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Generalization of Hall planes of odd order

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Some properties of projective planes having a certain type of collineation group are proved, and a class of these planes which properly contains the class of all Hall planes of odd order is explicitly constructed.

1. Introduction

The Hall plane satisfies the condition:

(1) Π is a translation plane, and it has a Baer subplane Π_0 fixed pointwise by a collineation group which is simply transitive on those points of the line at infinity which do not lie in Π_0. The line at infinity belongs to Π_0.

We call planes satisfying (1) 'generalized Hall planes'. We will show (among other things) that when such a plane has odd or zero characteristic then the subplane Π_0 is desarguesian; and we will construct a class of these planes which appears, to the author, to include some new finite planes.

2. Properties of generalized Hall planes

Let Π be a generalized Hall plane. Then Π may be coordinatized by a (right distributive) V.-W. system F which contains a subsystem F_0 corresponding to Π_0. (Π_0 is a translation plane since it contains

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the line at infinity of Π . We choose the coordinate quadrangle to lie in Π_0 .)

We shall use Greek letters to denote elements of F_0 .

THEOREM 1. *If* Π *is a plane of odd or zero characteristic satisfying* (1), *then* F_0 *is a skew field and* F *is a right vector space of dimension* 2 *over* F_0 .

COROLLARY. Π_0 *is desarguesian.*

Proof of Theorem 1. Choose an element z of $F \backslash F_0$. Let w be any other element of F . Then, since Π_0 is a Baer subplane, the point (z, w) lies on some line $y = x\alpha + \beta$ of Π_0 , that is $w = z\alpha + \beta$ for some $\alpha, \beta \in F$.

The collineations fixing Π_0 pointwise correspond to automorphisms of F which fix F_0 elementwise. So $(z\rho)\sigma = z\alpha + \beta$ for some α and β which depend only on ρ and σ . Also $((z+1)\rho)\sigma = (z+1)\alpha + \beta$, so $\alpha = \rho\sigma$ and $(z\rho)\sigma = z(\rho\sigma) + \beta$. Furthermore $((2z)\rho)\sigma = (2z)(\rho\sigma) + \beta$ and $2 \in \mathrm{Kern} F$, so $\beta = 0$. Thus $(z\rho)\sigma = z(\rho\sigma)$ for all $\rho, \sigma \in F_0$. Similarly $z(\rho+\sigma) = z\rho + z\sigma$. [Start with $z\rho + z\sigma = z\lambda + \mu$.]

For any $\rho, \sigma, \tau \in F_0$, we have:

$$\big((z\rho)\sigma\big)\tau = (z\rho)(\sigma\tau) = z\big(\rho(\sigma\tau)\big) ,$$

and

$$\big((z\rho)\sigma\big)\tau = \big(z(\rho\sigma)\big)\tau = z\big((\rho\sigma)\tau\big) .$$

Thus $\rho(\sigma\tau) = (\rho\sigma)\tau$; similarly $\rho(\sigma+\tau) = \rho\sigma + \rho\tau$. This completes the proof of the theorem.

From the multiplication operation in F we obtain two mappings f and g of $F_0 \times F_0$ onto F_0 , defined by:

$$(z\alpha+\beta)z = zf(\alpha, \beta) + g(\alpha, \beta) .$$

The V.-W. system F may be described as follows:

(2) F is a right vector space of dimension 2 over a skew field F_0 embedded in it in the usual way, with a multiplication operation

$$x.\alpha = x\alpha \text{ (multiplication by a scalar) } \forall\, x \in F,\ \alpha \in F_0 ,$$

$$(z\alpha+\beta).z = zf(\alpha,\ \beta) + g(\alpha,\ \beta) ,\ \forall\, z \in F \backslash F_0 ;\ \alpha,\ \beta \in F_0 ,$$

where f and g are mappings of $F_0 \times F_0$ onto F_0 .

The mappings f and g in (2) are of course not arbitrary.

THEOREM 2. *A finite system* $(F, +, \cdot)$ *satisfying* (2) *is a* V.-W. *system if and only if*

(a) f *and* g *are additive homomorphisms with* $f(0, 1) = 1$ *and* $g(0, 1) = 0$,

(b) for any given γ *and* δ *, the equation* $\big(f(\alpha, \beta), g(\alpha, \beta)\big) = (\gamma, \delta)$ *has exactly one solution* (α, β) *, and*

(c) the equation $\big(f(\alpha, \beta), g(\alpha, \beta)\big) = (\alpha\gamma, \beta\gamma+\delta)$ *has exactly one solution* (α, β) *, given* γ *and* δ *; also, for this solution,* $\alpha = 0$ *if and only if* $\delta = 0$.

Proof. The necessity of *(a)* and *(b)* follows immediately from the right distributivity of F and the requirement that $\cdot$ be a loop operation on F^* . This loop requirement also implies *(c)*. For consider the equation $z(z\alpha+\beta) = z\gamma + \delta$. Now, if $\alpha \neq 0$,

$$\begin{aligned} z(z\alpha+\beta) &= \left[(z\alpha+\beta)\left(\alpha^{-1}\right)-\beta\alpha^{-1}\right](z\alpha+\beta) \\ &= (z\alpha+\beta)f\left(\alpha^{-1},\ -\beta\alpha^{-1}\right) + g\left(\alpha^{-1},\ -\beta\alpha^{-1}\right) . \end{aligned}$$

If we replace α^{-1} by α and $-\beta\alpha^{-1}$ by β , the requirement that $zw = t$ has exactly one solution w yields condition *(c)*.

The sufficiency of *(a)*, *(b)*, *(c)* is now evident, since, F being finite, we merely need to show that these imply that F is right distributive and that $\cdot$ is a loop operation on F^* .

A more complicated necessary and sufficient condition that F be a (planar) V.-W. system is easily calculated for the case where F is allowed to be infinite.

We note that a V.-W. system satisfying (2) necessarily possesses a group of automorphisms which is transitive on $F \backslash F_0$ while fixing F_0 elementwise.

3. A construction

We start with an arbitrary finite field F_0 of odd order. Let ν be any non-square in F_0 and let θ and φ be any two (possibly trivial, and possibly equal) automorphisms of F_0. We now construct a V.-W. system $(F, +, \cdot)$ from F_0, ν, θ and φ.

Let F be a right vector space of dimension 2 over F_0. Suppose F_0 is embedded in F in the usual way. Addition is to be the same as vector addition, and multiplication to be given by the rules stated in (2) above, with the mappings f and g defined by:

$$(3) \qquad f(\alpha, \beta) = \beta^\theta \quad g(\alpha, \beta) = \alpha^\varphi \nu .$$

Conditions *(a)*, *(b)*, *(c)* are easily verified, so that $(F, +, \cdot)$ is a V.-W. system. The plane Π over F is a generalized Hall plane.

When $\theta = \varphi = 1$, F is the Hall system determined by F_0 and the polynomial $x^2 - \nu$. Since Hall systems of the same order coordinatize isomorphic planes [5], the generalized Hall planes we have constructed include all Hall planes of odd order.

As in the Hall system for F_0 and ν, we have for all F:

$z^2 = \nu$, $\forall z \in F \backslash F_0$. But $\alpha z = z\alpha^\theta$ and $(z\alpha)z = \alpha^\varphi \nu$ when $z \in F \backslash F_0$ and $\alpha \in F_0$.

In the case where $F_0 = GF(9)$, $\theta = 1$ and φ equals the non-trivial automorphism of $GF(9)$, it is readily verified that Kern$F$ is the subfield of order 3 in $GF(9)$. Since the Kern of any Hall system of order 81 is $GF(9)$, the plane over F is not a Hall plane. By comparing the collineation group of the plane over F with that of each of the Foulser generalized André planes of order 81, it is not difficult to show that our class of planes is not a subclass of Foulser's: the Foulser planes of order 81 with Kern of order 3 all have a group of 10 (X, OY)-homologies, whereas our plane has no such group (of order 10), no matter how X and Y be chosen on the line at infinity.

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