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A NOTE ON POSITIVE $\mathcal{AN}$ OPERATORS

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Abstract

We show that positive absolutely norm attaining operators can be characterised by a simple property of their spectra. This result clarifies and simplifies a result of Ramesh. As an application we characterise weighted shift operators which are absolutely norm attaining.

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A bounded linear operator T on a complex Hilbert space H is said to be absolutely norm attaining if, given any nonzero subspace M of H, there exists $\boldsymbol{x}_0$ in the unit ball M_1 of M such that $\|T\boldsymbol{x}_0\| = \sup\{\|T\boldsymbol{x}\| : \boldsymbol{x} \in M_1\}$. The set of all absolutely norm attaining operators, which we shall denote by $\mathcal{AN}$, was introduced by Carvajal and Neves [1]. As was shown by Pandey and Paulsen, there are severe restrictions on the structure of such operators.

THEOREM 1 [4, Theorem 5.1]. *Suppose that T is a positive operator on H. Then $T \in \mathcal{AN}$ if and only if $T = \alpha I + K + F$ where $\alpha \geq 0$, K is a positive compact operator and F is a self-adjoint finite-rank operator.*

In [5], Ramesh proposes a different characterisation of positive $\mathcal{AN}$ operators. Unfortunately Theorem 2.4 of [5] is misstated, and is perhaps more complicated than it needs to be. The main point of Theorem 2 below is that one only needs to check two elementary properties of the spectrum of an operator to ensure that it is of the form described in Theorem 1.

Let $C(H) = B(H)/K(H)$ denote the Calkin algebra with quotient map π, and recall that the essential spectrum of an operator T, denoted by $\sigma_{\mathrm{ess}}(T)$, is the spectrum of $\pi(T)$ in $C(H)$.

THEOREM 2. *Suppose that T is a positive operator on an infinite-dimensional Hilbert space H. Then $T \in \mathcal{AN}$ if and only if $\sigma_{\mathrm{ess}}(T)$ contains a single point α and $\sigma(T)$ contains only finitely many elements less than α.*

Proof. The forward direction is a direct consequence of standard results about the invariance of the essential spectrum under compact perturbations (see, for example, [2, Section XI.4]), and the proof of Theorem 1 (or Theorem 3.25 of [1]).

Conversely, suppose that T is a positive operator and that $\sigma_{\mathrm{ess}}(T) = \{\alpha\}$. This implies that $\pi(T - \alpha I)$ is a quasinilpotent self-adjoint element of $C(H)$ and hence is zero. That is, $T = \alpha I + K$ where K is a compact self-adjoint operator. The spectral theorem for such operators says that $K = \sum_{n\in N} \lambda_n P_n$, where N is a countable set and P_n is the orthogonal finite-rank projection onto the eigenspace for the eigenvalue λ_n.

Let $N^- = \{n : \lambda_n < 0\}$ and $N^+ = \{n : \lambda_n \geq 0\}$. Since $\sigma(T) = \{\alpha\} \cup \{\alpha + \lambda_n : n \in N\}$, if $\sigma(T)$ contains only finitely many elements less than α then N^- is a finite set and hence $F = \sum_{n\in N^-} \lambda_n P_n$ is self-adjoint and of finite rank (with the convention that an empty sum is zero). The operator $K^+ = \sum_{n\in N^+} \lambda_n P_n$ is compact and positive. Since $T = \alpha I + K^+ + F$ we can apply Theorem 1 to deduce that $T \in \mathcal{AN}$. □

Pandey and Paulsen observed in [4, Lemma 6.2] that $T \in \mathcal{AN}$ if and only if $|T| = (T^*T)^{1/2} \in \mathcal{AN}$, so one may write a corresponding characterisation of general $\mathcal{AN}$ operators in terms of the spectral properties of $|T|$.

As a simple application, it follows that a (bounded) weighted shift operator on ℓ^2,

$$T(x_1, x_2, x_3, \dots) = (0, w_1x_1, w_2x_2, \dots),$$

is absolutely norm attaining if and only if either:

(i) the set $\{|w_n|\}_{n=1}^{\infty}$ has a unique limit point α,
(ii) for all $\beta \neq \alpha$, $|w_n| = \beta$ for only finitely many values of n, and
(iii) $|w_n| < \alpha$ for only finitely many values of n;

or

(i′) $\sigma(|T|) = \{|w_n|\}_{n=1}^{\infty}$ is a finite set, and
(ii′) there is only one value of $\beta \in \sigma(|T|)$ such that $|w_n| = \beta$ for infinitely many values of n.

Some related results can be found in [3].

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