

SOME NEW ORTHOGONAL DIAGONAL LATIN SQUARES

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Abstract

A recursive construction for orthogonal diagonal latin squares, using group divisible designs, is presented. In consequence the numbers of orders for which the existence of such squares is in question is reduced to 72.

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1. Introduction

A *latin square* of order v is a $v \times v$ array such that every row and every column contains each of the symbols $1, 2, \dots, v$ exactly once. A *transversal* in a latin square is a set of v positions, one in each row and column, which contain the symbols $1, 2, \dots, v$ in some order. A *transversal square* is a latin square whose leading diagonal is a transversal, and a *diagonal latin square* is one whose leading diagonal and back diagonal are both transversals.

Two latin squares (a_{ij}) and (b_{ij}) of order v are said to be *orthogonal* if the v^2 ordered pairs (a_{ij}, b_{ij}) include the v^2 ordered pairs on the symbols $1, 2, \dots, v$ exactly once each. One can then speak of *orthogonal transversal squares* and *orthogonal diagonal latin squares*.

For background on latin squares and orthogonality, and for a description of the relation between orthogonal diagonal latin squares and magic squares, the reader should consult [1]. In that book the following problem is posed: for what orders do orthogonal pairs of diagonal latin squares exist? Clearly, 2, 3 and 6 are

impossible. In [3] we reduced the set of unsolved orders to 129 members, namely,

- 10, 12, 14, 15, 18, 22, 24, 26, 30, 34, 38, 39,
- 42, 46, 54, 58, 60, 62, 66, 70, 74, 82, 86, 87,
- 90, 94, 98, 102, 110, 114, 118, 122, 126, 130, 138, 142,
- 146, 150, 154, 158, 170, 174, 178, 182, 186, 194, 198, 202,
- 206, 214, 222, 226, 230, 234, 238, 242, 254, 258, 262, 266,
- 270, 278, 282, 290, 294, 306, 310, 314, 318, 322, 326, 327,
- 334, 338, 354, 366, 370, 374, 378, 382, 394, 398, 402, 410,
- 418, 422, 426, 434, 446, 458, 482, 490, 494, 506, 522, 530,
- 538, 542, 566, 570, 578, 586, 594, 602, 614, 618, 626, 634,
- 642, 658, 662, 674, 682, 686, 690, 698, 710, 734, 758, 762,
- 786, 878, 882, 902, 926, 930, 978, 1026, 1074,

(For details and a bibliography of earlier studies, see [3].)

In the present paper we deduce we reduce the above list somewhat by a construction involving group divisible designs.

2. Group divisible designs

By a *group divisible design* $(\mathcal{V}; \mathcal{G}; \mathcal{K})$ is meant a partition \mathcal{G} (whose parts are called *groups*) of a finite set \mathcal{V} together with a collection \mathcal{K} of subsets (called *blocks*) of \mathcal{V} , such that two members of different groups occur together in precisely one block and two members of the same group do not occur together in any block. If G and K are sets of positive integers such that all group-sizes are in G and all block-sizes are in K , and if $v = |\mathcal{V}|$, then $(\mathcal{V}; \mathcal{G}; \mathcal{K})$ is called a $GD(v; G; K)$; if G has only one member g , then one writes $GD(v, g, K)$, and similarly if K is singleton.

THEOREM 1. *If there exists a set of $k - 2$ pairwise orthogonal latin squares of side g then there exists a $GD(gk; g; k)$.*

PROOF. Suppose the r -th latin square has (i, j) entry l_{ij}^r . Then the blocks

$$\{l_{ij}^1, l_{ij}^2, \dots, l_{ij}^{k-2}, i^{k-1}, j^k\}$$

for $1 \leq i \leq g, 1 \leq j \leq g$, form a $GD(gk; g; k)$ whose r -th group is $\{1^r, 2^r, \dots, g^r\}$.

Various generalizations of this easy result can be obtained; for example, one could delete one block completely, delete all but one element from one block, delete some elements from one group, and so on.

The reader will be familiar with balanced incomplete block designs, or BIBDs. The following construction is useful because BIBDs have been very widely studied.

THEOREM 2. *If there exists a balanced incomplete block design with parameters $(v, b, r, k, 1)$ then there exists a $GD(v - 1; k - 1; k)$.*

PROOF. One treatment is deleted from the balanced incomplete block design. The blocks which previously contained it become the *groups* of the new group divisible design, and the other blocks become the *blocks* of the new design.

3. A construction of orthogonal diagonal latin squares from group divisible designs

THEOREM 3. *Suppose there exists a $GD(v, G, K)$ in which at most one group has odd order. Suppose further that there exist orthogonal diagonal latin squares of order g for all $g \in G$, and there exist orthogonal transversal squares of order k for all $k \in K$. Then there exist orthogonal diagonal latin squares of order v .*

PROOF. First we order the groups in some way, so that no group except (possibly) the last has an odd number of elements. Say they have g_1, g_2, \dots, g_n elements respectively. Now relabel the elements of the group divisible design so that the first group becomes $\{1, v, 2, v - 1, \dots, \frac{1}{2} g_1, v - \frac{1}{2} g_1 + 1\}$, the second group becomes $\{\frac{1}{2} g_1 + 1, v - \frac{1}{2} g_1, \frac{1}{2} g_1 + 2, \dots, v - \frac{1}{2} g_1 - \frac{1}{2} g_2 + 1\}$, and so on.

Next we construct two latin squares L_1 and L_2 of size v . Given any block $B = \{b_1, b_2, \dots, b_k\}$ of size k , select two orthogonal transversal squares H_1 and H_2 of order k which have common diagonal $(1, 2, \dots, k)$. Define H_1^* from H_1 by replacing entry i by b_i at every occurrence, for $i = 1, 2, \dots, k$. Then the (b_i, b_j) entry of L_1 is set to be the (i, j) entry of H_1^* . H_2^* is defined similarly and used to define elements of L_2 . This process is carried out for every block. Next, given any group of size g , choose two orthogonal diagonal latin squares H_1 and H_2 of side g which have common diagonal $(1, 2, \dots, g)$, and proceed similarly.

Since every pair in $\{1, 2, \dots, v\}$ occurs either in exactly one block or in exactly one group, but not both, this process unambiguously defines $v \times v$ arrays L_1 and L_2 , which are easily seen to be orthogonal diagonal latin squares, as required.

4. Applications of the preceding construction

LEMMA 4.1. *There exist orthogonal diagonal latin squares of orders 39, 87 and 327.*

PROOF. Since there exist three pairwise orthogonal latin squares of all orders except 2, 3, 6, 10 and 14 [4], there exist $GD(5g; g; 5)$ for $g = 8, 20$ and 80 . So, by deleting some elements from one group, we can construct a $GD(4g_1 + g_2; g_1, g_2; 4, 5)$ with one group of size g_2 in the following cases:

$$\begin{aligned} g_1 = 8, \quad g_2 = 7, \quad 4g_1 + g_2 = 39; \\ g_1 = 20, \quad g_2 = 7, \quad 4g_1 + g_2 = 87; \\ g_1 = 80, \quad g_2 = 7, \quad 4g_1 + g_2 = 327. \end{aligned}$$

Now Theorem 3 applies.

REMARK. This method can be used to produce easy solutions of many other odd orders. For example, orders 219, 267, 303, 447 and 543 were solved in [3] by a method involving the construction of several vectors, either by hand computation or computer. They now follow easily, using

$$\begin{aligned} g_1 = 48, \quad g_2 = 27, \quad 4g_1 + g_2 = 219; \\ g_1 = 56, \quad g_2 = 43, \quad 4g_1 + g_2 = 267; \\ g_1 = 68, \quad g_2 = 31, \quad 4g_1 + g_2 = 303; \\ g_1 = 100, \quad g_2 = 57, \quad 4g_1 + g_2 = 447; \\ g_1 = 120, \quad g_2 = 63, \quad 4g_1 + g_2 = 543. \end{aligned}$$

LEMMA 4.2. *There exist orthogonal diagonal latin squares of orders 24 and 60.*

PROOF. It is shown in [2] that a balanced incomplete block design with $k = 5$ and $\lambda = 1$ exists whenever $v \equiv 1$ or $v \equiv 5 \pmod{20}$. So by Theorem 2 there exists a $GD(v; 4; 5)$ when $v \equiv 0$ or $4 \pmod{20}$. The result then follows from Theorem 3.

Note that the existence of orthogonal diagonal latin squares of order 24 implies the existence of such squares of order $254 = 11(24 - 1) + 1$, by Preliminary Theorem 4 or 5 of [3].

LEMMA 4.3. *There exist orthogonal diagonal latin squares of order $16n + d$ whenever $4n \geq d$, when $d = 50, 78, 106$ or 134 .*

PROOF. There exist three pairwise orthogonal latin squares of order $4n$ in all the cases concerned, so by Theorem 1 there is a $GD(20n; 4n; 5)$. So for each value of d we can obtain a $GD(16n + d; \{4n, d\}; \{5, 4\})$ by deleting $4n - d$ elements from each block, whenever $4n \geq d$. Since orthogonal diagonal latin squares of orders $4n$ and d exist in every case, orthogonal latin squares of order $16n + d$ exist.

From Lemma 4.3 we see that there exist orthogonal diagonal latin squares of order v where $v \equiv 2 \pmod{4}$ and $v \geq 666$, and there also exist such squares of orders

258,	290,	306,	322,	338,	354,	370,	398,	402,
418,	434,	446,	482,	494,	530,	538,	542,	570,
578,	586,	594,	602,	618,	626,	634,	642,	658.

To improve this list we need the following specialised construction of group divisible designs.

LEMMA 4.4. *If p is a prime power, then there exists a $GD(v; G; K)$ with*

$$v = (k - 2)(p - 1) + d + e,$$

$$G = \{p - 1, d, e\},$$

$$K = \{k, k - 1, k - 2, k - 3\}$$

where $k \leq p - 1$, $d \leq p$ and $e \leq p - 1$.

PROOF. Since p is a prime power and $k \leq p - 1$, there exist k pairwise orthogonal latin squares of order p . So a $GD(p; k; p; k)$ can be constructed using Theorem 1.

Delete all the elements of one block except for the element in group 1. Then delete $p - d$ elements from group 1 and $p - 1 - e$ elements from another group. The resulting design has the required parameters.

COROLLARY. *There exist orthogonal diagonal latin squares of orders 490, 506, 522, 566, 614 and 662.*

PROOF. This follows from Lemma 4.4 and Theorem 3, using the identities

$$490 = 8 \times 52 + 50 + 24$$

$$506 = 8 \times 52 + 50 + 40$$

$$522 = 9 \times 52 + 50 + 4$$

$$566 = 9 \times 52 + 50 + 48$$

$$614 = 9 \times 60 + 50 + 24$$

$$662 = 8 \times 72 + 50 + 36.$$

Summarizing the results of this section we have

THEOREM 4. *There exist orthogonal diagonal latin squares of every order v where $v > 458$. Orders 2, 3 and 6 are impossible; the only orders for which the existence is undecided are:*

10, 12, 14, 15, 18, 22, 26, 30, 34, 38, 42, 46,
 54, 58, 62, 66, 70, 74, 82, 86, 90, 94, 98, 102,
 110, 114, 118, 122, 126, 130, 138, 142, 146, 150, 154, 158,
 170, 174, 178, 182, 186, 194, 198, 202, 206, 214, 222, 226,
 230, 234, 238, 242, 262, 266, 270, 278, 282, 294, 310, 314,
 318, 326, 334, 366, 374, 378, 382, 394, 410, 422, 426, 458.

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