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STABLE AND SEMISTABLE PROBABILITY MEASURES ON CONVEX CONE

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Abstract

The study concerns semistability and stability of probability measures on a convex cone, showing that the set $S(\mu)$ of all positive numbers $t > 0$ such that a given probability measure μ is t-semistable establishes a closed subgroup of the multiplicative group R^+; semistability and stability exponents of probability measures are positive numbers if and only if the neutral element of the convex cone coincides with the origin; a probability measure is (semi)stable if and only if its domain of (semi-)attraction is not empty; and the domain of attraction of a given stable probability measure coincides with its domain of semi-attraction.

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1. Introduction

The concepts of *stable* and *semistable* distributions on the real line were introduced in 1937 by Lévy [24]. A well-known complete description of stable distributions is given in the book [9] by Gnedenko and Kolmogorov. Together with stable distributions, semistable distributions have been the object of renewed interest in the last decades of the 20th century and in recent years. Stable distributions arise as solutions to central limit problems and have attracted very much attention in theoretical research [31, 37] as well as in applied research [7, 23, 27–29, 32]. Meanwhile, semistable distributions have proven to be a richer alternative than stable laws in stochastic modeling [1, 13, 19, 20, 34].

It is notable that the topic of stable and semistable distributions was studied mainly on linear algebraic structures like Euclidian spaces [4, 10–12, 15–17, 26, 33], Hilbert spaces [5, 14, 21], or Banach spaces [8, 22, 25, 35, 36].

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Recently, the fundamental paper of Davydov *et al.* [6] can be noticed as a pioneering work in investigation of stable distributions on a more abstract structure of semigroups. With the same approach, this study provides some results regarding stability and semistability related to domains of attraction or semi-attraction of probability measures on convex cones defined as topological semigroups endowed with continuous automorphisms playing the role of multiplication by positive scalars.

2. Preliminaries and notation

Throughout the paper we will use the terminology given in [6]. Let $\mathbf{K}$ be a complete separable metrizable *convex cone*, which means that:

(i) $\mathbf{K}$ is a topological abelian semigroup with the *neutral element* $\boldsymbol{e}$ satisfying $\boldsymbol{x} + \boldsymbol{e} = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbf{K}$, where $+$ denotes a commutative and associative continuous binary operation in $\mathbf{K}$;

(ii) $\mathbf{K}$ is equipped with a continuous operation $(\boldsymbol{x}, a) \mapsto a\boldsymbol{x}$ of *multiplication by positive scalars* so that for $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{K}$, for all positive numbers a and b, the following conditions are satisfied:

$$\begin{aligned} a(\boldsymbol{x} + \boldsymbol{y}) &= a\boldsymbol{x} + a\boldsymbol{y}, \\ a(b\boldsymbol{x}) &= (ab)\boldsymbol{x}, \\ 1\boldsymbol{x} &= \boldsymbol{x}, \\ a\boldsymbol{e} &= \boldsymbol{e}; \end{aligned}$$

(iii) $\mathbf{K}$ is a *pointed set* with respect to the multiplication, which means that there is a unique element $\mathbf{0}$ called the *origin* such that $a\boldsymbol{x} \to \mathbf{0}$ as $a \downarrow 0$ for any $\boldsymbol{x} \in \mathbf{K} \setminus \{\boldsymbol{e}\}$;

(iv) $\mathbf{K} \setminus \{\boldsymbol{e}\}$ (or $\mathbf{K}$, if $\boldsymbol{e} = \mathbf{0}$) is a complete separable metric space.

It is worth noting that the neutral element of $\mathbf{K}$ does not necessarily coincide with the origin.

Let $B(\mathbf{K})$ denote the σ-algebra of all Borel subsets of $\mathbf{K}$ and $P(\mathbf{K})$ the space of all probability measures (p.m.s) defined in $(\mathbf{K}, B(\mathbf{K}))$. It is well known that every p.m. from $P(\mathbf{K})$ is a Radon measure and $P(\mathbf{K})$ with the topology of weak convergence is a complete separable metric space [30, Theorems II.3.2 and II.6.2].

The semigroup structure of $\mathbf{K}$ leads to a *convolution* in $P(\mathbf{K})$. Namely, let $\kappa : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ be the continuous mapping given by $\kappa(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y}$. For $\boldsymbol{p}, \boldsymbol{q} \in P(\mathbf{K})$ the convolution $\boldsymbol{p} * \boldsymbol{q}$ is by definition the image measure of the Radon product measure $\boldsymbol{p} \otimes \boldsymbol{q}$ under the mapping κ,

$$\boldsymbol{p} * \boldsymbol{q}(\mathbf{B}) = \boldsymbol{p} \otimes \boldsymbol{q}(\kappa^{-1}(\mathbf{B}))$$

for $\mathbf{B} \in B(\mathbf{K})$. Then $P(\mathbf{K})$ is a separable metrizable abelian semigroup with neutral element $\boldsymbol{\delta}(\boldsymbol{e})$, the p.m. concentrated at the neutral element $\boldsymbol{e}$ of $\mathbf{K}$. Besides, $P(\mathbf{K})$ is endowed with the family of continuous automorphisms $\boldsymbol{T}_a : P(\mathbf{K}) \to P(\mathbf{K})$ indexed by positive real numbers a, defined by

$$\boldsymbol{T}_a\boldsymbol{p}(\mathbf{B}) = \boldsymbol{p}(a^{-1}\mathbf{B}) \quad \text{for } \boldsymbol{p} \in P(\mathbf{K}), \quad \mathbf{B} \in B(\mathbf{K}),$$

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where $a^{-1}\mathbf{B} = \{\boldsymbol{x} \in \mathbf{K} : a\boldsymbol{x} \in \mathbf{B}\}$. Then $P(\mathbf{K})$ with the convolution $*$ and the family $\{\boldsymbol{T}_a, a > 0\}$, taken as the multiplication by positive scalars, is a pointed convex cone with the origin $\boldsymbol{\delta}(\mathbf{0})$, the p.m. concentrated at the origin $\mathbf{0}$ of $\mathbf{K}$.

A p.m. $\boldsymbol{p}$ is called *infinitely divisible* (inf. div.) if for every positive integer n there exists a p.m. $\boldsymbol{p}_n$ such that

$$\boldsymbol{p} = \boldsymbol{p}_n^{*n} := \boldsymbol{p}_n \underbrace{* \cdots *}_{n} \boldsymbol{p}_n.$$

Then we can denote $\boldsymbol{p}_n$ as $\boldsymbol{p}^{1/n}$, and define $\boldsymbol{p}^{m/n} = \boldsymbol{p}_n^{*m}$ for every positive integer m. In consequence, we have the following remark.

Remark 2.1. The power $\boldsymbol{p}^r$ is well defined for every positive rational number r, and it is an inf. div. p.m. In Lemma 2.3 below, we see that the power $\boldsymbol{p}^s$ is also well defined for arbitrary positive real numbers s.

A *convolution semigroup* in $P(\mathbf{K})$ is a family $\{\boldsymbol{\mu}_t, t > 0\}$ of p.m.s such that $t \mapsto \boldsymbol{\mu}_t$ is a continuous homomorphism from the semigroup $(R^+, +)$ into the semigroup $(P(\mathbf{K}), *)$, such that:

(i) $\boldsymbol{\mu}_0 = \boldsymbol{\delta}(\boldsymbol{e})$;
(ii) $\boldsymbol{\mu}_{t+s} = \boldsymbol{\mu}_t * \boldsymbol{\mu}_s$ for all $s, t \in R^+$;
(iii) $t \mapsto \boldsymbol{\mu}_t$ is weakly continuous.

Assume that the semigroup $\mathbf{K}$ is equipped with involution, a continuous map $^* : \mathbf{K} \to \mathbf{K}$ satisfying $(\boldsymbol{x} + \boldsymbol{y})^* = \boldsymbol{x}^* + \boldsymbol{y}^*$ and $(\boldsymbol{x}^*)^* = \boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{K}$. Assume also that $(a\boldsymbol{x})^* = a\boldsymbol{x}^*$ for every $\boldsymbol{x} \in \mathbf{K}$ and for every positive number a. Note that the involution can be the identical map.

A function χ that maps $\mathbf{K}$ into the unit disk D on the complex plane is called a *character* if $\chi(\boldsymbol{e}) = 1$, $\chi(\boldsymbol{x} + \boldsymbol{y}) = \chi(\boldsymbol{x})\chi(\boldsymbol{y})$, and $\chi(\boldsymbol{x}^*) = \overline{\chi(\boldsymbol{x})}$ (the complex conjugate of $\chi(\boldsymbol{x})$) for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{K}$. The set $\hat{\mathbf{K}}$ of all characters (with the pointwise multiplication operation), endowed with the topology of pointwise convergence, is called the restricted dual semigroup to $\mathbf{K}$. The character $\mathbf{1}$ (identically equal to 1) is the neutral element in $\hat{\mathbf{K}}$. For each $\boldsymbol{x} \in \mathbf{K}$, the projection $\pi_{\boldsymbol{x}} : \chi \mapsto \chi(\boldsymbol{x})$ becomes a continuous function from $\hat{\mathbf{K}}$ into D. The multiplication by a in $\mathbf{K}$ induces the multiplication operation $\chi \mapsto a \circ \chi$ in $\hat{\mathbf{K}}$ given by $(a \circ \chi)(\boldsymbol{x}) = \chi(a\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbf{K}$.

The *Laplace transform* of a p.m. $\boldsymbol{p}$ on $\mathbf{K}$ is defined by

$$L(\boldsymbol{p}, \chi) = \int_{\mathbf{K}} \chi(\boldsymbol{x}) \boldsymbol{p}(d\boldsymbol{x})$$

for every Borel measurable character $\chi \in \hat{\mathbf{K}}$. Then

$$L(\boldsymbol{p} * \boldsymbol{q}, \chi) = L(\boldsymbol{p}, \chi) \cdot L(\boldsymbol{q}, \chi)$$

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for any $\boldsymbol{p}, \boldsymbol{q} \in P(\mathbf{K})$. The Laplace transform $L(\boldsymbol{p}, \chi)$ is positive definite. In fact, for complex numbers $c_1, c_2, \ldots, c_n$ and Borel measurable characters $\chi_1, \chi_2, \ldots, \chi_n$,

$$\sum_{i,j=1}^{n} c_i \bar{c}_j L(\boldsymbol{p}, \chi_i \bar{\chi}_j) = \sum_{i,j=1}^{n} c_i \bar{c}_j \int_K \chi_i \bar{\chi}_j \, d\boldsymbol{p} = \int_K \sum_{i,j=1}^{n} c_i \bar{c}_j \chi_i \bar{\chi}_j \, d\boldsymbol{p}$$
$$= \int_K \Big| \sum_{i=1}^{n} c_i \chi_i \Big|^2 \, d\boldsymbol{p} \geq 0.$$

A sub-semigroup $\tilde{\mathbf{K}}$ of characters generates a $\tilde{\mathbf{K}}$-weak topology on $\mathbf{K}$ by declaring $\boldsymbol{x}_n \to_w \boldsymbol{x}$ if and only if $\chi(\boldsymbol{x}_n) \to \chi(\boldsymbol{x})$ for all $\chi \in \tilde{\mathbf{K}}$. The $\tilde{\mathbf{K}}$-weak topology is the weakest topology that makes all characters from $\tilde{\mathbf{K}}$ continuous. Let $B(\mathbf{K}, \tilde{\mathbf{K}})$ denote the σ-algebra of all Borel subsets of $\mathbf{K}$ in the sense of the $\tilde{\mathbf{K}}$-weak topology. Then $B(\mathbf{K}, \tilde{\mathbf{K}}) \subset B(\mathbf{K})$. Let $F(\mathbf{K}, \tilde{\mathbf{K}})$ be the smallest σ-algebra on $\mathbf{K}$ that makes all $\chi \in \tilde{\mathbf{K}}$ measurable. The set $\tilde{\mathbf{K}}$ is called separating if for any two distinct elements $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{K}$ there exists a character $\chi \in \tilde{\mathbf{K}}$ such that $\chi(\boldsymbol{x}) \neq \chi(\boldsymbol{y})$. Henceforth, we always suppose that $\hat{\mathbf{K}}$ has a separating sub-semigroup $\tilde{\mathbf{K}}$ such that $F(\mathbf{K}, \tilde{\mathbf{K}}) = B(\mathbf{K}, \tilde{\mathbf{K}}) = B(\mathbf{K})$. Then every p.m. $\boldsymbol{p} \in P(\mathbf{K})$ is uniquely determined by its Laplace transform [6, Theorem 5.4].

Remark 2.2. With the condition $B(\mathbf{K}, \tilde{\mathbf{K}}) = B(\mathbf{K})$, if $(\boldsymbol{p}_n)$ is a sequence of p.m.s weakly convergent to a p.m. $\boldsymbol{p}_0$, then $L(\boldsymbol{p}_n, \chi) \to L(\boldsymbol{p}_0, \chi)$ for all $\chi \in \tilde{\mathbf{K}}$.

The following lemma plays a crucial role in the study.

Lemma 2.3. *Let $\boldsymbol{p} \in P(\mathbf{K})$ be an inf. div. p.m. such that $\boldsymbol{p} \neq \boldsymbol{\delta}(\mathbf{0})$; then there exists a convolution semigroup $\{\boldsymbol{\mu}_t, t > 0\}$ in $P(\mathbf{K})$ such that $\boldsymbol{p} = \boldsymbol{\mu}_1$. Moreover, for every positive number s, the convolution power $\boldsymbol{p}^s = \boldsymbol{\mu}_s$ is well defined and the map $s \mapsto \boldsymbol{p}^s$ is weakly continuous.*

Proof. Denote by $\mathbf{K}^\# = \hat{\tilde{\mathbf{K}}}$ the restricted dual semigroup to $\tilde{\mathbf{K}}$. We equip $\mathbf{K}^\#$ with the topology of pointwise convergence. Define the evaluation map $\epsilon : \mathbf{K} \to \tilde{\mathbf{K}}$ by associating every element $\boldsymbol{x} \in \mathbf{K}$ with $\pi = \pi_{\boldsymbol{x}} \in \mathbf{K}^\#$ such that $\pi_{\boldsymbol{x}}(\chi) = \boldsymbol{x}$ for all $\chi \in \tilde{\mathbf{K}}$ [18, Section 20]. The evaluation map ϵ is injective, as $\tilde{\mathbf{K}}$ is separating. Then we can consider $\mathbf{K}$ as a subset of $\mathbf{K}^\#$.

For every natural number n, the convolution power $\boldsymbol{p}^{1/n}$ is well defined. Besides, both the Laplace transforms $L(\boldsymbol{p}, \chi)$ and $L(\boldsymbol{p}^{1/n}, \chi) = L(\boldsymbol{p}, \chi)^{1/n}$ are positive-definite functions of χ. The results on inf. div. functions in semigroups [2, Theorem 3.2.2 and Proposition 4.3.1] imply that

$$L(\boldsymbol{p}, \chi) = \exp(-\varphi(\chi)), \quad \chi \in \tilde{\mathbf{K}},$$

where φ is a negative-definite complex-valued function on $\tilde{\mathbf{K}}$ with $\mathrm{Re}\varphi \in [0, \infty)$ and $\varphi(1) = 0$. As a consequence, [2, Theorem 4.3.7] provides the existence of a convolution semigroup $\{\boldsymbol{\mu}_t, t > 0\}$ on $\mathbf{K}^\#$ such that

$$\hat{\boldsymbol{\mu}}_t(\chi) := \int_{\mathbf{K}^\#} \eta(\chi) \, d\boldsymbol{\mu}_t(\eta) = \exp(-t\varphi(\chi)) \quad \text{for } \chi \in \tilde{\mathbf{K}}, \quad t \geq 0.$$

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On the other hand, for every positive rational number r, the convolution power $\boldsymbol{p}^r$ is well defined (see Remark 2.1) and

$$L(\boldsymbol{p}^r, \chi) = \exp(-r\varphi(\chi)) = \hat{\boldsymbol{\mu}}_r(\chi) \quad \text{for } \chi \in \tilde{\mathbf{K}}.$$

Then $\boldsymbol{\mu}_r = \boldsymbol{p}^r$ by virtue of [2, Theorem 4.2.11] and $\boldsymbol{\mu}_r$ is supported on $\mathbf{K}$. Consequently, $\boldsymbol{\mu}_t$ is supported on $\mathbf{K}$ for every positive real number t because of the weak continuity of the convolution semigroup $\{\boldsymbol{\mu}_t, t > 0\}$. The lemma is proved by putting $\boldsymbol{p}^s = \boldsymbol{\mu}_s$ for each positive irrational number s. □

3. Strictly semistable and strictly stable probability measures

A p.m. $\boldsymbol{\mu}$ is called *strictly* (r, α)*-semistable* for given $r > 0$ and $\alpha \neq 0$ if $\boldsymbol{\mu}$ is inf. div. and

$$\boldsymbol{\mu}^r = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\mu}. \tag{3.1}$$

Then we say briefly that $\boldsymbol{\mu}$ is $St(r, \alpha)SS$ and denote $\boldsymbol{\mu} \in \boldsymbol{St}(r, \alpha)\boldsymbol{SS}$.

Lemma 3.1. *Let* $\mathrm{Triv}(\mathbf{K}) := \{\boldsymbol{\tau}(a) = (1 - a)\boldsymbol{\delta}(\mathbf{0}) + a\boldsymbol{\delta}(\boldsymbol{e}), 0 \leq a \leq 1\}$ *denote the class of all trivial p.m.s concentrated on the subset* $\{\mathbf{0}, \boldsymbol{e}\}$ *of* $\mathbf{K}$. *Then we have the following statements.*

(i) *Every p.m.* $\boldsymbol{\tau}(a)$ *from* $\mathrm{Triv}(\mathbf{K})$ *is inf. div.*
(ii) *If* $\mathbf{0} \neq \boldsymbol{e}$ *and* $0 < a < 1$, *then* $\boldsymbol{\tau}(a) \notin St(r, \alpha)SS$ *for all numbers* r *and* α *such that* $0 < r \neq 1$, $\alpha \neq 0$.
(iii) *If* $\boldsymbol{T}_c\boldsymbol{\mu} = \boldsymbol{\mu}$, $c > 0$, *and* $\boldsymbol{\mu} \notin \mathrm{Triv}(\mathbf{K})$, *then* $c = 1$.
(iv) *For all p.m.s* $\boldsymbol{\mu}$ *from* $P(\mathbf{K})$, *we have* $\boldsymbol{T}_0\boldsymbol{\mu} = \boldsymbol{\tau}(\boldsymbol{\mu}(\{\boldsymbol{e}\})) \in \mathrm{Triv}(\mathbf{K})$.

Proof. (i) It is easy to verify that $\boldsymbol{\tau}(a) * \boldsymbol{\tau}(b) = \boldsymbol{\tau}(ab)$ for all $a, b \in [0; 1]$ and then $\boldsymbol{\tau}(a) = \boldsymbol{\tau}(a^{1/n})^{*n}$ for every natural n, which confirms that $\boldsymbol{\tau}(a)$ is inf. div.

(ii) Suppose that $\boldsymbol{\tau}(a) \in \boldsymbol{St}(r, \alpha)\boldsymbol{SS}$,

$$(\boldsymbol{\tau}(a))^r = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\tau}(a).$$

Then

$$\begin{aligned}\boldsymbol{\tau}(a^r) = (\boldsymbol{\tau}(a))^r = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\tau}(a) &= \boldsymbol{T}_{r^{1/\alpha}}[(1 - a)\delta(\mathbf{0}) + a\delta(\boldsymbol{e})] \\ &= (1 - a)\delta(\mathbf{0}) + a\delta(\boldsymbol{e}) = \boldsymbol{\tau}(a).\end{aligned}$$

This implies that $a^r = a$, which is true only when $a = 0$ or $a = 1$, which contradicts the condition $0 < a < 1$.

(iii) For $\boldsymbol{\mu} \notin \mathrm{Triv}(\mathbf{K})$, we have $\boldsymbol{\mu}(\mathbf{K}\backslash\{\mathbf{0}, \boldsymbol{e}\}) = \varepsilon > 0$. Let ρ be a metric in $\mathbf{K}\backslash\{\boldsymbol{e}\}$ and denote

$$\mathbf{B}_r := \{\boldsymbol{x} \in \mathbf{K} : \rho(\mathbf{0}, \boldsymbol{x}) < r\}$$

for every given positive number r. Then

$$\boldsymbol{\mu}(\mathbf{B}_k\backslash\{\mathbf{0}\}) \nearrow \boldsymbol{\mu}(\mathbf{K}\backslash\{\mathbf{0}, \boldsymbol{e}\}) = \varepsilon$$

when $k \nearrow \infty$ and there exists M such that

$$\mu(\mathbf{B}_M \backslash \{\mathbf{0}\}) \geq \varepsilon/2 > 0. \tag{3.2}$$

On the other hand, if $c \neq 1$, then without loss of generality we can assume that $c > 1$, which implies that $c^{-n}\mathbf{B}_M \searrow \{\mathbf{0}\}$ as $n \to \infty$. Then the condition $\boldsymbol{T}_c\boldsymbol{\mu} = \boldsymbol{\mu}$ yields

$$\mu(\mathbf{B}_M) = T_{c^n}\mu(\mathbf{B}_M) = \mu(c^{-n}\mathbf{B}_M) \to \mu(\{\mathbf{0}\}),$$

which leads to $\mu(\mathbf{B}_M \backslash \{\mathbf{0}\}) = 0$, which contradicts (3.2). Part (iii) is proved.

Finally, part (iv) is an immediate consequence of the fact that $0.\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$ from $\mathbf{K}\backslash\{\boldsymbol{e}\}$ (as $\mathbf{K}$ is a pointed set). □

Proposition 3.2. *Let $\boldsymbol{\mu}$ be a p.m., $\boldsymbol{\mu} \neq \boldsymbol{\delta}(\mathbf{0})$, r and s two positive numbers, and α and β two real numbers. Suppose that $\boldsymbol{\mu}$ is simultaneously $St((r,\alpha))SS$ and $St((s,\beta))SS$. Then $\alpha = \beta$.*

Proof. From the assumption,

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{-1/\alpha}}\boldsymbol{\mu}^{r} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{\mu}^{s}.$$

First, for the case when $s = r^k$, $k = 1, 2, \ldots,$

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{T}_{r^{-1/\alpha}}\boldsymbol{\mu}^{r} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{\mu}^{r^k} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{T}_{r^{-1/\alpha}}\boldsymbol{\mu}^{r^k} = \boldsymbol{T}_{s^{-1/\beta}}(\boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\mu})^{r^{k-1}}\\ &= \cdots = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{T}_{r^{k/\alpha}}\boldsymbol{\mu} = \boldsymbol{T}_{r^{-k/\beta}.r^{k/\alpha}}\boldsymbol{\mu} = \boldsymbol{T}_{r^{(-k/\beta+k/\alpha)}}\boldsymbol{\mu}.\end{aligned}$$

This, together with (ii) and (iii) of Lemma 3.1, yields $r^{(k/\alpha-k/\beta)} = 1$ and $k/\alpha - k/\beta = 0$, which means that $\alpha = \beta$.

Secondly, for $s = r^{1/m}$, $m = 1, 2, \ldots$, by symmetry we also have $\alpha = \beta$. Therefore, the equality $\alpha = \beta$ is correct for the case when $s = r^{k/m}$, which is the case of $s = r^q$ with a positive rational number q.

Finally, let $s = r^c$ with $c > 0$. Then $c = \lim_n q_n$ for some increasing sequence (q_n) of rational numbers with $0 < q_n < c$ for all n. We have $c = q_n + d_n$ with $d_n > 0$ and $d_n \to 0+$ as $n \to \infty$. In that case,

$$\boldsymbol{\mu} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{\mu}^{s} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{\mu}^{r^c} = \boldsymbol{T}_{s^{-1/\beta}}\boldsymbol{\mu}^{r^{q_n+d_n}} = \boldsymbol{T}_{s^{-1/\beta}}(\boldsymbol{T}_{r^{q_n/\alpha}}\boldsymbol{\mu})^{r^{d_n}}.$$

It is clear that $r^{d_n} \to 1$ when $n \to \infty$. Then

$$L(\boldsymbol{\mu},\chi) = (L(\boldsymbol{\mu},(s^{1/\beta}\cdot r^{-q_n/\alpha})\circ\chi))^{r^{d_n}} \to L(\boldsymbol{\mu},(s^{1/\beta}\cdot r^{-c/\alpha})\circ\chi)$$

for all $\chi \in \tilde{\mathbf{K}}$. Therefore, [6, Theorem 5.4] ensures that

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{(-c/\beta+c/\alpha)}}\boldsymbol{\mu}.$$

Applying Lemma 3.1 again,

$$r^{(-c/\beta+c/\alpha)} = 1,$$

which means that $\alpha = \beta$, so the proof is complete. □

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After the above proposition, α is called the *semistability exponent* of $\boldsymbol{\mu}$. The next important lemma (Lemma 3.3) follows immediately from continuity of the mapping $(\boldsymbol{x}, a) \to \boldsymbol{T}_a \boldsymbol{x}$ and the Skorokhod representation theorem (see [3, page 70]).

Lemma 3.3. *Let $(\boldsymbol{p}_n) \subset P(\mathbf{K})$, $(\boldsymbol{p}_0) \in P(\mathbf{K})$, and (a_n) be a sequence of positive numbers. Suppose that $\boldsymbol{p}_n \to_{\mathrm{w}} \boldsymbol{p}_0$ and $a_n \to a_0$. Then*

$$\boldsymbol{T}_{a_n} \boldsymbol{p}_n \to_{\mathrm{w}} \boldsymbol{T}_{a_0} \boldsymbol{p}_0.$$

Proposition 3.4. *Let $\boldsymbol{\mu}$ be an inf. div. p.m. on $\mathbf{K}$, $\alpha \neq 0$, and define*

$$S(\boldsymbol{\mu}) := \{t > 0 : \boldsymbol{\mu} \in \boldsymbol{St}(t, \alpha)\boldsymbol{SS}\}.$$

Then $S(\boldsymbol{\mu})$ is a closed subgroup of the multiplicative group $(R^+; \cdot)$.

Proof. For any $t_1, t_2 \in S(\boldsymbol{\mu})$,

$$\boldsymbol{\mu}^{t_1 \cdot t_2} = (\boldsymbol{\mu}^{*t_1})^{t_2} = (\boldsymbol{T}_{t_1^{1/\alpha}} \boldsymbol{\mu})^{t_2} = \boldsymbol{T}_{t_1^{1/\alpha}} (\boldsymbol{\mu}^{t_2}) = \boldsymbol{T}_{t_1^{1/\alpha}} \boldsymbol{T}_{t_2^{1/\alpha}} \boldsymbol{\mu} = \boldsymbol{T}_{(t_1 \cdot t_2)^{1/\alpha}} \boldsymbol{\mu},$$

which means that $t_1 \cdot t_2 \in S(\boldsymbol{\mu})$. Moreover, it is obvious that $1, t_1^{-1} \in S(\boldsymbol{\mu})$. Hence, $S(\boldsymbol{\mu})$ is a subgroup of $(R^+; \cdot)$.

Let $t_n \in S(\boldsymbol{\mu})$ and $t_n \to t_0$, $t_0 > 0$. Then, by virtue of the weak continuity of the convolution semigroup $\{\boldsymbol{\mu}_t = \boldsymbol{\mu}^t, t > 0\}$,

$$\boldsymbol{\mu}^{t_n} \to_{\mathrm{w}} \boldsymbol{\mu}^{t_0}. \tag{3.3}$$

On the other hand, $\boldsymbol{\mu}^{t_n} = \boldsymbol{T}_{t_n^{1/\alpha}} \boldsymbol{\mu}$. Therefore, Lemma 3.3 implies that

$$\boldsymbol{\mu}^{t_n} = \boldsymbol{T}_{t_n^{1/\alpha}} \mu \to_{\mathrm{w}} \boldsymbol{T}_{t_0^{1/\alpha}} \boldsymbol{\mu}. \tag{3.4}$$

For $\chi \in \tilde{\mathbf{K}}$, (3.3) and (3.4) give

$$L(\boldsymbol{\mu}^{t_n}, \chi) = \int_{\boldsymbol{K}} \chi(\boldsymbol{x}) \boldsymbol{\mu}^{t_n}(d\boldsymbol{x}) \to \int_{\boldsymbol{K}} \chi(\boldsymbol{x}) \boldsymbol{\mu}^{t_0}(d\boldsymbol{x}) = L(\boldsymbol{\mu}^{t_0}, \chi)$$

and

$$\begin{aligned} L(\boldsymbol{\mu}^{t_n}, \chi) &= L(\boldsymbol{T}_{t_n^{1/\alpha}} \boldsymbol{\mu}, \chi) = \int_{\boldsymbol{K}} \chi(\boldsymbol{x}) \boldsymbol{T}_{t_n^{1/\alpha}} \boldsymbol{\mu}(d\boldsymbol{x}) = \int_{\boldsymbol{K}} \chi(t_n^{1/\alpha} \boldsymbol{y}) \boldsymbol{\mu}(d\boldsymbol{y}) \\ &\to \int_{\boldsymbol{K}} \chi(t_0^{1/\alpha} \boldsymbol{y}) \boldsymbol{\mu}(d\boldsymbol{y}) = \int_{\boldsymbol{K}} \chi(\boldsymbol{x}) \boldsymbol{T}_{t_0^{1/\alpha}} \boldsymbol{\mu}(d\boldsymbol{x}) = L(\boldsymbol{T}_{t_0^{1/\alpha}} \boldsymbol{\mu}, \chi). \end{aligned}$$

Thus, $L(\boldsymbol{\mu}^{t_0}, \chi) = L(\boldsymbol{T}_{t_0^{1/\alpha}} \boldsymbol{\mu}, \chi)$. Then, by virtue of [6, Theorem 5.4], we get $\boldsymbol{\mu}^{t_0} = \boldsymbol{T}_{t_0^{1/\alpha}} \boldsymbol{\mu}$ and the proof is completed. □

After the above proposition, we see that $S(\boldsymbol{\mu})$ is either a discrete subgroup generated by some positive r_0 greater than 1, or equal to R^+. In the first case, $S(\boldsymbol{\mu}) = \{r = r_0^m, m \in Z\}$, $r_0 > 1$, the p.m. $\boldsymbol{\mu}$ is called *strictly core* (r_0, α)*-semistable*, or briefly $StC(r_0, \alpha)SS$. In the last case, the p.m. $\boldsymbol{\mu}$ is called *strictly α-stable*, $\alpha \neq 0$, if $\boldsymbol{\mu}$ is inf. div. and

$$\boldsymbol{\mu}^r = \boldsymbol{T}_{r^{1/\alpha}} \boldsymbol{\mu} \tag{3.5}$$

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for every positive number r. Then we say briefly that $\boldsymbol{\mu}$ is $St\alpha S$ and denote it by $\boldsymbol{\mu} \in St\alpha S$; the parameter α is called the *stability exponent* of $\boldsymbol{\mu}$. For convenience, sometimes we omit the term α (or (r, α)) and say that a given p.m. $\boldsymbol{\mu}$ is *stable* (or *semistable*, respectively).

Proposition 3.5. *Let $\boldsymbol{\mu}$ be an inf. div. p.m. on $\mathbf{K}$ and α be given, $\alpha \neq 0$. Then the following conditions are equivalent:*

(i) *the p.m. $\boldsymbol{\mu}$ is strictly α-stable;*

(ii) *for $n = 2, 3, \ldots,$*

$$\boldsymbol{\mu}^n = \boldsymbol{T}_{n^{1/\alpha}}\boldsymbol{\mu}; \tag{3.6}$$

(iii) *for every $a > 0$, $b > 0$,*

$$\boldsymbol{T}_{a^{1/\alpha}}\boldsymbol{\mu} * \boldsymbol{T}_{b^{1/\alpha}}\boldsymbol{\mu} = \boldsymbol{T}_{(a+b)^{1/\alpha}}\boldsymbol{\mu}. \tag{3.7}$$

Proof. It is clear that (i) immediately implies (ii). We claim that (ii) entails (i). Indeed, let r be any positive number. Without loss of generality, we can suppose that $r \in (0; 1)$. Then there exists a subsequence (n_k) of natural numbers such that $n_k/n_{k+1} \to r$. From (3.6),

$$\boldsymbol{\mu} = \boldsymbol{T}_{n^{-1/\alpha}}\boldsymbol{\mu}^n \quad \text{and} \quad \boldsymbol{\mu} = \boldsymbol{T}_{n^{1/\alpha}}\boldsymbol{\mu}^{1/n}$$

for every natural n. Consequently,

$$\boldsymbol{\mu} = \boldsymbol{T}_{n_k^{1/\alpha}}\boldsymbol{\mu}^{1/n_k} = \boldsymbol{T}_{n_k^{1/\alpha}}(\boldsymbol{T}_{n_{k+1}^{-1/\alpha}}\boldsymbol{\mu}^{n_{k+1}})^{1/n_k} = \boldsymbol{T}_{(n_k/n_{k+1})^{1/\alpha}}(\boldsymbol{\mu})^{(n_{k+1}/n_k)}.$$

Then Lemmas 2.3 and 3.3 and the convergence $n_k/n_{k+1} \to r$ yield

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\mu}^{1/r},$$

which ensures that (3.5) is true.

We will show that (i) implies (iii). Namely, if (3.5) is valid, then

$$\boldsymbol{T}_{a^{1/\alpha}}\boldsymbol{\mu} * \boldsymbol{T}_{b^{1/\alpha}}\boldsymbol{\mu} = \boldsymbol{\mu}^a * \boldsymbol{\mu}^b = \boldsymbol{\mu}^{(a+b)} = \boldsymbol{T}_{(a+b)^{1/\alpha}}\boldsymbol{\mu},$$

which means that (3.7) holds.

To complete the proof, we suppose that (3.7) is true and conduct an induction by n to show (3.6). For $n = 2$, let us apply (3.7) with $a = b = 1$ to get

$$\boldsymbol{\mu}^2 = \boldsymbol{\mu} * \boldsymbol{\mu} = \boldsymbol{T}_{1^{1/\alpha}}\boldsymbol{\mu} * \boldsymbol{T}_{1^{1/\alpha}}\boldsymbol{\mu} = \boldsymbol{T}_{2^{1/\alpha}}\boldsymbol{\mu}.$$

Let (3.6) hold for n; then, applying (3.7) with $a = n^{1/\alpha}$ and $b = 1$,

$$\boldsymbol{\mu}^{n+1} = \boldsymbol{\mu}^n * \boldsymbol{\mu} = \boldsymbol{T}_{n^{1/\alpha}}\boldsymbol{\mu} * \boldsymbol{T}_{1^{1/\alpha}}\boldsymbol{\mu} = \boldsymbol{T}_{(n+1)^{1/\alpha}}\boldsymbol{\mu}.$$

Therefore, (3.6) is true for $n + 1$. The proposition is proved. □

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EXAMPLE. Let $\mathbf{K}^+ = [0; +\infty)$ and $\mathbf{K}^- = [-\infty; 0]$, both endowed with $\vee$ (maximum) as semigroup operator, and the ordinary multiplication by positive numbers. Then $\mathbf{K}^+$ is a convex cone in which the neutral element coincides with the origin of the cone; simultaneously $\mathbf{K}^-$ is a convex cone with neutral element different from the cone origin. Let α be a positive number, β be a negative number, and k be a natural number greater than 1. It is well known that the Fréchet function

$$\Phi_\alpha(x) = \begin{cases} \exp(-x^{-\alpha}) & x > 0, \\ 0 & x = 0 \end{cases}$$

is the distribution function of an α-stable random variable taking values in $\mathbf{K}^+$, and the Weibull function

$$\Psi_\beta(x) = \exp(-(-x)^{-\beta}), \quad x \le 0$$

is the distribution function of a β-stable random variable taking values in $\mathbf{K}^-$.

For semistability of p.m.s, we are concerned with the functions

$$F_{1,\alpha,k}(x) = \exp\Big[-\sum_{n=-\infty}^{\infty} x^{-\alpha}\mathbf{1}_{\mathbf{A}_{\alpha,n}}(x)\Big], \quad x > 0;$$

$$F_{2,\beta,k}(x) = \exp\Big[-\sum_{n=-\infty}^{\infty} (-x)^{-\beta}\mathbf{1}_{\mathbf{B}_{\beta,n}}(x)\Big], \quad x < 0,$$

where $\mathbf{1}_{\mathbf{C}}(x)$ denotes the indicator function of the set $\mathbf{C}$, $\mathbf{A}_{\alpha,n} = (k^{n/\alpha}; k^{(n+1)/\alpha}]$, $\mathbf{B}_{\beta,n} = (-k^{n/\beta}; -k^{(n+1)/\beta}]$, $n = 0, \pm1, \pm2, \ldots$. It is clear that the distribution functions $F_{1,\alpha,k}(x)$ and $F_{2,\beta,k}(x)$ satisfy $F^k_{1,\alpha,k}(x) = F_{1,\alpha,k}(k^{-1/\alpha}x)$ for all $x \in \mathbf{K}^+$ and $F^k_{2,\beta,k}(x) = F_{2,\beta,k}(k^{-1/\beta}x)$ for all $x \in \mathbf{K}^-$. This means that the function $F_{1,\alpha,k}(x)$ is a distribution function of a (k,α)-semistable random variable with values in $\mathbf{K}^+$ and $F_{2,\beta,k}(x)$ is one of a (k,β)-semistable random variable with values in $\mathbf{K}^-$.

PROPOSITION 3.6. *Let $\boldsymbol{\mu} \in P(\mathbf{K})$ be an inf. div. p.m. Suppose that there exist numbers α_1, α_2, r_1, and r_2 such that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $r_1 > 0$, $r_2 > 0$, and $\ln r_1/\ln r_2$ is an irrational number. Assume in addition that $\boldsymbol{\mu}$ is both $St(r_1,\alpha_1)SS$ and $St(r_2,\alpha_2)SS$. Then $\alpha_1 = \alpha_2 = \alpha$ and $\boldsymbol{\mu}$ is strictly α-stable.*

PROOF. The equality $\alpha_1 = \alpha_2$ is an immediate consequence of Proposition 3.2. Moreover, from Proposition 3.4, we see that $S(\boldsymbol{\mu})$ is a closed subgroup of $(R^+; \cdot)$. In the meantime, because $r_1, r_2 \in S(\boldsymbol{\mu})$, it implies from the irrationality of $\ln r_1/\ln r_2$ that $S(\boldsymbol{\mu})$ must be R^+, which means that $\boldsymbol{\mu}$ is strictly α-stable. The proof is completed. □

PROPOSITION 3.7. *Let $\boldsymbol{\mu} \in P(\mathbf{K})$ and numbers α and r be given such that $\alpha \neq 0$, $r > 0$, and $r \neq 1$. Assume that $\boldsymbol{\delta}(\mathbf{0}) \neq \boldsymbol{\mu} \neq \boldsymbol{\delta}(\boldsymbol{e})$ and $\boldsymbol{\mu}$ is $St(r,\alpha)SS$. Then the semistability exponent α is positive if and only if the neutral element in $\mathbf{K}$ coincides with the origin, $\boldsymbol{e} = \mathbf{0}$ (equivalently, α is negative if and only if the neutral element in $\mathbf{K}$ and the origin are different, $\boldsymbol{e} \neq \mathbf{0}$).*

Proof. Without loss of generality, we can suppose that $r > 1$. Then, from (3.1),

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{n/\alpha}}\boldsymbol{\mu}^{1/r^n} \tag{3.8}$$

for all natural n.

From $r > 1$, it is evident that $1/r^n \to 0$; then Lemma 2.3 implies that

$$\boldsymbol{\mu}^{1/r^n} \to_{\mathrm{w}} \boldsymbol{\delta}(\boldsymbol{e}). \tag{3.9}$$

Suppose that $\boldsymbol{e} = \mathbf{0}$ and $\alpha < 0$; then $r^{n/\alpha} \to 0$. In that case, Lemma 3.3 together with (3.9) yields $\boldsymbol{\mu} = \boldsymbol{\delta}(\boldsymbol{e})$. This contradicts the assumption and α must be positive.

Conversely, if $\boldsymbol{e} \neq \mathbf{0}$ and $\alpha > 0$, then $r^{-n/\alpha} \to 0$; from (3.8),

$$\boldsymbol{T}_{r^{-n/\alpha}}\boldsymbol{\mu} = \boldsymbol{\mu}^{1/r^n}. \tag{3.10}$$

The left-hand side of (3.10) weakly converges to $\boldsymbol{T}_0\boldsymbol{\mu} = \boldsymbol{\tau}(\boldsymbol{\mu}(\{\boldsymbol{e}\}))$ by virtue of Lemma 3.3; meanwhile, $\boldsymbol{\tau}(\boldsymbol{\mu}(\{\boldsymbol{e}\})) \neq \boldsymbol{\delta}(\boldsymbol{e})$ because $\boldsymbol{\mu} \neq \boldsymbol{\delta}(\boldsymbol{e})$ by assumption. This contradicts (3.9). Hence, α must be a negative number. □

The following corollary is an immediate result of the above proposition.

Corollary 3.8. *Assume that $\alpha \neq 0$ and a p.m. $\boldsymbol{\mu}$ is St(α)S, $\boldsymbol{\delta}(\mathbf{0}) \neq \boldsymbol{\mu} \neq \boldsymbol{\delta}(\boldsymbol{e})$. Then:*

(a) *the stability exponent α is positive if the neutral element in $\mathbf{K}$ coincides with the origin, $\boldsymbol{e} = \mathbf{0}$;*

(b) *the stability exponent α is negative if the neutral element in $\mathbf{K}$ and the origin are different, $\boldsymbol{e} \neq \mathbf{0}$.*

4. Domain of attraction and domain of semi-attraction

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P(\mathbf{K})$, we say that $\boldsymbol{\lambda}$ belongs to the *domain of strict attraction* of $\boldsymbol{\mu}$ (in symbols $\boldsymbol{\lambda} \in DStA(\boldsymbol{\mu})$) if

$$\boldsymbol{T}_{a_n}\boldsymbol{\lambda}^{*n} \to_{\mathrm{w}} \boldsymbol{\mu}$$

for some sequence of positive numbers a_n. Then we have the following theorem.

Theorem 4.1. *Let $\boldsymbol{\mu} \in P(\mathbf{K})$ be a p.m. Then:*

(a) *if $\boldsymbol{\mu}$ is strictly stable, then $DStA(\boldsymbol{\mu}) \neq \emptyset$;*

(b) *suppose that $\boldsymbol{\mu} \notin \mathrm{Triv}(\mathbf{K})$ and $\boldsymbol{\mu}$ is not a nilpotent element of $(P(\mathbf{K}), *)$, which means that $\boldsymbol{\mu} \neq \boldsymbol{\mu}^{*n}$ for every $n = 2, 3, \ldots$. In that case, if $DStA(\boldsymbol{\mu}) \neq \emptyset$, then $\boldsymbol{\mu}$ is strictly stable.*

Proof. Part (a) is an immediate consequence of Proposition 3.5, because the condition (3.5) implies $\boldsymbol{\mu} \in DStA(\boldsymbol{\mu})$.

To prove (b), let $\boldsymbol{T}_{c_m}\boldsymbol{\lambda}^{*m} \to_{\mathrm{w}} \boldsymbol{\mu}$. Then, for any fixed natural number n greater than 1,

$$\boldsymbol{T}_{c_{k.n}}\boldsymbol{\lambda}^{*k.n} \to_{\mathrm{w}} \boldsymbol{\mu} \tag{4.1}$$

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when $k \to \infty$. On the other hand,

$$\boldsymbol{T}_{c_{k.n}}\boldsymbol{\lambda}^{*kn} = \boldsymbol{T}_{c_{k.n}/c_k}((\boldsymbol{T}_{c_k}\boldsymbol{\lambda}^{*k})\underbrace{*\cdots*}_{n}(\boldsymbol{T}_{c_k}\boldsymbol{\lambda}^{*k})). \tag{4.2}$$

Let us denote the set of all cluster points of the real number sequence $(c_{k.n}/c_k)$ by $LIM[(c_{k.n}/c_k)]$. Then $0, \infty \notin LIM[(c_{k.n}/c_k)]$. Indeed, if there exists a subsequence (k_j) of natural numbers such that $c_{k_j.n}/c_{k_j} \to 0$, then (4.1) and (4.2) together with Lemmas 3.1 and 3.3 yield

$$\boldsymbol{\mu} = \boldsymbol{T}_0\boldsymbol{\mu}^{*n} \in \mathrm{Triv}(\mathbf{K}),$$

which contradicts the assumption.

If there exists a subsequence (k_j) of natural numbers with $c_{k_j.n}/c_{k_j} \to \infty$, then $c_{k_j}/c_{k_j.n} \to 0$. Meanwhile, from (4.2),

$$\boldsymbol{T}_{c_{k_j}/c_{n.k_j}}(\boldsymbol{T}_{c_{k_j.n}}\boldsymbol{\lambda}^{*k_j.n}) = (\boldsymbol{T}_{c_{k_j}}\boldsymbol{\lambda}^{*k_j})\underbrace{*\cdots*}_{n}(\boldsymbol{T}_{c_{k_j}}\boldsymbol{\lambda}^{*k_j}),$$

which, together with Lemmas 3.1 and 3.3, entails $\boldsymbol{\mu}^{*n} \in \mathrm{Triv}(\mathbf{K})$, which yields $\boldsymbol{\mu} \in \mathrm{Triv}(\mathbf{K})$, and we have a contradiction.

Let $a, b \in LIM[(c_{k.n}/c_k)]$. Using the same argument as above, we can conclude that $\boldsymbol{\mu} = \boldsymbol{T}_a\boldsymbol{\mu}^{*n} = \boldsymbol{T}_b\boldsymbol{\mu}^{*n}$. Then $\boldsymbol{T}_{a/b}\boldsymbol{\mu}^{*n} = \boldsymbol{\mu}^{*n}$; this, together with Lemma 3.1, implies that $a = b = b_n$ and $LIM[(c_{k.n}/c_k)] = \{b_n\}$ for some positive number b_n dependent on n. Consequently, $\boldsymbol{\mu} = \boldsymbol{T}_{b_n}\boldsymbol{\mu}^{*n}$, which ensures that $\boldsymbol{\mu}$ is inf. div. and $b_n \neq 1$ as $\boldsymbol{\mu}$ is not a nilpotent element. Therefore, $\boldsymbol{\mu}$ is $St(1/n, \alpha_n)SS$ with $\alpha_n = -1/\log_n(b_n) \neq 0$.

For another natural number m greater than 1, $\boldsymbol{\mu}$ is also $St(1/m, \alpha_m)SS$ by the same argument. Then Proposition 3.2 ensures that $\alpha_n = \alpha_m = \alpha \neq 0$, where the number α is independent of n. Consequently, the condition (3.5) holds for every natural number n, and $\boldsymbol{\mu}$ is strictly α-stable by virtue of Proposition 3.5. The theorem is proved. □

For $\boldsymbol{\mu}, \boldsymbol{\lambda} \in P(\mathbf{K})$ and $r \in (0; 1)$, we say that $\boldsymbol{\lambda}$ belongs to the *domain of strict r-semi-attraction* of $\boldsymbol{\mu}$ (denoted $\boldsymbol{\lambda} \in DStSA(r, \boldsymbol{\mu})$ if

$$\boldsymbol{T}_{a_k}\boldsymbol{\lambda}^{*n_k} \to_{\mathrm{w}} \boldsymbol{\mu} \tag{4.3}$$

for a subsequence (n_k) of natural numbers and some sequence (a_k) of positive numbers such that

$$n_k/n_{k+1} \to r \in (0; 1) \tag{4.4}$$

when $k \to \infty$.

Lemma 4.2.

(i) *Let $\boldsymbol{\lambda}$ be a p.m. on $\mathbf{K}$ and (a_n) be a sequence of positive numbers. If $L(\boldsymbol{T}_{a_n}\boldsymbol{\lambda}, \chi) \to 1$ as $n \to \infty$ for every $\chi \in \tilde{\mathbf{K}}$, then $\boldsymbol{T}_{a_n}\boldsymbol{\lambda} \to_{\mathrm{w}} \boldsymbol{\delta}(\boldsymbol{e})$ as $n \to \infty$.*

(ii) *Let $\boldsymbol{\mu} \in P(\mathbf{K})$; $(\boldsymbol{\lambda}_k)$ be a sequence of inf. div. p.m.s and (n_k) and (m_k) be sequences of natural numbers such that $m_k/n_k \to r$ and $\boldsymbol{\lambda}_k^{*n_k} \to_{\mathrm{w}} \boldsymbol{\mu}$. Then $\boldsymbol{\lambda}_k^{*m_k} \to_{\mathrm{w}} \boldsymbol{\mu}^r$.*

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Proof. (i) Analogously as in the proof of Lemma 2.3, we can consider $\mathbf{K}$ as a subset of $\mathbf{K}^{\#}$ and consider $\boldsymbol{\lambda}$, $\boldsymbol{T}_{a_n}\boldsymbol{\lambda}$, and $\boldsymbol{\delta}(\boldsymbol{e})$ as p.m.s on $\mathbf{K}^{\#}$, concentrated on $\mathbf{K}$. Then the first part of the lemma follows immediately from [2, Theorem 4.2.11].

(ii) Using the same idea as the first part, concerning $\boldsymbol{\lambda}_k$, $\boldsymbol{\mu}$, $\boldsymbol{\lambda}_k^{*n_k}$, $\boldsymbol{\lambda}_k^{*m_k}$, and $\boldsymbol{\mu}^r$ as p.m.s on $\mathbf{K}^{\#}$, concentrated on $\mathbf{K}$, we can conclude the result from [2, Theorem 4.2.11] and the equality

$$L(\boldsymbol{\lambda}_k^{*m_k}, \chi) = L(\boldsymbol{\lambda}_k, \chi)^{m_k} = L(\boldsymbol{\lambda}_k^{*n_k}, \chi)^{m_k/n_k}$$

for $\chi \in \tilde{\mathbf{K}}$. □

Theorem 4.3. *Suppose that $\boldsymbol{\mu} \in P(\mathbf{K})$, $\boldsymbol{\mu} \neq \boldsymbol{\delta}(\boldsymbol{e})$, and $\boldsymbol{\mu}(\{\mathbf{0}\}) = 0$. Then $\boldsymbol{\mu}$ is strictly r-semistable if and only if $DStSA(r, \boldsymbol{\mu}) \neq \emptyset$.*

Proof. To prove the necessity, from $\boldsymbol{\mu}^r = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\mu}$, without loss of generality, we can suppose that $r > 1$; then $r^k \to \infty$ as $k \to \infty$. We define $n_k := [r^k]$, where $[a]$ denotes the integer part of a real number a. Then it is evident that

$$n_k/n_{k+1} = [r^k]/[r^{k+1}] \to 1/r$$

when $k \to \infty$. Taking $a_k = r^{n/\alpha}$,

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{-1/\alpha}}\boldsymbol{\mu}^r = \boldsymbol{T}_{r^{-1/\alpha}}(\boldsymbol{T}_{r^{-1/\alpha}}\boldsymbol{\mu}^r)^r = \boldsymbol{T}_{r^{-2/\alpha}}\boldsymbol{\mu}^{r^2} = \cdots = \boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{r^k}.$$

Therefore,

$$\boldsymbol{\mu} = \boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{[r^k]} * \boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{r^k-[r^k]}. \tag{4.5}$$

In the case when $\boldsymbol{e} = \mathbf{0}$, by virtue of Proposition 3.7, we have $\alpha > 0$, and $r^{-k/\alpha} \to 0$ as $k \to \infty$. Then, because $r^k - [r^k] \in [0; 1)$, each subsequence (k_i) of (k) contains another subsequence $(m_j) \subset (k_i)$ such that $r^{m_j} - [r^{m_j}] \to c$ as $j \to \infty$, for some $c \in [0; 1]$. Then Lemmas 3.1 and 3.3 entail

$$\boldsymbol{T}_{r^{-m_j/\alpha}}\boldsymbol{\mu}^{(r^{m_j}-[r^{m_j}])} \to_{\mathrm{w}} \boldsymbol{\delta}(\boldsymbol{e}).$$

Consequently, every subsequence of $(\boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{r^k-[r^k]})$ contains another subsequence weakly convergent to $\boldsymbol{\delta}(\boldsymbol{e})$. Then, because $P(\mathbf{K})$ is a metric space,

$$\boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{r^k-[r^k]} \to_{\mathrm{w}} \boldsymbol{\delta}(\boldsymbol{e}). \tag{4.6}$$

The conditions (4.5) and (4.6) yield the convergence

$$\boldsymbol{T}_{r^{-k/\alpha}}\boldsymbol{\mu}^{*[r^k]} \to_{\mathrm{w}} \boldsymbol{\mu},$$

which means that $\boldsymbol{\mu} \in DStSA(r, \boldsymbol{\mu})$.

In the case when $\boldsymbol{e} \neq \mathbf{0}$, Proposition 3.7 ensures that $\alpha < 0$, and $r^{-k/\alpha} \to \infty$ as $k \to \infty$. Besides, $\mathbf{B}_k \uparrow \mathbf{K}\backslash\{e\}; \mathbf{B}_{1/k} \downarrow \{0\}$ as $k \to \infty$, and the condition $\boldsymbol{\mu}(\{\mathbf{0}\}) = 0$ implies $\boldsymbol{\mu}^{(r^k-[r^k])}(\{\mathbf{0}\}) = 0$ for every natural number k.

For any subsequence $(k_j) \subset (k)$ with $r^{k_j} - [r^{k_j}] \to c$ as $j \to \infty$ for some $c \in [0; 1]$, by virtue of Lemma 2.3, we have $\boldsymbol{\mu}^{(r^{k_j}-[r^{k_j}])} \to_{\mathrm{w}} \boldsymbol{\mu}^c$. Then it is easy to show that $\boldsymbol{\mu}^c(\{\mathbf{0}\}) = 0$

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and, for every positive number ε, there is a large number m such that $\mu^c(\mathbf{B}_{1/m}) < \varepsilon$. Moreover, we can choose a positive number s less than $1/m$ such that $\mathbf{B}_s$ is a μ^c-continuity set. Then [30, Theorem II.6.1] ensures that $\mu^{(r^{k_j}-[r^{k_j}])}(\mathbf{B}_s) \to \mu^c(\mathbf{B}_s) < \varepsilon$. Therefore, for every given large positive number t,

$$T_{r^{-k_j/\alpha}}\mu^{(r^{k_j}-[r^{k_j}])}(\mathbf{B}_t) = \mu^{(r^{k_j}-[r^{k_j}])}(\mathbf{B}_{t.r^{k_j/\alpha}}) \le \varepsilon \tag{4.7}$$

whenever $t.r^{k_j/\alpha} \le s$, which holds for all sufficiently large j.

The condition (4.7) ensures that $T_{r^{-k_j/\alpha}}\mu^{(r^{k_j}-[r^{k_j}])} \to_{\mathrm{w}} \delta(e)$ as $j \to \infty$, which confirms that every subsequence of $(T_{r^{-k/\alpha}}\mu^{(r^k-[r^k])})$ contains another subsequence weakly convergent to $\delta(e)$; therefore, $T_{r^{-k/\alpha}}\mu^{(r^k-[r^k])} \to_{\mathrm{w}} \delta(e)$ as $k \to \infty$. This, together with (4.5), entails $\mu \in DStSA(r,\mu)$. The necessity is proved.

For the sufficiency, let (4.3) and (4.4) hold. First, we will show that μ is inf. div. Namely, let m be an arbitrary natural number. From (4.3),

$$L(T_{a_k}\lambda,\chi)^{n_k-1} \cdot L(T_{a_k}\lambda,\chi) = L(T_{a_k}\lambda,\chi)^{n_k} = L(T_{a_k}\lambda^{*n_k},\chi) \to L(\mu,\chi)$$

as $k \to \infty$ for every $\chi \in \tilde{\mathbf{K}}$. However, since $(n_k-1)/n_k \to 1$ as $k \to \infty$,

$$L(T_{a_k}\lambda,\chi)^{n_k-1} \to L(\mu,\chi).$$

Therefore, $L(T_{a_k}\lambda,\chi) \to 1$ as $k \to \infty$ for every $\chi \in \tilde{\mathbf{K}}$. Hence, Lemma 4.2 entails $T_{a_k}\lambda \to_{\mathrm{w}} \delta(e)$; in consequence, $T_{a_k}\lambda^{*l} \to_{\mathrm{w}} \delta(e)$ as $k \to \infty$ for each positive integer l not greater than m. This, together with (4.3), implies that

$$T_{a_k}\lambda^{*m.[n_k/m]} = T_{a_k}\lambda^{*[n_k/m]} \underbrace{*\cdots*}_{m} T_{a_k}\lambda^{*[n_k/m]} \to_{\mathrm{w}} \mu$$

and

$$L(T_{a_k}\lambda^{*[n_k/m]},\chi)^m \to L(\mu,\chi) \tag{4.8}$$

as $k \to \infty$ for every $\chi \in \tilde{\mathbf{K}}$.

We consider $\mathbf{K}$ again as a subset of $\mathbf{K}^\#$ and consider $T_{a_k}\lambda^{*[n_k/m]}$ and μ as p.m.s on $\mathbf{K}^\#$, concentrated on $\mathbf{K}$. Then it follows immediately from (4.8) that $L(T_{a_k}\lambda^{*[n_k/m]},\chi) \to L(\mu,\chi)^{1/m}$ and we can apply [2, Theorem 4.2.11] to confirm the existence of a p.m. $\mu_{1/m}$ concentrated on $\mathbf{K}$ such that $L(\mu_{1/m},\chi) = L(\mu,\chi)^{1/m}$ for all $\chi \in \tilde{\mathbf{K}}$ and $T_{a_k}\lambda^{*[n_k/m]} \to_{\mathrm{w}} \mu_{1/m}$ as $k \to \infty$. In consequence, $\mu_{1/m}^{*m} = \mu$. The last equality holds for every natural number m; this means that μ is inf. div.

Now we show that

$$\frac{a_k}{a_{k+1}} \to a_0 \tag{4.9}$$

for some positive number a_0. Indeed, for every $\chi \in \tilde{\mathbf{K}}$, from (4.3),

$$L(T_{a_k}\lambda^{*n_k},\chi) \to L(\mu,\chi); \quad L(T_{a_{k+1}}\lambda^{*n_{k+1}},\chi) \to L(\mu,\chi) \tag{4.10}$$

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and

$$L(\boldsymbol{T}_{a_k}\boldsymbol{\lambda}^{*n_k},\chi) = (L(\boldsymbol{T}_{a_{k+1}}\boldsymbol{\lambda}^{*n_{k+1}},(a_k/a_{k+1})\circ\chi))^{(n_k/n_{k+1})}. \tag{4.11}$$

If 0 is a cluster point of the sequence (a_k/a_{k+1}), then (4.10), (4.11), and Lemma 3.1 imply that

$$L(\boldsymbol{\mu},\chi) = L(\boldsymbol{\delta}(\mathbf{0})^r,\chi) = L(\boldsymbol{\delta}(\mathbf{0}),\chi).$$

This, together with [6, Theorem 5.4], yields $\boldsymbol{\mu} = \boldsymbol{\delta}(\mathbf{0})$, which contradicts the assumption $\boldsymbol{\mu}(\{\mathbf{0}\}) = 0$.

If ∞ is a cluster point of the sequence (a_k/a_{k+1}), then 0 is a cluster point of the sequence (a_{k+1}/a_k), $n_{k+1}/n_k \to 1/r$, and instead of (4.11) we have

$$L(\boldsymbol{T}_{a_{k+1}}\boldsymbol{\lambda}^{*n_{k+1}},\chi) = L(\boldsymbol{T}_{a_k}\boldsymbol{\lambda}^{*n_k},(a_{k+1}/a_k)\circ\chi)^{(n_{k+1}/n_k)}. \tag{4.11$'$}$$

In that case, repeating the same argument as above, (4.10) and (4.11′) also lead to a contradiction.

In conclusion, all cluster points of the sequence (a_k/a_{k+1}) are positive and finite. Let a_{01} and a_{02} be two such cluster points. Then, in the same way as above, we can show that (4.10) and (4.11) imply that

$$L(\boldsymbol{\mu},\chi) = L(\boldsymbol{T}_{a_{01}}\boldsymbol{\mu},\chi)^r = L(\boldsymbol{T}_{a_{02}}\boldsymbol{\mu},\chi)^r$$

for all $\chi \in \tilde{\mathbf{K}}$, which results in $\boldsymbol{T}_{a_{01}}\boldsymbol{\mu} = \boldsymbol{T}_{a_{02}}\boldsymbol{\mu}$ by virtue of [6, Theorem 5.4]. The last equation together with Lemma 3.1 ensures that $a_{01} = a_{02}$. Consequently, (4.9) is valid.

Finally, combining (4.3), (4.4), and (4.9) and Lemmas 3.3 and 4.2,

$$\boldsymbol{T}_{a_k}\boldsymbol{\lambda}^{*n_k} \to_{\mathrm{w}} \boldsymbol{\mu}\,; \quad \boldsymbol{T}_{a_{k+1}}\boldsymbol{\lambda}^{*n_{k+1}} \to_{\mathrm{w}} \boldsymbol{\mu}$$

and

$$\boldsymbol{T}_{a_k}\boldsymbol{\lambda}^{*n_k} = \boldsymbol{T}_{a_k/a_{k+1}}(\boldsymbol{T}_{a_{k+1}}\boldsymbol{\lambda}^{*n_{k+1}})^{*n_k/n_{k+1}} \to_{\mathrm{w}} \boldsymbol{T}_{a_0}\boldsymbol{\mu}^{*r},$$

which imply that $\boldsymbol{\mu} = \boldsymbol{T}_{a_0}\boldsymbol{\mu}^{*r}$, which means that $\boldsymbol{\mu}$ is strictly r-semistable. The proof is completed. □

Theorem 4.4. *Suppose that $\boldsymbol{\mu} \in P(\mathbf{K})$ is strictly stable. Then*

$$DStSA(r,\boldsymbol{\mu}) = DStA(\boldsymbol{\mu})$$

for every $r \in (0;1)$.

Proof. Let $r \in (0;1)$ be given and $\boldsymbol{\mu}$ be a strictly stable p.m., which means that

$$\boldsymbol{\mu}^{*r} = \boldsymbol{T}_{r^{1/\alpha}}\boldsymbol{\mu} \tag{4.12}$$

for every $r \in R^+$. For the inclusion $DStSA(r,\boldsymbol{\mu}) \subset DStA(\boldsymbol{\mu})$, assume that $\boldsymbol{\lambda} \in DStSA(r,\boldsymbol{\mu})$; then there exists a sequence of positive numbers (b_{n_k}) such that

$$\boldsymbol{T}_{b_{n_k}}\boldsymbol{\lambda}^{*n_k} \to_{\mathrm{w}} \boldsymbol{\mu} \tag{4.13}$$

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with $n_k/n_{k+1} \to r$ as $k \to \infty$. We extend the sequence (b_{n_k}) to a sequence (a_n) indexed by the whole sequence of natural numbers so that

$$\boldsymbol{T}_{a_n}\boldsymbol{\lambda}^{*n} \to_{\mathrm{w}} \boldsymbol{\mu}. \tag{4.14}$$

Namely, without loss of generality, we can suppose that $n_1 = 1$ and $b_1 = 1$. Then, for each k and every natural number m such that $n_k \le m < n_{k+1}$, we define $a_m = b_{n_k}(m/n_k)^{-1/\alpha}$ and claim that every subsequence of $(\boldsymbol{T}_{a_n}\boldsymbol{\lambda}^{*n})$ contains another subsequence weakly convergent to $\boldsymbol{\mu}$, which, together with the fact of $P(\mathbf{K})$ being a separable metric space, ensures the validity of (4.14).

Indeed, let (m') be any subsequence of natural numbers; then, for each $m' \in (m')$, one can find a natural number $j_{m'}$ such that

$$n_{j_{m'}} \le m' < n_{j_{m'}+1}.$$

Hence, $n_{j_{m'}}/n_{j_{m'}+1} < n_{j_{m'}}/m' \le 1$ and, because $n_k/n_{k+1} \to r$, we can pick from (m') another subsequence (m'') such that

$$n_{j_{m''}}/m'' \to u \in [r, 1]. \tag{4.15}$$

Then, combining Lemmas 3.3 and 4.2 and (4.12), (4.13), and (4.15),

$$\begin{aligned}\boldsymbol{T}_{a_{m''}}\boldsymbol{\lambda}^{*m''} &= \boldsymbol{T}_{(m''/n_{j_{m''}})^{-1/\alpha}}(\boldsymbol{T}_{b_{n_{j_{m''}}}}\boldsymbol{\lambda}^{*n_{j_{m''}}})^{*m''/n_{j_{m''}}}\\ &\to_{\mathrm{w}} \boldsymbol{T}_{u^{1/\alpha}}\boldsymbol{\mu}^{*1/u} = (\boldsymbol{\mu}^{u})^{1/u} = \boldsymbol{\mu}.\end{aligned}$$

In consequence, (4.14) is true and $DStSA(r, \boldsymbol{\mu}) \subset DStA(\boldsymbol{\mu})$.

To prove $DStSA(r, \boldsymbol{\mu}) \supset DStA(\boldsymbol{\mu})$, let $\boldsymbol{\lambda} \in DStA(\boldsymbol{\mu})$. Then (4.14) holds for some sequence of positive numbers (a_n) and there exists a subsequence (n_k) of the sequence of natural numbers such that $n_k/n_{k+1} \to r$ as $k \to \infty$. Therefore, $\boldsymbol{T}_{a_{n_k}}\boldsymbol{\lambda}^{*n_k} \to_{\mathrm{w}} \boldsymbol{\mu}$, which means that $\boldsymbol{\lambda} \in DStSA(r, \boldsymbol{\mu})$. The theorem is proved. □

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