ where $j+k \geqq n$ and $k \geqq i+1$. The required result is the statement S_{n-1}. It is given that S_0 is true. For

$$i \in \{1, \cdots, n-1\}$$

suppose S_{i-1} is true, then

$$[d, (n-i+1)ab, (i-1)b]^{f(i-1)}\pi_i(ab, b) \\ \{[d, (n-i+1)a, (i-1)b]^{f(i-1)}\pi_i(a, b)\}^{-1} = e.$$

Hence, using (7) and (4), S_i is true and the result follows.

Proof of lemma 2.3. The first step of the proof is similar to the proof of Lemma 2.1: replace a, b by $a_1 \cdots a_n, b_1 \cdots b_r$ repectively, write as a product of commutators in $a_1, \cdots, a_n, b_1, \cdots, b_r$, put $a_1, \cdots, a_n, b_1, \cdots, b_r$ in turn equal to e and use (5) to permute the entries in the 3rd to $(n+r)$-th place. This gives

$$\prod_{i=1}^{n}\prod_{j=1}^{r}[a_i, b_j, a_1, \cdots, \\ a_{i-1}, a_{i+1}, \cdots, a_n, b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_r]^{(n-1)!(r-1)!} = e.$$

for all $a_1, \cdots, a_n, b_1, \cdots, b_r$ in G. Let $w(a_1, b_1)$ denote the left-hand side of this relation, then

$$w(a_1, b_1)w(b_1, a_1)^{-1} = e.$$

Using (1), (3), (4), (5) this gives

$$[a_1, b_1, a_2, \cdots, a_n, b_2, \cdots, b_r]^{(n-1)!(r-1)!(n+r)} = e$$

as required.

Proof of lemma 2.4. We have, as in the proof of Lemma 2.3,

$$\prod_{i=1}^{n}\prod_{j=1}^{r}[a_i, b_j, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, \\ b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_r, c]^{(n-1)!(r-1)!} = e$$

for all $a_1, \cdots, a_n, b_1, \cdots, b_r, c$ in G. Let $w(a_1, c)$ denote the left-hand side of this relation, then

$$w(a_1, c)w(c, a_1)^{-1} = e.$$

Using (1), (3), (4), (5) this gives

$$[a_1, c, a_2, \cdots, a_n, b_1, \cdots, b_r]^{(n-1)!(r-1)!r} = e.$$

A similar operation with b_1 and c gives

$$[b_1, c, a_1, \cdots, a_n, b_2, \cdots, b_r]^{(n-1)!(r-1)!n} = e$$

which on interchanging a_1 and b_1 and using (5) gives

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$$[a_1, c, a_2, \cdots, a_n, b_1, \cdots, b_r]^{(n-1)!(r-1)!n} = e.$$

The result follows from combining these two.

3. Examples

(3.1) For every integer $m \geqq 3$, there is a group K with the following properties:

(a) nilpotent-of-class-2-by-abelian,

(b) abelian-by-nilpotent-of-class-2,

(c) nilpotent of class $m+1$,

(d) 3 generators,

(e) every 2-generator subgroup has class at most m,

(f) $\gamma_{m+1}(K)$ contains an element of infinite order.

DETAILS. Let H be a nilpotent-of-class-2 group generated by elements $a_1, \cdots, a_m, b_1, \cdots, b_m$ satisfying the following relations (and their consequences but no others)

$$\begin{array}{lll}
[a_i, a_j] = [b_i, b_j] = e & \text{for all } i, j & \text{in } \{1, \cdots, m\} \\
[a_i, b_j] = e & \text{for all } i, j & \text{in } \{3, \cdots, m\} \\
[a_2, b_j] = [a_j, b_2] & \text{for all } j & \text{in } \{3, \cdots, m\} \\
[a_2, b_m] = e & & \\
[a_1, b_i] = [a_i, b_1] & \text{for all } i & \text{in } \{2, \cdots, m\} \\
[a_1, b_m]^{m+1}[a_2, b_{m-1}]^{\frac{1}{2}(m+3)(m-2)} = e. & &
\end{array}$$

Clearly $[a_1, b_m]$ has infinite order in H.

It is routine to check that there is an automorphism ξ of H which maps a_i to $a_i a_{i+1}$, b_i to $b_i b_{i+1}$ for all i in $\{1, \cdots, m-1\}$ and fixes a_m, b_m. Let K be the splitting extension of H by an infinite cyclic group $gp\{x\}$, with x inducing ξ in H. Clearly K has properties (a) and (d). Let $c_{i+1} = [a_1, b_i]$ for all i in $\{1, \cdots, m\}$ and $d_{j+2} = [a_2, b_j]$ for all j in $\{2, \cdots, m-1\}$, then

$$\begin{array}{lll}
[c_2, x] = c_3^2 d_4 & & \\
[c_i, x] = c_{i+1} d_{i+1} d_{i+2} & \text{for all } i & \text{in } \{3, \cdots, m-1\} \\
[c_m, x] = c_{m+1} d_{m+1} & & \\
[c_{m+1}, x] = e & & \\
[d_j, x] = d_{j+1} & \text{for all } j & \text{in } \{4, \cdots, m\} \\
[d_{m+1}, x] = e. & &
\end{array}$$

It follows that $\gamma_r(K)$ is generated by all the a's b's, c's and d's whose subscripts are at least r. From this properties (b), (c), (f) follow at once and it

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remains to verify (*e*). Every element of K can be written in the form $x^r a_1^s b_1^t y$ where y belongs to $\gamma_2(K)$. Therefore, since K is nilpotent of class $m+1$, in order to show that every 2-generator subgroup has class at most m, it suffices to show that every left-normed commutator of weight $m+1$ in $x^r a_1^s b_1^t$ and $x^u a_1^v b_1^w$ is trivial for all choices of r, s, t, u, v, w. There is no loss of generality in considering only those left-normed commutators of weight $m+1$ whose first three entries are $x^r a_1^s b_1^t$, $x^u a_1^v b_1^w$, $x^u a_1^v b_1^w$ respectively. Now $[x^r a_1^s b_1^t, x^u a_1^v b_1^w] = a_2^{su-rv} b_2^{tu-rw} c_2^{sw-tv} y_3$ where y_3 belongs to $\gamma_3(K)$, so

$$[x^r a_1^s b_1^t, x^u a_1^v b_1^w, x^u a_1^v b_1^w] = a_3^{u(su-rv)} b_3^{u(tu-rw)} c_3^{3u(sw-tv)} y_4$$

where y_4 belongs to $\gamma_4(K)$. It is now routine to prove inductively for all i in $\{4, \cdots, m+1\}$ that every left-normed commutator of weight i in $x^r a_1^s b_1^t$ and $x^u a_1^v b_1^w$ starting as above has the form

$$a_i^{z(su-rv)} b_i^{z(tu-rw)} c_i^{zi(sw-tv)} d_i^{\frac{1}{2}(i+2)(i-3)z(sw-tv)} y_{i+1} \text{ where } y_{i+1}$$

belongs to $\gamma_{i+1}(K)$ and $a_{m+1} = b_{m+1} = e$. It follows from the last relation given for H that K has property (*e*) as required.

(3.2) For every odd prime p, there is a group N with the following properties:

(a) metabelian,

(b) nilpotent of class p,

(c) 3 generators,

(d) every 2-generator subgroup has class at most $p-1$,

(e) $\gamma_p(N)$ contains an element of order p.

DETAILS. For $p = 3$ the 3-generator free group of exponent 3 has all these properties. For $p \geqq 5$ let K be the group constructed in 3.1 with $m = p-1$, then the second derived subgroup $D = \gamma_2(\gamma_2(G))$ is generated by $d_4, \cdots, d_p$. Take N to be K/D, then clearly N is metabelian and properties (b), (c), (d) follow from the properties of K. Finally property (e) holds because $c_p D$ has order p.

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