J. Appl. Prob. **50,** 464–474 (2013)
Printed in England

STOCHASTIC COMPARISONS OF SYMMETRIC SUPERMODULAR FUNCTIONS OF HETEROGENEOUS RANDOM VECTORS

ANTONIO DI CRESCENZO,* *Università di Salerno*
ESTHER FROSTIG,** *University of Haifa*
FRANCO PELLEREY,*** *Politecnico di Torino*

Abstract

Consider random vectors formed by a finite number of independent groups of independent and identically distributed random variables, where those of the last group are stochastically smaller than those of the other groups. Conditions are given such that certain functions, defined as suitable means of supermodular functions of the random variables of the vectors, are supermodular or increasing directionally convex. Comparisons based on the increasing convex order of supermodular functions of such random vectors are also investigated. Applications of the above results are then provided in risk theory, queueing theory, and reliability theory, with reference to (i) net stop-loss reinsurance premiums of portfolios from different groups of insureds, (ii) closed cyclic multiclass Gordon–Newell queueing networks, and (iii) reliability of series systems formed by units selected from different batches.

Keywords: Supermodular function; directionally convex function; increasing convex order; risks portfolio; cyclic queueing network; reliability; series system

2010 Mathematics Subject Classification: Primary 60E15
Secondary 62P05; 90B25

1. Introduction

In a previous paper [12] the reliability of series or parallel systems has been studied when the system components are randomly chosen from two different batches, the components of the first batch being more reliable than those of the second. It has been proved that the system's reliability increases, in the usual stochastic order sense, when the random number of components chosen from the first batch increases in the increasing convex order. As a consequence, the randomness in the number of components extracted from the two batches improves the reliability of the series system.

Stimulated by the previous research, in this paper we aim to obtain similar results involving stochastic systems described by more general mathematical structures than series or parallel systems. The starting idea is to consider random vectors composed of a finite number of independent groups of independent and identically distributed (i.i.d.) random variables, where the sizes of the groups are random themselves, and where the random variables of the last

Received 21 June 2012; revision received 13 September 2012.
* Postal address: Dipartimento di Matematica, Università di Salerno, Via Giovanni Paolo II, 132, Fisciano (SA) 84084, Italy. Email address: adicrescenzo@unisa.it
** Postal address: Department of Statistics, University of Haifa, Mount Carmel, Haifa 31905, Israel.
Email address: frostig@stat.haifa.ac.il
*** Postal address: Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino 10129, Italy. Email address: franco.pellerey@polito.it

group are stochastically smaller than those of the other groups. Roughly speaking, we propose to show that if the randomness in the sizes of the groups increases in some stochastic sense, then suitable measures of the random vectors, expressed by supermodular functions, increase in the 'increasing convex' stochastic order.

We recall that recent contributions oriented to the stochastic comparisons of random vectors, and involving increasing directionally convex transformations of the vectors, are given in [5]. See also [1], where increasing convex comparisons of generalized order statistics are given, and are extended to the increasing directionally convex comparisons of random vectors of generalized order statistics.

The plan of the paper is as follows. Section 2 is devoted to recalling some useful notions that will be used in the sequel, such as certain properties of multidimensional functions, and the definitions of various one-dimensional and multidimensional stochastic orders. The two main results of the paper are given in Section 3, where conditions are given such that a suitable function of the sizes of the groups is supermodular or is increasing directionally convex, and where stochastic comparisons are shown for pairs of supermodular functions of the underlying random vectors. In Section 4 we present some applications of the main results. The first case deals with risk theory, and involves stochastic comparisons of the total claim amount from a possible portfolio of risks, where the number of insureds of each group is random. In the second case we consider closed cyclic multiclass Gordon–Newell queueing networks with FCFS (first-come–first-served) service discipline, where some units of the network provide a service that gains a profit and other units supply auxiliary services originating a cost. We stochastically compare the total profit gained by the activity of the multiclass queueing network in the equilibrium state, by emphasizing the dependence on the random sizes of the units' classes. The last application involves the reliability of series systems formed by units selected from different batches. Comparisons are given for the availability and the lifetimes of different systems.

Throughout the paper, the terms 'increasing' and 'decreasing' are used in the nonstrict sense.

2. Preliminary notions

Hereafter we recall some necessary notions and useful properties of n-dimensional functions.

Let '$\le$' denote the coordinatewise ordering in $\mathbb{R}^n$. Given a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, we recall that it is said to be

- *directionally convex* if, for any $\boldsymbol{x}_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, such that $\boldsymbol{x}_1 \le \boldsymbol{x}_2 \le \boldsymbol{x}_4$, $\boldsymbol{x}_1 \le \boldsymbol{x}_3 \le \boldsymbol{x}_4$, and $\boldsymbol{x}_1 + \boldsymbol{x}_4 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, we have
$$\varphi(\boldsymbol{x}_2) + \varphi(\boldsymbol{x}_3) \le \varphi(\boldsymbol{x}_1) + \varphi(\boldsymbol{x}_4);$$
- *coordinatewise convex* if it is convex in each coordinate when the remaining ones are fixed;
- *supermodular* if, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, it satisfies
$$\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{y}) \le \varphi(\boldsymbol{x} \wedge \boldsymbol{y}) + \varphi(\boldsymbol{x} \vee \boldsymbol{y}), \tag{1}$$
where the operators '$\wedge$' and '$\vee$' respectively denote the coordinatewise minimum and maximum;
- *symmetric supermodular* if it is supermodular and satisfies
$$\varphi(x_1, \dots, x_n) = \varphi(x_{\pi(1)}, \dots, x_{\pi(n)})$$
for all permutations $(\pi(1), \dots, \pi(n))$ of $(1, \dots, n)$.

Remark 1. We note (see Section 7.A.8 of [19]) that in the multivariate case directional convexity neither implies, nor is implied by, conventional convexity, whereas in the univariate case the two notions are identical. Moreover, directionally convexity implies supermodularity. We also recall that a function is directionally convex if and only if it is both supermodular and coordinatewise convex.

Remark 2. We note that if a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is supermodular, and if $\psi\colon \mathbb{R} \to \mathbb{R}$ is increasing, then the composition $\psi(\varphi)$ is not necessarily supermodular. Indeed, for $n = 2$, let us consider the supermodular function $\varphi(x_1, x_2) = (x_1 + x_2)^2$, and the increasing function $\psi(u) = u^c$, $u \geq 0$, for $0 < c < \frac{1}{2}$. When $\boldsymbol{x} = (0, 1)$ and $\boldsymbol{y} = (1, 0)$, we have $\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{y}) = 1$, $\varphi(\boldsymbol{x} \wedge \boldsymbol{y}) = 0$, and $\varphi(\boldsymbol{x} \vee \boldsymbol{y}) = 4$, so that (1) is satisfied. However, $\psi(\varphi)$ is not supermodular, since

$$2 = \psi[\varphi(\boldsymbol{x})] + \psi[\varphi(\boldsymbol{y})] > \psi[\varphi(\boldsymbol{x} \wedge \boldsymbol{y})] + \psi[\varphi(\boldsymbol{x} \vee \boldsymbol{y})] = 4^c$$

for $0 < c < \frac{1}{2}$.

Various properties of the above notions are given in [19], where their use in the definition of suitable stochastic orders is also pinpointed. Chapter 6 of [15] contains useful examples of supermodular functions, denominated as 'L-superadditive' functions. We remark that supermodular functions play a significant role in applied fields, such as optimization and game theory (see [8] for instance). Recent contributions dealing with applications of directional convexity and supermodularity to risk management, insurance, queueing, and macroeconomic dynamics are given, for instance, in [7], [9], [10], [16], [17], and [18].

Let us now recall the definitions of four stochastic orders that will be used later (see [19] for other details).

(i) A random variable X is said to be larger than Y in the usual stochastic order (denoted by $X \geq_{\text{st}} Y$) if $\mathbb{P}(X > t) \geq \mathbb{P}(Y > t)$ for all $t \in \mathbb{R}$ or, equivalently, if $\mathbb{E}[\psi(X)] \geq \mathbb{E}[\psi(Y)]$ for all increasing functions $\psi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

(ii) A random variable X is said to be larger than Y in the increasing convex order (denoted by $X \geq_{\text{icx}} Y$) if $\mathbb{E}[\psi(X)] \geq \mathbb{E}[\psi(Y)]$ for all increasing convex functions $\psi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

(iii) A d-dimensional random vector $\boldsymbol{X}$ is said to be larger than $\boldsymbol{Y}$ in the increasing directionally convex order (denoted by $\boldsymbol{X} \geq_{\text{idcx}} \boldsymbol{Y}$) if $\mathbb{E}[\psi(\boldsymbol{X})] \geq \mathbb{E}[\psi(\boldsymbol{Y})]$ for all increasing directionally convex functions $\psi : \mathbb{R}^d \to \mathbb{R}$ for which the expectations exist.

(iv) A d-dimensional random vector $\boldsymbol{X}$ is said to be larger than $\boldsymbol{Y}$ in the supermodular order (denoted by $\boldsymbol{X} \geq_{\text{sm}} \boldsymbol{Y}$) if $\mathbb{E}[\psi(\boldsymbol{X})] \geq \mathbb{E}[\psi(\boldsymbol{Y})]$ for all supermodular functions $\psi : \mathbb{R}^d \to \mathbb{R}$ for which the expectations exist.

We remark that the supermodular order strictly implies the increasing directionally convex order. However, the supermodular order compares only the dependence structure of vectors with fixed equal marginals, whereas the increasing directionally convex order also compares the marginals both in variability and location, where the marginals are possibly different. Moreover, when $d = 1$, the increasing directionally convex order reduces to the increasing convex order.

3. The main results

We first recall a preliminary result on the monotonicity of a function expressed in terms of the means of a supermodular function, and of stochastically ordered random variables. The proof of this statement may be found, for example, in Theorem 3.1 of [14].

Lemma 1. *Let $X \geq_{\mathrm{st}} Y$. If $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is a supermodular function then*

$$h(z) = \mathbb{E}[\varphi(X, z) - \varphi(Y, z)]$$

is an increasing function of z, provided that the above means are finite.

In the following, given the random variables X_i, $i = 1, 2, \ldots, d$, and Y, we consider n-dimensional random vectors of the form

$$(X_{1,1}, \ldots, X_{1,k_1}, \ldots, X_{d,1}, \ldots, X_{d,k_d}, Y_1, \ldots, Y_{n-k^*}), \tag{2}$$

where we have set $k^* := k_1 + \cdots + k_d$, and where $X_{i,j}$ and Y_j for $i = 1, 2, \ldots, d$ and $j \geq 1$ denote independent copies of X_i and Y, respectively. Independence among all the random variables appearing in these vectors is also assumed from now on. Moreover, we set

$$\mathcal{D}_n := \{(k_1, \ldots, k_d) \in \mathbb{N}^d \colon k^* \leq n\}. \tag{3}$$

The following theorem shows that if X_i and Y are stochastically ordered, then the mean of a symmetric supermodular (respectively increasing symmetric supermodular) function of (2) is supermodular (respectively increasing directionally convex) in the sizes of the groups of i.i.d. variables.

Theorem 1. *If $X_i \geq_{\mathrm{st}} Y$ for all $i = 1, 2, \ldots, d$ and if $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a symmetric supermodular (respectively increasing symmetric supermodular) function, then, for $1 \leq d \leq n$, the function*

$$\psi(k_1, \ldots, k_d) := \mathbb{E}[\varphi(X_{1,1}, \ldots, X_{1,k_1}, \ldots, X_{d,1}, \ldots, X_{d,k_d}, Y_1, \ldots, Y_{n-k^*})] \tag{4}$$

is supermodular (respectively increasing directionally convex) in $(k_1, \ldots, k_d) \in \mathcal{D}_n$, provided the expectation is finite.

Proof. To prove the supermodularity of φ, it is enough to consider the case $d = 2$. Indeed, due to (1), the supermodularity of a d-dimensional function follows from the supermodularity with respect to any pair of its coordinates. Hence, we have to prove that

$$\psi(k_1, k_2) - \psi(k_1 + 1, k_2) - \psi(k_1, k_2 + 1) + \psi(k_1 + 1, k_2 + 1) \geq 0 \tag{5}$$

for positive integers k_1 and k_2 such that $k_1 + k_2 + 2 \leq n$. Consider the $(n - 2)$-dimensional vector

$$\boldsymbol{V} = (X_{1,1}, \ldots, X_{1,k_1}, X_{2,1}, \ldots, X_{2,k_2}, Y_1, \ldots, Y_{n-k_1-k_2-2}).$$

It is enough to prove inequality (5) under the conditional expectation, namely given $\boldsymbol{V} = \boldsymbol{v}$. Since φ is symmetric by assumption, we have to prove that

$$\mathbb{E}[\varphi(Y_1, Y_2, \boldsymbol{v}) - \varphi(Y_1, X_2, \boldsymbol{v})] \geq \mathbb{E}[\varphi(X_1, Y_2, \boldsymbol{v}) - \varphi(X_1, X_2, \boldsymbol{v})]. \tag{6}$$

Since φ is supermodular, for $x > y$,

$$\varphi(x, t, \boldsymbol{v}) - \varphi(y, t, \boldsymbol{v})$$

is monotone increasing in t. From the assumption that $X_2 \geq_{\mathrm{st}} Y_2$ we thus have, for $x > y$,

$$\mathbb{E}[\varphi(x, X_2, \boldsymbol{v}) - \varphi(y, X_2, \boldsymbol{v})] \geq \mathbb{E}[\varphi(x, Y_2, \boldsymbol{v}) - \varphi(y, Y_2, \boldsymbol{v})].$$

Hence,

$$\mathbb{E}[\varphi(x, X_2, \boldsymbol{v}) - \varphi(x, Y_2, \boldsymbol{v})]$$

is increasing in x. This implies (6), by again using the assumption that $X_1 \geq_{\text{st}} Y_1$. The proof for supermodularity of φ is thus completed. Let us now prove that ψ is increasing directionally convex when φ is increasing symmetric supermodular. Due to Remark 1, it is now sufficient to prove that ψ is coordinatewise convex, for instance, in the first coordinate. Hence, we need to prove that

$$\psi(k_1 + 1, \boldsymbol{k}) - \psi(k_1, \boldsymbol{k}) \geq \psi(k_1, \boldsymbol{k}) - \psi(k_1 - 1, \boldsymbol{k}), \tag{7}$$

where $\boldsymbol{k} = (k_2, \dots, k_d)$ and $k_1 + \cdots + k_d \leq n - 1$. Let us consider the $(n-2)$-dimensional vector

$$\boldsymbol{W} = (X_{1,1}, \dots, X_{1,k_1-1}, X_{2,1}, \dots, X_{2,k_2}, \dots, X_{d,1}, \dots, X_{d,k_d}, Y_1, \dots, Y_{n-k^*-1}),$$

where $k^* := k_1 + \cdots + k_d$. It is now enough to prove (7) under the conditional expectation, i.e. given $\boldsymbol{W} = \boldsymbol{w}$. Since φ is a symmetric function, we have

$$\begin{aligned}\psi(k_1 + 1, \boldsymbol{k}) - \psi(k_1, \boldsymbol{k}) &= \mathbb{E}[\varphi(X_{1,k_1}, X_{1,k_1+1}, \boldsymbol{w}) - \varphi(X_{1,k_1}, Y_{n-k^*}, \boldsymbol{w})] \\ &\geq \mathbb{E}[\varphi(Y_{n-k^*}, X_{1,k_1}, \boldsymbol{w}) - \varphi(Y_{n-k^*}, Y_{n-k^*+1}, \boldsymbol{w})] \\ &= \psi(k_1, \boldsymbol{k}) - \psi(k_1 - 1, \boldsymbol{k}).\end{aligned}$$

Indeed, the above inequality immediately follows from Lemma 1 and recalling the assumption that $X_1 \geq_{\text{st}} Y$. The proof is thus completed by noting that the monotonicity of ψ can be proved by means of the same arguments.

Suitable examples of symmetric supermodular functions that satisfy the assumptions of Theorem 1 are

(i) $\varphi(x_1, \dots, x_n) = -\max\{x_1, \dots, x_n\}$;

(ii) $\varphi(x_1, \dots, x_n) = \gamma(x_1 + \cdots + x_n)$ if $\gamma(\cdot)$ is convex;

(iii) $\varphi(x_1, \dots, x_n) = x_1 \cdots x_n$.

Examples of increasing symmetric supermodular functions that satisfy the assumptions of Theorem 1 are

(i) $\varphi(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}$;

(ii) $\varphi(x_1, \dots, x_n) = \gamma(x_1 + \cdots + x_n)$ if $\gamma(\cdot)$ is increasing convex;

(iii) $\varphi(x_1, \dots, x_n) = x_1 \cdots x_n$ if $x_1, \dots, x_n \geq 0$.

Let $\boldsymbol{K} = (K_1, \dots, K_d)$ and $\boldsymbol{M} = (M_1, \dots, M_d)$ be random vectors taking values in $\mathcal{D}_n$. Let

$$Z_{\boldsymbol{K}} = \varphi(X_{1,1}, \dots, X_{1,K_1}, \dots, X_{d,1}, \dots, X_{d,K_d}, Y_1, \dots, Y_{n-K^*}), \tag{8}$$

whose distribution is expressed by

$$\mathbb{P}(Z_{\boldsymbol{K}} \leq t) = \int_{\mathcal{D}_n} F_{\boldsymbol{k}}(t) \,\mathrm{d}P_{\boldsymbol{K}}(\boldsymbol{k}),$$

where $F_{\boldsymbol{k}}$ is the distribution of $\varphi(X_{1,1}, \dots, X_{1,k_1}, \dots, X_{d,1}, \dots, X_{d,k_d}, Y_1, \dots, Y_{n-k^*})$ and $P_{\boldsymbol{K}}$ is the distribution of $\boldsymbol{K}$. We define $Z_{\boldsymbol{M}}$ and its distribution similarly.

Hereafter we provide a comparison result that follows from Theorem 1.

Theorem 2. *Let $X_i \geq_{\text{st}} Y$ for all $i = 1, 2, \dots, d$.*

(i) *If*

$$\boldsymbol{K} \geq_{\text{sm}} \boldsymbol{M} \tag{9}$$

and $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is a symmetric supermodular function, then

$$\mathbb{E}[Z_{\boldsymbol{K}}] \geq \mathbb{E}[Z_{\boldsymbol{M}}],$$

provided the expectations exist.

(ii) *If*

$$\boldsymbol{K} \geq_{\text{idcx}} \boldsymbol{M} \tag{10}$$

and $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is an increasing symmetric supermodular function, then

$$Z_{\boldsymbol{K}} \geq_{\text{icx}} Z_{\boldsymbol{M}}.$$

Proof. (i) Let $\psi(\boldsymbol{k})$ be defined as in (4). Thus, recalling (8) and the fact that φ is symmetric supermodular,

$$\mathbb{E}[Z_{\boldsymbol{K}}] = \mathbb{E}[\mathbb{E}[Z_{\boldsymbol{K}} \mid \boldsymbol{K}]] = \mathbb{E}[\psi(\boldsymbol{K})] \geq \mathbb{E}[\psi(\boldsymbol{M})] = \mathbb{E}[\mathbb{E}[Z_{\boldsymbol{M}} \mid \boldsymbol{M}]] = \mathbb{E}[Z_{\boldsymbol{M}}],$$

where the inequality follows from Theorem 1 and assumption (9).

(ii) Let $h\colon \mathbb{R} \to \mathbb{R}$ denote any increasing convex function. We recall that (see [3] for instance), under the given assumptions, the composition $h \circ \varphi$ is increasing symmetric supermodular. Thus, defining

$$\psi(\boldsymbol{k}) = \mathbb{E}[h \circ \varphi(X_{1,1}, \dots, X_{1,k_1}, \dots, X_{d,1}, \dots, X_{d,k_d}, Y_1, \dots, Y_{n-k^*})],$$

and setting $K^* := K_1 + \cdots + K_d$ and $M^* := M_1 + \cdots + M_d$, it follows that

$$\begin{aligned}
\mathbb{E}[h(Z_{\boldsymbol{K}})] &= \mathbb{E}[h \circ \varphi(X_{1,1}, \dots, X_{1,K_1}, \dots, X_{d,1}, \dots, X_{d,K_d}, Y_1, \dots, Y_{n-K^*})] \\
&= \mathbb{E}[\mathbb{E}[h \circ \varphi(X_{1,1}, \dots, X_{1,K_1}, \dots, X_{d,1}, \dots, X_{d,K_d}, Y_1, \dots, Y_{n-K^*}) \mid \boldsymbol{K}]] \\
&= \mathbb{E}[\psi(\boldsymbol{K})] \\
&\geq \mathbb{E}[\psi(\boldsymbol{M})] \\
&= \mathbb{E}[h(Z_{\boldsymbol{M}})],
\end{aligned}$$

where the inequality follows from Theorem 1 and assumption (10).

We note that a result similar to Theorem 2(ii), with $Z_{\boldsymbol{K}} \geq_{\text{st}} Z_{\boldsymbol{M}}$ instead of $Z_{\boldsymbol{K}} \geq_{\text{icx}} Z_{\boldsymbol{M}}$, cannot be obtained, due to Remark 2.

Hereafter we give an example of random vectors $\boldsymbol{K}$ and $\boldsymbol{M}$ satisfying conditions (9) and (10).

Example 1. Let $\boldsymbol{K}$ and $\boldsymbol{M}$ be bivariate random vectors such that

$$\mathbb{P}(\boldsymbol{K} = (0, 0)) = \mathbb{P}(\boldsymbol{K} = (1, 1)) = \tfrac{1}{2}, \qquad \mathbb{P}(\boldsymbol{M} = (0, 1)) = \mathbb{P}(\boldsymbol{M} = (1, 0)) = \tfrac{1}{2}.$$

Hence, since any supermodular function η can be expressed as

$$\eta(\boldsymbol{x}) = \sum_i a_i \eta_i(\boldsymbol{x}),$$

where $a_i \geq 0$ for all i, and $\eta_i(\boldsymbol{x}) = \mathbf{1}_{\mathcal{U}_i \cup \mathcal{L}_i}(\boldsymbol{x})$, where $\mathcal{U}_i$ and $\mathcal{L}_i$ are suitable upper and lower orthants, respectively, it is not hard to verify that $\boldsymbol{K} \geq_{\text{sm}} \boldsymbol{M}$, which, in turn, implies that $\boldsymbol{K} \geq_{\text{idcx}} \boldsymbol{M}$.

Remark 3. We point out that, since in the univariate case the increasing directionally convex order reduces to the increasing convex order, if $d = 1$ then inequality (10) can be replaced by $K \geq_{\text{icx}} M$ in the statement of Theorem 2, and everywhere in the subsequent section.

4. Applications

This section deals with applications of the previous results in various fields.

4.1. Application to risk theory

We consider a finite population of n insureds divided into $d + 1$ different groups with sizes $k_1, k_2, \dots, k_d, n - k^*$, where $k^* = k_1 + k_2 + \cdots + k_d \leq n$. Let

$$(X_{1,1}, \dots, X_{1,k_1}, \dots, X_{d,1}, \dots, X_{d,k_d}, Y_1, \dots, Y_{n-k^*}) \tag{11}$$

be a possible portfolio of risks, where $X_{i,j}$ is a nonnegative random variable denoting the amount of claim caused by the jth insured of the ith group ($i = 1, 2, \dots, d$ and $j = 1, 2, \dots, k_i$). Assume that claims within groups are identically distributed, the variables $X_{i,j}$ are independent, and $X_i \geq_{\text{st}} Y$. This situation can happen, for example, in a bonus malus system where insured drivers are classified according to their potential risk, so that good drivers having low damage experience pay lower premiums than drivers with a higher probability for damage. For example, drivers living in regions with low traffic or careful drivers have a low probability for damage. In this case the risk for these drivers is stochastically smaller than the risk of the other drivers. This phenomena also occurs in health insurance where people with 'good' genetics or a healthy lifestyle have smaller risk. Examples of models where risks are divided into more groups is also provided in [4]. We note that this paper also presented some comparison results involving the supermodular ordering and the symmetric supermodular ordering.

Let us now assume that the size of each group is random, and denote by $\boldsymbol{K} = (K_1, K_2, \dots, K_d)$ the random vector of the first d group sizes, so that the last group, formed by an insured causing stochastically smaller claims, has size $n - K^*$. Consider the total claim amount

$$S_{\boldsymbol{K}} = \sum_{i=1}^{d} \sum_{j=1}^{K_i} X_{i,j} + \sum_{j=1}^{n-K^*} Y_j; \tag{12}$$

similarly define the total claim amount $S_{\boldsymbol{M}}$ when the portfolio has distinct group sizes $\boldsymbol{M} = (M_1, M_2, \dots, M_d)$.

Proposition 1. *Let* $\boldsymbol{K} = (K_1, \dots, K_d)$ *and* $\boldsymbol{M} = (M_1, \dots, M_d)$ *be random vectors taking values in the set* $\mathcal{D}_n$*. If*

$$\boldsymbol{K} \geq_{\text{idcx}} \boldsymbol{M}$$

then

$$S_{\boldsymbol{K}} \geq_{\text{icx}} S_{\boldsymbol{M}}.$$

Proof. The proof follows from Theorem 2(ii), since $X_i \geq_{\text{st}} Y$ and

$$\varphi(x_{1,1}, \dots, x_{1,k_1}, \dots, x_{d,1}, \dots, x_{d,k_d}, y_1, \dots, y_{n-k^*}) = \sum_{i=1}^{d} \sum_{j=1}^{k_i} x_{i,j} + \sum_{j=1}^{n-k^*} y_j$$

is an increasing symmetric supermodular function.

We remark that (12) gives the total claim amount when the variability in the insureds group sizes is described by $\boldsymbol{K}$. Proposition 1 thus shows that the total claim amount increases in the icx-order as the variability in the group sizes increases in the idcx-order.

Remark 4. Consider the net stop-loss reinsurance premium of portfolio (11), i.e.

$$\pi_{\boldsymbol{K}}(a) = \mathbb{E}\bigg[\bigg(\sum_{i=1}^{d}\sum_{j=1}^{K_i} X_{i,j} + \sum_{j=1}^{n-K^*} Y_j - a\bigg)_+\bigg], \qquad a > 0,$$

where, as usual, $(w)_+ := \max\{w, 0\}$; similarly define $\pi_{\boldsymbol{M}}(a)$ when the portfolio has distinct group sizes $\boldsymbol{M}$. Hence, under the assumptions of Proposition 1, in particular we have

$$\pi_{\boldsymbol{K}}(a) \geq \pi_{\boldsymbol{M}}(a), \qquad a > 0,$$

provided the expectations exist.

4.2. Application to a closed cyclic multiclass queueing network

Consider a closed Gordon–Newell network of n queues with exponential service times, and with a total population of $\ell > 0$ customers. For the definition of closed Gordon–Newell queues, see, for instance, the seminal paper by Gordon and Newell [13] or the book by Breuer and Baum [6]. Assume that the network is cyclic, so that every customer completing service at queue i moves to queue $i + 1$ if $1 \leq i < n$ or to queue 1 if $i = n$. The state space of the network is given by

$$S(\ell, n) = \bigg\{(x_1, \dots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = \ell \text{ and } x_i \geq 0 \text{ for all } i = 1, 2, \dots, n\bigg\},$$

where x_i represents the number of customers in the ith queue. A model of the network is shown in Figure 1.

Let us now introduce a multiclass nature in the network units. Assume that the network units may perform two kinds of activity: some units provide a service that gains a certain profit, whereas the remaining units supply auxiliary services, such as recovery or repair activity, originating a cost. Specifically, we assume that the n queues of the network are partitioned into $d + 1$ classes, each class being formed by a random number of units. The ith class for $i = 1, 2, \dots, d$ is formed by K_i units. Each customer of the jth unit of the ith class provides a random profit of $A_{i,j} \geq 0$ assets (due to some performance of the job on that station). Note that such profit is identical for each customer in the queue. Moreover, the $(d + 1)$th class is formed by $n - \sum_{i=1}^{d} K_i$ units. Each customer of the jth unit of the $(d + 1)$th class causes a random cost $C_j \leq 0$, which is identical for each customer in the queue. Hence, only the

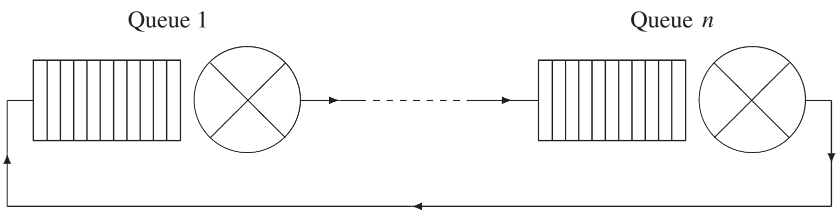

FIGURE 1: Closed cyclic network with n queues.

https://doi.org/10.1239/jap/1371648954 Published online by Cambridge University Press

first d classes are 'producing classes'. The random vector $(K_1, K_2, \dots, K_d)$ describes the sizes of the d producing classes, and takes values in $\mathcal{D}_n$ (see (3)).

Moreover, we assume that the service discipline at all queues is FCFS, and the exponential service times of the queues within each class are identically distributed. We thus denote by μ_i the parameter of the service times of the queues of class i for $i = 1, 2, \dots, d+1$. Denoting by $\tilde{X}_{i,j}$ the number of customers in the jth queue of the ith class in the equilibrium state, and by $\tilde{Y}_j$ the number of customers in the jth queue of the $(d+1)$th class in the equilibrium state, then the state of the network in the equilibrium state is described by the n-dimensional random vector

$$(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}) = (\tilde{X}_{1,1}, \dots, \tilde{X}_{1,K_1}, \dots, \tilde{X}_{d,1}, \dots, \tilde{X}_{d,K_d}, \tilde{Y}_1, \dots, \tilde{Y}_{n-K^*}),$$

where $K^* = \sum_{i=1}^{d} K_i$, with state space $S(\ell, n)$. The equilibrium state probability distribution of the network conditional on $(K_1, K_2, \dots, K_d) = (k_1, k_2, \dots, k_d) \in \mathcal{D}_n$ is given, for $(\boldsymbol{x}, \boldsymbol{y}) \in S(\ell, n)$, by

$$\pi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{G(\ell, n)} \prod_{i=1}^{d} \prod_{j=1}^{k_i} \left(\frac{1}{\mu_i}\right)^{x_{i,j}} \prod_{j=1}^{n-k^*} \left(\frac{1}{\mu_{d+1}}\right)^{y_j},$$

where $k^* = \sum_{i=1}^{d} k_i$ and

$$G(\ell, n) = \sum_{(\boldsymbol{x},\boldsymbol{y}) \in S(\ell,n)} \prod_{i=1}^{d} \prod_{j=1}^{k_i} \left(\frac{1}{\mu_i}\right)^{x_{i,j}} \prod_{j=1}^{n-k^*} \left(\frac{1}{\mu_{d+1}}\right)^{y_j}$$

is the normalizing constant. We remark that subvectors $(X_{i,1}, \dots, X_{i,K_i})$ for $i = 1, 2, \dots, d$, and $(Y_1, \dots, Y_{n-K^*})$ are formed by i.i.d. random variables.

Let us now define the following random variables:

$$X_{i,j} = A_{i,j} \tilde{X}_{i,j}, \qquad i = 1, 2, \dots, d, \; j = 1, 2, \dots, K_i,$$
$$Y_j = C_j \tilde{Y}_j, \qquad j = 1, 2, \dots, n - K^*.$$

Hence, $X_{i,j}$ represents the profit gained due to the customers in the jth unit of the ith class, whereas Y_j gives the cost caused by the customers in the jth unit of the $(d+1)$th class. We note that $X_{i,j} \geq_{\text{st}} Y_j$. Let

$$S_{\boldsymbol{K}} = \sum_{i=1}^{d} \sum_{j=1}^{K_i} X_{i,j} + \sum_{j=1}^{n-K^*} Y_j$$

be the total profit gained by the activity of the multiclass queueing network in the equilibrium state when the d producing classes have random sizes $K_1, \dots, K_d$. Similarly define $S_{\boldsymbol{M}}$ when the producing classes have random sizes $M_1, \dots, M_d$. Hence, as for Proposition 1, if $\boldsymbol{K} \geq_{\text{idcx}} \boldsymbol{M}$ then $S_{\boldsymbol{K}} \geq_{\text{icx}} S_{\boldsymbol{M}}$. This implies that the total profit increases in the icx-order as the variability in the group sizes increases.

4.3. Application to the reliability of series systems

Results involving the fact that increasing the randomness in the structure of reliability systems causes a stochastical improvement of the system have been given recently in [11], where it was shown that the lifetime of a series or of a parallel system may be stochastically improved by means of suitable mixtures. See also [12], where the reliability of a system improves by introducing randomness in the number of system components extracted from different batches.

Along the above lines, hereafter we consider a series system formed by n units randomly selected from $d+1$ different batches. Specifically, assume that $K_1, K_2, \dots, K_d, n-K^*$ units are selected from the first, second, $\dots$, dth, $(d+1)$th batch, where the nonnegative integer-valued random variables K_i are such that $K^* = K_1 + K_2 + \cdots + K_d \le n$. For $i = 1, 2, \dots, d$ and $j = 1, 2, \dots, K_i$, let $X_{i,j}(t)$ denote the Bernoulli random variable that describes the functioning at time $t \ge 0$ of the jth unit selected from the ith batch. Similarly, $Y_j(t)$ is the Bernoulli random variable describing the functioning at time $t \ge 0$ of the jth unit selected from the $(d+1)$th batch, which is assumed to contain weaker units. Hence, denoting by $p_{i,j}(t) = \mathbb{P}(X_{i,j}(t) = 1)$ and $q_j(t) = \mathbb{P}(Y_j(t) = 1)$ the availability of $X_{i,j}(t)$ and $Y_j(t)$, respectively, we have

$$p_{i,j}(t) \ge q_j(t), \qquad t \ge 0,$$

so that $X_{i,j}(t) \ge_{\mathrm{st}} Y_j(t)$. Moreover, we assume that the random variables that describe the functioning of the components within each batch are i.i.d. The overall functioning of the series system at time $t \ge 0$ is given by (see [2])

$$R_{\boldsymbol{K}}(t) = \prod_{i=1}^{d} \prod_{j=1}^{K_i} X_{i,j}(t) \prod_{j=1}^{n-K^*} Y_j(t),$$

and $R_{\boldsymbol{M}}(t)$ is similarly expressed when $M_1, M_2, \dots, M_d, n-M^*$ are the random numbers of units.

Proposition 2. *Let $\boldsymbol{K} = (K_1, \dots, K_d)$ and $\boldsymbol{M} = (M_1, \dots, M_d)$ be random vectors taking values in $\mathcal{D}_n$. If*

$$\boldsymbol{K} \ge_{\mathrm{idcx}} \boldsymbol{M}$$

then

$$R_{\boldsymbol{K}}(t) \ge_{\mathrm{icx}} R_{\boldsymbol{M}}(t)$$

for all $t \ge 0$.

The proof of Proposition 2 is similar to that of Proposition 1, since $X_{i,j}(t) \ge_{\mathrm{st}} Y_j(t)$, and since $\varphi(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{d} \prod_{j=1}^{k_i} x_{i,j} \prod_{j=1}^{n-k^*} y_j$ is an increasing symmetric supermodular function for $x_{i,j}, y_j \ge 0$.

Remark 5. Let $S_{\boldsymbol{K}}$ be the lifetime of the above series system under randomization of the number of units described by $\boldsymbol{K}$, and similarly for $S_{\boldsymbol{M}}$. Under the above assumptions, due to Proposition 2, we also have

$$\mathbb{P}(S_{\boldsymbol{K}} > t) = \mathbb{E}[R_{\boldsymbol{K}}(t)] \ge \mathbb{E}[R_{\boldsymbol{M}}(t)] = \mathbb{P}(S_{\boldsymbol{M}} > t) \quad \text{for all } t \ge 0,$$

that is, $S_{\boldsymbol{K}} \ge_{\mathrm{st}} S_{\boldsymbol{M}}$. This shows that the system lifetime stochastically increases as the variability of the group sizes increases.

Acknowledgements

The research of A. Di Crescenzo was performed under partial support from the MIUR-PRIN 2008 'Mathematical models and computation methods for information processing and transmission in neuronal systems subject to stochastic dynamics', while F. Pellerey was partially supported by MIUR-PRIN 2008 'Probability and Finance'. F. Pellerey is grateful to the Department of Mathematics of Salerno University for partial support during a visit in March 2012.

References

[1] Balakrishnan, N., Belzunce, F., Sordo, M. A. and Suárez-Llorens, A. (2012). Increasing directionally convex orderings of random vectors having the same copula, and their use in comparing ordered data. *J. Multivariate Anal.* **105,** 45–54.

[2] Barlow, R. E. and Proschan, F. (1996). *Mathematical Theory of Reliability*. (Classics Appl. Math. **17**). SIAM, Philadelphia, PA.

[3] Bäuerle, N. (1997). Inequalities for stochastic models via supermodular orderings. *Commun. Statist. Stoch. Models* **13,** 181–201.

[4] Bäuerle, N. and Müller, A. (1998). Modeling and comparing dependencies in multivariate risk portfolios. *ASTIN Bull.* **28,** 59–76.

[5] Belzunce, F., Suárez-Llorens, A. and Sordo, M. A. (2012). Comparison of increasing directionally convex transformations of random vectors with a common copula. *Insurance Math. Econom.* **50,** 385–390.

[6] Breuer, L. and Baum, D. (2005). *An Introduction to Queueing Theory and Matrix-Analytic Methods*. Springer, Dordrecht.

[7] Cai, J. and Li, H. (2007). Dependence properties and bounds for ruin probabilities in multivariate compound risk models. *J. Multivariate Anal.* **98,** 757–773.

[8] Carter, M. (2001). *Foundations of Mathematical Economics*. MIT Press, Cambridge, MA.

[9] Datta, M., Mirman, L. J., Morand, O. F. and Reffett K. L. (2002). Monotone methods for Markovian equilibrium in dynamic economies. Stochastic equilibrium problems in economics and game theory. *Ann. Operat. Res.* **114,** 117–144.

[10] Denuit, M., Frostig, E. and Levikson, B. (2007). Supermodular comparison of time-to-ruin random vectors. *Methodology Comput. Appl. Prob.* **9,** 41–54.

[11] Di Crescenzo, A. and Pellerey, F. (2011). Improving series and parallel systems through mixtures of duplicated dependent components. *Naval Res. Logistics* **58,** 411–418.

[12] Di Crescenzo, A. and Pellerey, F. (2011). Stochastic comparisons of series and parallel systems with randomized independent components. *Operat. Res. Lett.* **39,** 380–384.

[13] Gordon, W. J. and Newell, G. F. (1967). Closed queueing systems with exponential servers. *Operat. Res.* **15,** 254–265.

[14] Kijima, M. and Ohnishi, M (1996). Portfolio selection problems via the bivariate characterization of stochastic dominance relations. *Math. Finance* **6,** 237–277.

[15] Marshall, A. W. and Olkin, W. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

[16] Meester, L. E. and Shanthikumar, J. G. (1999). Stochastic convexity on general space. *Math. Operat. Res.* **24,** 472–494.

[17] Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance Math. Econom.* **21,** 219–223.

[18] Öner, K. B., Kiesmüller, G. P. and van Houtum, G. J. (2009). Monotonicity and supermodularity results for the Erlang loss system. *Operat. Res. Lett.* **37,** 265–268.

[19] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.