

# WEAK CONVERGENCE TO DERIVATIVES OF FRACTIONAL BROWNIAN MOTION

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It is well known that, under suitable regularity conditions, the normalized fractional process with fractional parameter  $d$  converges weakly to fractional Brownian motion (fBm) for  $d > \frac{1}{2}$ . We show that, for any nonnegative integer  $M$ , derivatives of order  $m = 0, 1, \dots, M$  of the normalized fractional process with respect to the fractional parameter  $d$  jointly converge weakly to the corresponding derivatives of fBm. As an illustration, we apply the results to the asymptotic distribution of the score vectors in the multifractional vector autoregressive model.

## 1. INTRODUCTION

The  $p$ -dimensional fractionally integrated process of Type II (e.g., Marinucci and Robinson, 1999) is given by

$$\Delta_+^{-d}\xi_t = (1-L)_+^{-d}\xi_t = \sum_{n=0}^{t-1} \pi_n(d)\xi_{t-n} = \sum_{n=1}^t \pi_{t-n}(d)\xi_n, \quad t = 1, 2, \dots \quad (1)$$

This expression defines the operator  $\Delta_+^{-d} = (1-L)_+^{-d}$  as a finite sum, and the fractional coefficients  $\pi_n(d)$  are defined by the binomial expansion of  $(1-z)^{-d}$ . That is,

$$\pi_n(d) = (-1)^n \binom{-d}{n} = d(d+1)\dots(d+n-1)/n! \sim cn^{d-1}$$

with “ $\sim$ ” denoting that the ratio of the left- and right-hand sides converges to 1. The parameter  $d$  is called the memory parameter, which we assume satisfies  $d > \frac{1}{2}$ . Throughout,  $\xi_t$  is a  $p$ -dimensional linear process,

$$\xi_t = C(L)\varepsilon_t = \sum_{j=-\infty}^{\infty} C_j\varepsilon_{t-j}, \quad (2)$$

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for some  $p \times p$  coefficient matrices  $C_j$  and a  $p$ -dimensional innovation sequence,  $\varepsilon_t$ , which is independently and identically distributed (i.i.d.) with mean zero and variance matrix  $\Sigma$  (precise conditions will be given in Section 3).

We define the normalized process  $Z_{\lfloor Tr \rfloor}(d) = T^{1/2-d} \Delta_+^{-d} \xi_{\lfloor Tr \rfloor}$  for  $d > \frac{1}{2}$  and  $r \in [0, 1]$ , where  $\lfloor \cdot \rfloor$  denotes the integer part of the argument. The functional central limit theorem<sup>1</sup> for  $Z_{\lfloor Tr \rfloor}(d)$  was proved by Akonom and Gouriéroux (1987) for autoregressive moving-average processes  $\xi_t$ , and by Marinucci and Robinson (2000) for linear processes  $\xi_t$  with coefficients satisfying a summability condition (see Assumption 1). In particular, these authors showed that

$$Z_{\lfloor Tr \rfloor}(d) = T^{1/2-d} \sum_{n=0}^{\lfloor Tr \rfloor-1} \pi_n(d) \xi_{\lfloor Tr \rfloor-n} \Rightarrow \Gamma(d)^{-1} \int_0^r (r-s)^{d-1} dW(s) = W(r; d), \tag{3}$$

where  $\Gamma(\cdot)$  is the Gamma function,  $W$  is Brownian motion with variance matrix  $C(1)\Sigma C(1)'$ ,  $C(1) = \sum_{j=-\infty}^{\infty} C_j$ , and “ $\Rightarrow$ ” denotes weak convergence in the space of càdlàg functions on  $[0, 1]$  endowed with the Skorokhod topology (see Billingsley, 1968, for a general treatment). That is, the normalized process  $Z_{\lfloor Tr \rfloor}(d)$  converges weakly to fractional Brownian motion (fBm),  $W(r; d)$ , which is also of Type II (see Marinucci and Robinson, 1999, for a detailed comparison of Types I and II fBm).

In fact, the results in Marinucci and Robinson (2000) also imply weak convergence of the derivative of  $\Delta_+^{-d} \xi_t$ , suitably normalized. We use  $D_d^m$  to denote the  $m$ th-order derivative with respect to  $d$ . Differentiating term by term, we find  $D_d \pi_n(d) = \pi_n(d) \sum_{k=0}^{n-1} (k+d)^{-1}$  (see Appendix A of Johansen and Nielsen, 2016, for additional details on the fractional coefficients and their derivatives). With this notation, Marinucci and Robinson (2000) proved that

$$\begin{aligned} Z_{\lfloor Tr \rfloor}^*(d) &= T^{1/2-d} (\log T)^{-1} D_d \Delta_+^{-d} \xi_{\lfloor Tr \rfloor} \\ &= T^{1/2-d} (\log T)^{-1} \sum_{n=1}^{\lfloor Tr \rfloor-1} \pi_n(d) \sum_{k=0}^{n-1} (k+d)^{-1} \xi_{\lfloor Tr \rfloor-n} \Rightarrow W(r; d). \end{aligned} \tag{4}$$

Thus, because of the factor  $\sum_{k=0}^{n-1} (k+d)^{-1} \sim \log n$ , a different normalization is needed, but the weak limit is still fBm.

Related to (3) and (4), Hualde (2012) showed the limit result<sup>2</sup>

$$H_{\lfloor Tr \rfloor}(d) = T^{1/2-d} \sum_{n=0}^{\lfloor Tr \rfloor-1} \pi_n(d) \left( - \sum_{k=n}^T (k+d)^{-1} \right) \xi_{\lfloor Tr \rfloor-n} \Rightarrow A(r; d), \tag{5}$$

<sup>1</sup>Even earlier results were available for the so-called Type I process (e.g., Davydov, 1970; Taqqu, 1975).

<sup>2</sup>There is a missing minus sign in either (6) or (8) in Hualde (2012). Of course, this is irrelevant for the marginal distribution of  $A(r; d)$  because  $A(r; d)$  is a zero-mean Gaussian process. However, the sign is critical when considering the joint distribution of  $A(r; d)$  and  $W(r; d)$ , for example.

where  $A(r; d) = \Gamma(d)^{-1} \int_0^r \log(r-s)(r-s)^{d-1} dW(s)$  was denoted a “modified fBm.” The derivation of (5) was motivated by a regression analysis of so-called “unbalanced cointegration,” where the process  $A(r; d)$  enters in the asymptotic distribution theory (see Hualde, 2012, 2014). Note, however, that  $A(r; d) = \Gamma^{-1}(d)D_d(\Gamma(d)W(r; d))$  is not the derivative of fBm.

In this paper, we prove related results for weak convergence of the derivatives with respect to  $d$  of  $Z_{\lfloor Tr \rfloor}(d)$  to corresponding derivatives of fBm. Differentiating term by term as in (4), we find

$$D_d Z_t(d) = \sum_{n=0}^{t-1} D_d(T^{1/2-d} \pi_n(d)) \xi_{t-n} = T^{1/2-d} \sum_{n=0}^{t-1} \left( -\log T + \sum_{k=0}^{n-1} \frac{1}{k+d} \right) \pi_n(d) \xi_{t-n}. \quad (6)$$

In the general case, the coefficients in the linear representation of  $D_d^n Z_t(d)$  will be calculated by recursion (see Section 4 and Lemma 1). Note the relation

$$D_d Z_t(d) = (\log T)(Z_t^*(d) - Z_t(d)). \quad (7)$$

In recent work, Johansen and Nielsen (2021) generalize earlier work on statistical inference in the fractionally cointegrated vector autoregressive model (Johansen and Nielsen, 2012b) to allow each variable in the multivariate process to have its own fractional parameter (integration order). They call this the “multifractional” cointegrated vector autoregressive model. One interpretation of this model is a generalization of Hualde’s (2012) bivariate unbalanced cointegrated regression model to a multivariate system framework. Johansen and Nielsen (2021) show that, in this setting, the derivative  $D_d Z_{\lfloor Tr \rfloor}(d)$  and its weak limit  $D_d W(r; d)$  play an important role in the asymptotic distribution theory for the maximum likelihood estimators of the fractional parameters. We present some details of this analysis in Section 5 to motivate and apply our results.

In Section 3, we show that the result (5) of Hualde (2012) can be generalized to allow for weights  $(-\sum_{k=n}^T (k+d)^{-1})^m$  for any integer  $m \geq 0$ . In Section 4, we use this result together with (3) of Marinucci and Robinson (2000) to show weak convergence of  $D_d^m Z_{\lfloor Tr \rfloor}(d)$  to derivatives of fBm. The application of our results to the multifractional cointegration model is given in Section 5, and some concluding remarks are given in Section 6. In the next section, however, we first consider  $m = 1$ , because the arguments simplify substantially in that case.

## 2. WEAK CONVERGENCE OF THE DERIVATIVE $D_d Z_{\lfloor Tr \rfloor}(d)$

In this section, we apply the results of Marinucci and Robinson (2000) in (3) and Hualde (2012) in (5) to show that the first derivative of the fractional process, i.e.,  $D_d Z_{\lfloor Tr \rfloor}(d)$ , converges weakly to  $D_d W(r; d)$ . Precise conditions under which the results hold will be stated in Section 3 before we give the general results.

The derivative  $D_d Z_{\lfloor Tr \rfloor}(d)$  is rewritten, using (6) and  $\sum_{k=0}^{n-1} (k+d)^{-1} = \sum_{k=0}^T (k+d)^{-1} - \sum_{k=n}^T (k+d)^{-1}$ , as

$$\begin{aligned}
 D_d Z_{\lfloor Tr \rfloor}(d) &= \sum_{n=0}^{\lfloor Tr \rfloor - 1} (D_d T^{1/2-d} \pi_n(d)) \xi_{\lfloor Tr \rfloor - n} \\
 &= \sum_{n=0}^{\lfloor Tr \rfloor - 1} T^{1/2-d} \pi_n(d) \left( -\log T + \sum_{k=0}^T (k+d)^{-1} \right) \xi_{\lfloor Tr \rfloor - n} \\
 &\quad + \sum_{n=0}^{\lfloor Tr \rfloor - 1} T^{1/2-d} \pi_n(d) \left( -\sum_{k=n}^T (k+d)^{-1} \right) \xi_{\lfloor Tr \rfloor - n} \\
 &= \left( -\log T + \sum_{k=0}^T (k+d)^{-1} \right) Z_{\lfloor Tr \rfloor}(d) + H_{\lfloor Tr \rfloor}(d). \tag{8}
 \end{aligned}$$

Here,  $Z_{\lfloor Tr \rfloor}(d) \Rightarrow W(r; d)$  and  $H_{\lfloor Tr \rfloor}(d) \Rightarrow A(r; d)$  by (3) and (5), respectively. Strictly speaking, (3) and (5) need to hold jointly, but that is a consequence of Theorem 1.

To evaluate the factor  $-\log T + \sum_{k=0}^T (k+d)^{-1}$  in (8), recall the following definition and series expansion of the Digamma function,

$$\psi(d) = D_d \log \Gamma(d) = -\gamma - \sum_{k=0}^{\infty} ((k+d)^{-1} - (k+1)^{-1}) \text{ for } d \neq 0, -1, \dots,$$

where  $\gamma = \lim_{T \rightarrow \infty} (\sum_{k=1}^T k^{-1} - \log T) = 0.577\dots$  is the Euler–Mascheroni constant (see Abramowitz and Stegun, 1972, eqns. (6.3.1) and (6.3.16)). We then find that

$$\begin{aligned}
 -\log T + \sum_{k=0}^T \frac{1}{k+d} &= -\left( \log T - \sum_{k=1}^T k^{-1} \right) - \left( \sum_{k=0}^{T-1} (k+1)^{-1} - \sum_{k=0}^T (k+d)^{-1} \right) \\
 &\rightarrow \gamma - \sum_{k=0}^{\infty} ((k+1)^{-1} - (k+d)^{-1}) = -\psi(d). \tag{9}
 \end{aligned}$$

Finally, we prove that

$$D_d \int_0^r (r-s)^{d-1} dW(s) = \int_0^r \log(r-s) (r-s)^{d-1} dW(s) \text{ for } d > 1/2. \tag{10}$$

By definition of the derivative,

$$D_d \int_0^r (r-s)^{d-1} dW(s) - \int_0^r \log(r-s) (r-s)^{d-1} dW(s) = \lim_{\delta \rightarrow 0} K_d(\delta),$$

where

$$K_d(\delta) = \delta^{-1} \int_0^r (r-s)^{d-1} ((r-s)^\delta - 1 - \delta \log(r-s)) dW(s).$$

By the mean value theorem,

$$(r - s)^\delta - 1 - \delta \log(r - s) = \frac{1}{2} \delta^2 \log^2(r - s)(r - s)^{\delta^*} \text{ for } |\delta^*| \leq |\delta|.$$

Hence, we find, using the Frobenius norm  $\|A\| = (\text{tr}\{A'A\})^{1/2}$ ,

$$\begin{aligned} \|\text{Var}(K_d(\delta))\| &= \frac{1}{4} \delta^2 \|\text{Var}(W(1))\| \int_0^r \log^4(r - s)(r - s)^{2d-2+2\delta^*} ds \\ &\leq c\delta^2 \int_0^r \log^4(r - s)(r - s)^{2d-2-2|\delta|} ds = c\delta^2 \int_0^r (\log^4 s) s^{|\delta|} s^{2d-2-3|\delta|} ds \\ &\leq c\delta^2 \int_0^r s^{2d-2-3|\delta|} ds = c\delta^2 r^{2d-1-3|\delta|} \rightarrow 0 \text{ as } \delta \rightarrow 0, \end{aligned}$$

because  $2d - 1 > 0$ . This proves (10).

Combining these results, it follows that

$$\begin{aligned} D_d Z_{\lfloor Tr \rfloor}(d) &\Rightarrow -\psi(d)W(r; d) + A(r; d) \\ &= \Gamma(d)^{-1} \int_0^r (-\psi(d) + \log(r - s))(r - s)^{d-1} dW(s) \\ &= \int_0^r D_d(\Gamma(d)^{-1}(r - s)^{d-1}) dW(s) = D_d W(r; d). \end{aligned} \tag{11}$$

Thus, the first derivative of the fractional process  $Z_{\lfloor Tr \rfloor}(d)$  converges weakly to the first derivative of the fBm  $W(r; d)$ . Interestingly, the above arguments leading to (11) required only the weak convergences in (3) and (5) (jointly) together with some well-known results regarding the Digamma function. Consequently, our result (11) holds whenever (3) and (5) hold jointly. In the next two sections, we will prove the corresponding result for derivatives of any order under precisely stated conditions.

### 3. A GENERALIZATION OF THE RESULT OF HUALDE (2012)

In this section, we generalize the result (5) of Hualde (2012). To this end, we define the processes

$$\begin{aligned} H_{m, \lfloor Tr \rfloor}(d) &= T^{1/2-d} \sum_{n=0}^{\lfloor Tr \rfloor - 1} \pi_n(d) \left( - \sum_{k=n}^T (k + d)^{-1} \right)^m \xi_{\lfloor Tr \rfloor - n} \\ &= T^{1/2-d} \sum_{n=1}^{\lfloor Tr \rfloor} \pi_{\lfloor Tr \rfloor - n}(d) \left( - \sum_{k=\lfloor Tr \rfloor - n}^T (k + d)^{-1} \right)^m \xi_n, \quad m = 0, 1, 2, \dots, \end{aligned} \tag{12}$$

so that  $Z_{\lfloor Tr \rfloor}(d) = H_{0, \lfloor Tr \rfloor}(d)$  and  $H_{\lfloor Tr \rfloor}(d) = H_{1, \lfloor Tr \rfloor}(d)$ . In Theorem 1, we find the joint weak limit of  $H_{m, \lfloor Tr \rfloor}(d)$ ,  $m = 0, 1, \dots, M$ , for any nonnegative integer  $M$ , but first we state our assumptions.

**Assumption 1.** The  $p$ -dimensional process  $\xi_t$  is such that

$$\xi_t = \sum_{j=-\infty}^{\infty} C_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} (\|C_k\|^2 + \|C_{-k}\|^2) < \infty,$$

where the  $C_j$  are  $p \times p$  deterministic matrices and  $C(1) = \sum_{j=-\infty}^{\infty} C_j$  has full rank,  $p$ .

**Assumption 2.** The  $p$ -dimensional process  $\varepsilon_t$  in Assumption 1 is i.i.d. with

$$E(\varepsilon_t) = 0, \quad E(\varepsilon_t \varepsilon_t') = \Sigma, \quad E \|\varepsilon_t\|^q < \infty,$$

for some  $q > \max\{2, 2/(2d - 1)\}$ ,  $d > 1/2$ , and  $\Sigma$  positive-definite.

We note that the moment condition in Assumption 2 is in fact necessary (see Johansen and Nielsen, 2012a). The rank conditions in Assumptions 1 and 2 ensure that the long-run variance of  $\xi_t$  is positive-definite.

Assumptions 1 and 2 are identical to the corresponding conditions in Marinucci and Robinson (2000) and Hualde (2012). Thus, (3), (5), and the results in Section 2, and in particular the weak convergence in (11), all hold under Assumptions 1 and 2.

**THEOREM 1.** Under Assumptions 1 and 2, it holds that, for  $m = 0, 1, 2, \dots$ ,

$$H_{m, [Tr]}(d) \Rightarrow A_m(r; d), \tag{13}$$

where  $A_m(r; d) = \Gamma(d)^{-1} \int_0^r (\log(r - s))^m (r - s)^{d-1} dW(s)$ . For any nonnegative integer  $M$ , the convergence in (13) holds jointly for  $m = 0, 1, \dots, M < \infty$ .

**Proof.** The main steps of the proof are identical to those in Marinucci and Robinson (2000) and Hualde (2012), so we focus on the relevant differences. We give the proof for a fixed  $m$ . Joint convergence follows by application of the Cramér–Wold device and the same proof.

Marinucci and Robinson (2000) generalize the results of Einmahl (1989) to short-range dependent variables, so they can construct copies in distribution of  $\xi_t$ , say  $\hat{\xi}_t$ , and independent  $w_t$  that are i.i.d.  $N(0, \Sigma)$  on the same probability space. We define  $S_j = \sum_{t=1}^j \hat{\xi}_t$ ,  $V_j = C(1) \sum_{t=1}^j w_t$ ,  $S_0 = V_0 = 0$ , and consider below the difference  $S_j - V_j$ , which is possible because  $S_j$  and  $V_j$  are defined on the same probability space. Specifically, based on results of Einmahl (1989, Theorems 1, 2, and 4), Marinucci and Robinson (2000, Lemma 2) show that  $\sup_{1 \leq j \leq T} |S_j - V_j| = o_p(T^{1/s})$  for  $2 < s < q$ , where  $q$  is given in Assumption 2. As in Hualde (2012), we define

$$\hat{H}_{m, [Tr]}(d) = T^{1/2-d} \sum_{n=1}^{[Tr]} \pi_{[Tr]-n}(d) \left( - \sum_{k=[Tr]-n}^T (k+d)^{-1} \right)^m \hat{\xi}_n, \quad m = 0, 1, 2, \dots$$

That is,  $\hat{H}_{m, \lfloor Tr \rfloor}(d)$  is defined exactly like  $H_{m, \lfloor Tr \rfloor}(d)$  in (12), but with  $\hat{\xi}_n$  replacing  $\xi_n$ . Because  $\hat{H}_{m, \lfloor Tr \rfloor}(d)$  is then a copy in distribution of  $H_{m, \lfloor Tr \rfloor}(d)$ , it suffices to show the required result for  $\hat{H}_{m, \lfloor Tr \rfloor}(d)$ .

We then decompose  $\hat{H}_{m, \lfloor Tr \rfloor}(d) = \sum_{i=1}^5 Q_{iT}(r)$ , where

$$\begin{aligned}
 Q_{1T}(r) &= \frac{1}{\Gamma(d)} T^{-1/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left(r - \frac{n}{T}\right)^{d-1} \left(\log\left(r - \frac{n}{T}\right)\right)^m (V_n - V_{n-1}) \mathbb{I}(\lfloor Tr \rfloor > 2), \\
 Q_{2T}(r) &= T^{1/2-d} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \pi_{\lfloor Tr \rfloor - n}(d) (S_n - S_{n-1} - (V_n - V_{n-1})) \left(-\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1}\right)^m \mathbb{I}(\lfloor Tr \rfloor > 2), \\
 Q_{3T}(r) &= T^{1/2-d} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left(\pi_{\lfloor Tr \rfloor - n}(d) \left(-\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1}\right)^m - \frac{(Tr-n)^{d-1}}{\Gamma(d)} \left(\log\left(r - \frac{n}{T}\right)\right)^m\right) \\
 &\quad \times (V_n - V_{n-1}) \mathbb{I}(\lfloor Tr \rfloor > 2), \\
 Q_{4T}(r) &= T^{1/2-d} \left(-\sum_{k=0}^T (k+d)^{-1}\right)^m (S_{\lfloor Tr \rfloor} - S_{\lfloor Tr \rfloor - 1}) \mathbb{I}(\lfloor Tr \rfloor > 2), \\
 Q_{5T}(r) &= T^{1/2-d} \sum_{n=1}^{\lfloor Tr \rfloor} \hat{\xi}_n \pi_{\lfloor Tr \rfloor - n}(d) \left(-\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1}\right)^m \mathbb{I}(\lfloor Tr \rfloor \leq 2),
 \end{aligned}$$

and  $\mathbb{I}(\cdot)$  denotes the indicator function. It suffices to show that

$$Q_{1T}(r) \Rightarrow A_m(r; d), \tag{14}$$

$$\sup_{0 \leq r \leq 1} \|Q_{iT}(r)\| \xrightarrow{P} 0 \quad \text{for } i = 2, \dots, 5. \tag{15}$$

Note that the only difference between our  $Q_{iT}(r)$  and the corresponding terms in Hualde (2012), aside from notational differences, is that instead of Hualde’s  $\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1}$  and  $\log(r - n/T)$ , we have  $(-\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1})^m$  and  $(\log(r - n/T))^m$ , respectively.

We first prove (14) and (15) for  $i = 2, 4, 5$ . These proofs follow nearly identically to the corresponding proofs of (24) and (25) in Hualde (2012), so we only outline the differences. First, we note that the bound established for  $m = 1$  in (26) of Hualde (2012) can easily be generalized to

$$|\log(r - n/T)|^m \leq K(r - n/T)^{-\alpha}, \quad n = 1, \dots, \lfloor Tr \rfloor - 1,$$

for any  $\alpha > 0$  and some positive constant  $K$  (if the bound applies for  $m = 1$  and any  $\alpha > 0$ , then clearly the bound also applies for any value of  $m$  on the left-hand side). Then the proof of (14) follows identically to that of the corresponding term in (24) of Hualde (2012). To prove (15) for  $i = 2, 4, 5$ , we can apply the same proofs as in Hualde (2012) except with

$$\left(\sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1}\right)^m \leq \left(\sum_{k=0}^T (k+d)^{-1}\right)^m \leq K(\log T)^m,$$

where Hualde has  $m = 1$ , and that change is inconsequential for the proofs.

It remains to prove (15) for the  $i = 3$  term, which is the term that involves the difference between the two factors  $(-\sum_{k=n}^T (k + d)^{-1})^m$  and  $\pi_n(d)$  and their corresponding limiting forms. We bound  $\sup_{0 \leq r \leq 1} \|Q_{3T}(r)\|$  by  $\sup_{1 \leq n \leq T} \|C(1)w_n\|$  times

$$\sup_{0 \leq r \leq 1} T^{-1/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left| \frac{\pi_n(d)}{T^{d-1}} - \frac{1}{\Gamma(d)} \left(\frac{n}{T}\right)^{d-1} \right| \left( \sum_{k=n}^T (k + d)^{-1} \right)^m \tag{16}$$

$$+ \frac{1}{\Gamma(d)} \sup_{0 \leq r \leq 1} T^{-1/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left| \left( \sum_{k=n}^T (k + d)^{-1} \right)^m - \left( \log \frac{n}{T} \right)^m \right| \left(\frac{n}{T}\right)^{d-1}. \tag{17}$$

For  $\lfloor Tr \rfloor > 2$  and any  $d \geq 0$ ,

$$\sup_{0 \leq r \leq 1} \sup_{1 \leq n \leq \lfloor Tr \rfloor - 1} \left( \sum_{k=n}^T (k + d)^{-1} \right)^m \leq \left( \sum_{k=1}^T (k + d)^{-1} \right)^m \sim (\log T)^m, \tag{18}$$

and thus the proof that (16) =  $o(1)$  is identical to that in (29) of Hualde (2012) except the logarithmic term is raised to the power  $m$ , which is inconsequential. Next, (17) is bounded by  $\Gamma(d)^{-1} \leq K$  times

$$\sup_{0 \leq r \leq 1} T^{-1/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left| \left( \sum_{k=n}^T (k + d)^{-1} \right)^m - \left( \sum_{k=n}^T k^{-1} \right)^m \right| \left(\frac{n}{T}\right)^{d-1} \tag{19}$$

$$+ \sup_{0 \leq r \leq 1} T^{-1/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left| \left( \sum_{k=n}^T k^{-1} \right)^m - \left( \int_n^T x^{-1} dx \right)^m \right| \left(\frac{n}{T}\right)^{d-1}. \tag{20}$$

To bound these terms, we use the identity  $x^m - y^m = (x - y) \sum_{j=0}^{m-1} x^j y^{m-1-j}$  and bound the first factor as

$$\sum_{k=n}^T \frac{1}{k} - \sum_{k=n}^T \frac{1}{k + d} = \sum_{k=n}^T \frac{d}{k(k + d)} \leq d \sum_{k=n}^T \frac{1}{k^2} \leq Kn^{-1}.$$

Using this bound together with (18), (19) is bounded by

$$\begin{aligned} K(\log T)^{m-1} \sup_{0 \leq r \leq 1} T^{-3/2} \sum_{n=1}^{\lfloor Tr \rfloor - 1} \left(\frac{n}{T}\right)^{d-2} &\leq K(\log T)^{m-1} T^{1/2-d} \sum_{n=1}^T n^{d-2} \\ &\leq K(\log T)^m T^{\max\{1/2-d, -1/2\}} \rightarrow 0. \end{aligned} \tag{21}$$

Similarly,

$$\sum_{k=n}^T \frac{1}{k} - \int_n^T x^{-1} dx \leq \sum_{k=n}^T \left( \frac{1}{k} - \frac{1}{k + 1} \right) = \frac{1}{n} - \frac{1}{T + 1} \leq n^{-1}$$



and  $\sup_{0 \leq r \leq 1} \sup_{1 \leq n \leq \lfloor Tr \rfloor - 1} (\int_n^T x^{-1} dx)^m \sim (\log T)^m$ , so that (20) is also bounded by (21). □

#### 4. WEAK CONVERGENCE OF $D_d^m Z_{\lfloor Tr \rfloor}(d)$

We next analyze the derivatives of the fractional process  $Z_{\lfloor Tr \rfloor}(d)$  with respect to the fractional parameter  $d$ , i.e.,  $D_d^m Z_{\lfloor Tr \rfloor}(d)$ . In terms of the fractional coefficients and their derivatives, we can define  $D_d^m Z_{\lfloor Tr \rfloor}(d)$  recursively as follows. We apply logarithmic differentiation and let

$$D_d^m Z_{\lfloor Tr \rfloor}(d) = \sum_{n=0}^{\lfloor Tr \rfloor - 1} D_d^m (T^{1/2-d} \pi_n(d)) \xi_{\lfloor Tr \rfloor - n} = \sum_{n=0}^{\lfloor Tr \rfloor - 1} T^{1/2-d} \pi_n(d) R_{Tn}^{(m)}(d) \xi_{\lfloor Tr \rfloor - n}, \tag{22}$$

where the coefficients  $R_{Tn}^{(m)}(d)$  are defined by the relation  $D_d^m (T^{1/2-d} \pi_n(d)) = T^{1/2-d} \pi_n(d) R_{Tn}^{(m)}(d)$ . We note that

$$D_d^{m+1} Z_{\lfloor Tr \rfloor}(d) = \sum_{n=0}^{\lfloor Tr \rfloor - 1} T^{1/2-d} \pi_n(d) (D_d R_{Tn}^{(m)}(d) + R_{Tn}^{(1)}(d) R_{Tn}^{(m)}(d)) \xi_{\lfloor Tr \rfloor - n},$$

so that the coefficients  $R_{Tn}^{(m)}(d)$  must satisfy the recursion

$$R_{Tn}^{(1)}(d) = D_d \log(T^{1/2-d} \pi_n(d)) = -\log T + \sum_{k=0}^{n-1} (k+d)^{-1}, \tag{23}$$

$$R_{Tn}^{(m+1)}(d) = D_d R_{Tn}^{(m)}(d) + R_{Tn}^{(1)}(d) R_{Tn}^{(m)}(d), \quad m = 1, 2, \dots \tag{24}$$

To illustrate the recursion, the next two terms of  $R_{Tn}^{(m)}(d)$  are

$$R_{Tn}^{(2)}(d) = -\sum_{k=0}^{n-1} (k+d)^{-2} + \left( -\log T + \sum_{k=0}^{n-1} (k+d)^{-1} \right)^2,$$

$$R_{Tn}^{(3)}(d) = 2 \sum_{k=0}^{n-1} (k+d)^{-3} - 3 \left( -\log T + \sum_{k=0}^{n-1} (k+d)^{-1} \right) \sum_{k=0}^{n-1} (k+d)^{-2} + \left( -\log T + \sum_{k=0}^{n-1} (k+d)^{-1} \right)^3.$$

There is a similar recursive definition of the derivatives of fBm. We define  $R^{(m)}(d)$  by the relation  $D_d^m (\Gamma(d)^{-1} (r-s)^{d-1}) = \Gamma(d)^{-1} (r-s)^{d-1} R^{(m)}(d)$  and find

$$D_d^m W(r; d) = \int_0^r D_d^m (\Gamma(d)^{-1} (r-s)^{d-1}) dW(s) = \Gamma(d)^{-1} \int_0^r R^{(m)}(d) (r-s)^{d-1} dW(s). \tag{25}$$

The first equality in (25) follows by the same proof as for (10). As in (23) and (24), we find that the functions  $R^{(m)}(d)$  must satisfy the recursion

$$R^{(1)}(d) = D_d \log(\Gamma(d)^{-1}(r-s)^{d-1}) = -\psi(d) + \log(r-s), \tag{26}$$

$$R^{(m+1)}(d) = D_d R^{(m)}(d) + R^{(1)}(d)R^{(m)}(d), \quad m = 1, 2, \dots \tag{27}$$

To compare with  $R_{Tn}^{(2)}(d)$  and  $R_{Tn}^{(3)}(d)$ , we find

$$R^{(2)}(d) = -\psi^{(1)}(d) + (-\psi(d) + \log(r-s))^2,$$

$$R^{(3)}(d) = -\psi^{(2)}(d) - 3(-\psi(d) + \log(r-s))\psi^{(1)}(d) + (-\psi(d) + \log(r-s))^3,$$

where  $\psi^{(j)}(d) = D_d^j \psi(d) = D_d^{j+1} \log \Gamma(d)$  denotes the polygamma function (see Abramowitz and Stegun, 1972, eqn. (6.4.1)). The recursive formulations in (24) and (27) are clearly much more tractable than direct calculation for larger values of  $m$ . We note, in particular, the strong similarity between the terms  $R_{Tn}^{(m)}(d)$  and  $R^{(m)}(d)$ . For example, for  $m = 1$  and with  $n$  replaced by  $[Tr] - [Ts]$ , we find that

$$\begin{aligned} R_{T, [Tr]-[Ts]}^{(1)}(d) &= -\log T + \sum_{k=0}^T (k+d)^{-1} - \sum_{k=[Tr]-[Ts]}^T (k+d)^{-1} \\ &\rightarrow -\psi(d) + \log(r-s) = R^{(1)}(d), \end{aligned}$$

as  $T \rightarrow \infty$  (cf. (9)).

We next derive the solutions to the recursions.

LEMMA 1. Let  $g(d) : \mathbb{R}^+ \rightarrow \mathbb{R}$  and assume that  $D^m g(d)$  exists for  $m = 1, 2, \dots$  and define  $G(d) = \int_0^d g(s) ds$ . Define recursively the functions  $g_m(d)$ ,  $m = 1, 2, \dots$ , by  $g_0(d) = 1$  and

$$g_{m+1}(d) = D_d g_m(d) + g(d)g_m(d). \tag{28}$$

The solution  $g_m(d)$  of (28) is given, for  $m = 1, 2, \dots$ , by

$$g_m(d) = e^{-G(d)} D_d^m e^{G(d)} = \sum_{(*)} c_{(*)} \prod_{i=1}^m (D_d^i G(d))^{j_i} = \sum_{(*)} c_{(*)} \prod_{i=1}^m (D_d^{i-1} g(d))^{j_i}, \tag{29}$$

where the summation  $\sum_{(*)}$  extends over all  $m$ -tuples of nonnegative integers  $(j_1, \dots, j_m)$  that satisfy  $\sum_{i=1}^m j_i = m$  and where  $c_{(*)} = m! \prod_{i=1}^m (j_i! (i!)^{j_i})^{-1}$ .

**Proof of Lemma 1.** The final equality in (29) follows easily because  $D_d^i G(d) = D_d^{i-1} g(d)$ . We multiply (28) by  $e^{G(d)}$  with derivative  $D_d e^{G(d)} = e^{G(d)} g(d)$  and find

$$\begin{aligned} e^{G(d)} g_{m+1}(d) &= e^{G(d)} D_d g_m(d) + e^{G(d)} g(d)g_m(d) = D_d (e^{G(d)} g_m(d)), \quad m = 0, 1, 2, \dots \end{aligned}$$

It follows by iteration that

$$e^{G(d)}g_{m+1}(d) = D_d(e^{G(d)}g_m(d)) = D_d^2(e^{G(d)}g_{m-1}(d)) = \dots = D_d^m(e^{G(d)}g(d)) = D_d^{m+1}(e^{G(d)}).$$

Dividing by  $e^{G(d)}$ , we have proved the first equality in (29). The next equality in (29) follows from the Faà di Bruno formula (see Roman, 1980, Theorem 2), which states that the derivatives of a composite function  $f(y)$ ,  $y = G(d)$ , are given by

$$\begin{aligned} D_d^m f(G(d)) &= \sum_{(*)} \frac{m!}{j_1!j_2! \dots j_m!} D_y^{j_1+\dots+j_m} f(y) \prod_{i=1}^m \left( \frac{D_d^i G(d)}{i!} \right)^{j_i} \\ &= \sum_{(*)} c_{(*)} D_y^{j_1+\dots+j_m} f(y) \prod_{i=1}^m (D_d^i G(d))^{j_i}. \end{aligned}$$

Inserting  $f(G(d)) = e^{G(d)}$  and noting that  $D_y^{j_1+\dots+j_m} f(y) = f(y)$ , we find (29). □

**COROLLARY 1.** *The solutions to the recursions (23)–(24) and (26)–(27) are given, for  $m = 1, 2, \dots$ , by*

$$R_{T,n}^{(m)}(d) = \sum_{(*)} c_{(*)} \prod_{i=1}^m (D_d^{i-1} R_{T,n}^{(1)}(d))^{j_i} \text{ and } R^{(m)}(d) = \sum_{(*)} c_{(*)} \prod_{i=1}^m (D_d^{i-1} R^{(1)}(d))^{j_i},$$

respectively, where, for  $i = 2, 3, \dots$ ,

$$D_d^{i-1} R_{T,n}^{(1)}(d) = (-1)^{i-1} (i-1)! \sum_{k=0}^{n-1} (k+d)^{-i} \text{ and } D_d^{i-1} R^{(1)}(d) = -\psi^{(i-1)}(d). \tag{30}$$

**Proof.** Apply Lemma 1 with initial functions  $g(d) = R_{T,n}^{(1)}(d) = -\log T + \sum_{k=0}^{n-1} (k+d)^{-1}$  and  $g(d) = R^{(1)}(d) = -\psi(d) + \log(r-s)$ , respectively. The solutions then follow from (29). □

We are now ready to give our main result.

**THEOREM 2.** *Under Assumptions 1 and 2, it holds that, for  $m = 0, 1, 2, \dots$ ,*

$$D_d^m Z_{[Tr]}(d) \Rightarrow D_d^m W(r; d),$$

where the derivatives are given in (22) and (25). For any nonnegative integer  $M$ , the convergence holds jointly for  $m = 0, \dots, M < \infty$ .

**Proof.** For  $m = 0$ , the result is given in (3), so we give the proof only for  $m \geq 1$ . Again, joint convergence follows by application of the Cramér–Wold device and the same proof.

We apply Corollary 1 and find that, in view of (22) and (25), it is enough to prove (joint) convergence for each  $(j_1, \dots, j_m)$  where  $j_i \geq 0$  :

$$\sum_{n=1}^{\lfloor Tr \rfloor} \prod_{i=1}^m (\mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d))^{j_i} T^{1/2-d} \pi_{\lfloor Tr \rfloor - n}(d) \xi_n \Rightarrow \int_0^r \prod_{i=1}^m (\mathbb{D}_d^{i-1} R^{(1)}(d))^{j_i} dW. \tag{31}$$

With this result, we can get the final result by taking the linear combination  $\sum_{(*)} c_{(*)}$  (see Lemma 1). Thus, we start by analyzing  $(\mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d))^j$  for some  $j \geq 1$ . We consider two cases.

The case  $i = 1$ : We find (see (9) and (23)), that

$$\begin{aligned} (R_{T, \lfloor Tr \rfloor - n}^{(1)})^j &= ((-\log T + \sum_{k=0}^T (k+d)^{-1}) - \sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1})^j \\ &= (-\psi(d) - \sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1})^j + o(1). \end{aligned} \tag{32}$$

The case  $i \geq 2$ : Adding and subtracting appropriately, we write  $\mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d)$  in (30) as

$$\begin{aligned} \mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d) &= (-1)^{i-1} (i-1)! \sum_{k=0}^T (k+d)^{-i} - (-1)^{i-1} (i-1)! \sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-i} \\ &= -\psi^{(i-1)}(d) + o(1) + u_{i,n}, \end{aligned}$$

where the convergence of the first term follows from Abramowitz and Stegun (1972, eqn. (6.4.10)) because  $i \geq 2$ , and where  $u_{i,n} = -(-1)^{i-1} (i-1)! \sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-i}$  satisfies  $|u_{i,n}| \leq K(\lfloor Tr \rfloor - n)^{-i+1} \leq K(\lfloor Tr \rfloor - n)^{-1}$  because  $i \geq 2$ . Thus, in the analysis of (31), we can use the approximation

$$(\mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d))^j = (-\psi^{(i-1)}(d))^j + o(1) + u_{i,n} \text{ for } i \geq 2. \tag{33}$$

Analysis of (31): We insert (32) and (33) into (31) and find, using (3) and Theorem 1, that

$$\begin{aligned} &\sum_{n=1}^{\lfloor Tr \rfloor} \prod_{i=1}^m (\mathbb{D}_d^{i-1} R_{T, \lfloor Tr \rfloor - n}^{(1)}(d))^{j_i} T^{1/2-d} \pi_{\lfloor Tr \rfloor - n}(d) \xi_n \\ &= \sum_{n=1}^{\lfloor Tr \rfloor} ((-\psi(d) - \sum_{k=\lfloor Tr \rfloor - n}^T (k+d)^{-1})^{j_1} \prod_{i=2}^m (-\psi^{(i-1)}(d))^{j_i} + o(1) + u_{i,n}) T^{1/2-d} \pi_{\lfloor Tr \rfloor - n}(d) \xi_n \\ &\Rightarrow \int_0^r (-\psi(d) + \log(r-s))^{j_1} \prod_{i=2}^m (-\psi^{(i-1)}(d))^{j_i} dW = \int_0^r \prod_{i=1}^m (\mathbb{D}_d^{i-1} R^{(1)}(d))^{j_i} dW \end{aligned}$$

(see (26) and (30)). This proves (31) and hence the desired result. □

## 5. APPLICATION TO THE MULTIFRACTIONAL COINTEGRATION MODEL

One motivation for the results on the weak convergence of derivatives of the fractional process comes from the analysis of the multifractional cointegrated vector autoregressive (MFCVAR) model (see Johansen and Nielsen, 2021). Let  $d = (d_1, \dots, d_p)'$  be a vector of fractional parameters, and let  $b$  be a scalar fractional parameter. The MFCVAR model with parameters  $\lambda = (d, b, \alpha, \beta, \Omega)$  and no lags is given by

$$\Lambda_+(d)X_t = \alpha\beta'(\Delta_+^{-b} - 1)\Lambda_+(d)X_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (34)$$

where the matrix differencing operator is  $\Lambda_+(d) = \text{diag}(\Delta_+^{d_1}, \dots, \Delta_+^{d_p})$  and  $\varepsilon_t$  satisfies Assumption 2. In particular,  $\varepsilon_t$  is i.i.d. with mean zero and variance  $\Omega$ .

The properties of the solution to these equations can be found from the corresponding result for the FCVAR model studied in Johansen and Nielsen (2012b). We denote true values by subscript zero, and in particular  $d_{0p}$  denotes the  $p$ th element of  $d_0$ . Now, if we define  $\tilde{X}_t$  by  $\Delta_+^{d_p}\tilde{X}_t = \Lambda_+(d)X_t$ , then  $\tilde{X}_t$  is given by the equations

$$\Delta_+^{d_p}\tilde{X}_t = \alpha\beta'(\Delta_+^{-b} - 1)\Delta_+^{d_p}\tilde{X}_t + \varepsilon_t, \quad t = 1, \dots, T. \quad (35)$$

These equations define the FCVAR model of Johansen and Nielsen (2012b) with scalar fractional parameters  $d_p$  and  $b$  together with  $(\alpha, \beta, \Omega)$ . It follows from Theorem 2 of Johansen and Nielsen (2012b) that the solution to (35), for  $(d_p, b, \alpha, \beta, \Omega) = (d_{0p}, b_0, \alpha_0, \beta_0, \Omega_0)$ , is

$$\tilde{X}_t = C_0\Delta_+^{-d_{0p}}\varepsilon_t + \Delta_+^{b_0-d_{0p}}Y_t,$$

where  $C_0 = \beta_{0\perp}(\alpha'_{0\perp}\beta_{0\perp})^{-1}\alpha'_{0\perp}$  and  $Y_t$  is a stationary linear process satisfying Assumption 1. Consequently, the solution to (34), for  $\lambda = \lambda_0$ , satisfies

$$\Lambda_+(d_0)X_t = C_0\varepsilon_t + \Delta_+^{b_0}Y_t. \quad (36)$$

This shows that  $\Delta_+^{d_{0i}}X_{it}$  is in general fractional of order zero, for  $i = 1, \dots, p$ , so that the model (34) allows each component of  $X_t$  to have its own fractional order, and is therefore called ‘‘multifractional.’’ Premultiplying (36) by  $\beta'_0$  shows that  $\Delta_+^{-b_0}\beta'_0\Lambda_+(d_0)X_t$  is also fractional of order zero; that is, some linear combinations of the processes  $\{\Delta_+^{d_{0i}-b_0}X_{it}\}_{i=1}^p$  are fractional of order zero and hence  $X_t$  is cointegrated.

We define the i.i.d. process  $\xi_t = (\alpha'_{0\perp}\beta_{0\perp})^{-1}\alpha'_{0\perp}\varepsilon_t$  such that  $C_0\varepsilon_t = \beta_{0\perp}\xi_t$ . The three processes  $Z_t(b_0)$ ,  $Z_t^*(b_0)$ , and  $D_{b_0}Z_t(b_0)$  are then defined in terms of  $\xi_t$  as in (3), (4), and (6), respectively. It follows from the above analysis that  $Z_{\lfloor Tr \rfloor}(b_0)$  and  $Z_{\lfloor Tr \rfloor}^*(b_0)$  converge weakly to fBm  $W(r; b_0)$ , that  $D_{b_0}Z_{\lfloor Tr \rfloor}(b_0)$  converges weakly to  $D_{b_0}W(r; b_0)$ , and that the processes converge jointly.

To simplify the subsequent analysis, we assume that  $\Omega = \Omega_0$ ,  $d_p = d_{0p}$ ,  $\alpha = \alpha_0$ , and  $b = b_0 > 1/2$ . This allows us to focus on the parameters that give rise

to “nonstandard” asymptotic distributions, and in particular to the application of  $D_{b_0} W(r; b_0)$ . Specifically, we define the parameters  $\theta = \beta'_{0\perp} \beta$  (or  $\beta = \beta_0 + \bar{\beta}_{0\perp} \theta$  with  $\bar{A} = A(A'A)^{-1}$  for any matrix  $A$  with full rank) and  $\gamma_i = d_i - d_{0i}$ , for  $i = 1, \dots, p$ , such that  $\gamma_p = 0$ . With this notation, we can define the residual, using (34) and (36), as

$$\varepsilon_t(\theta, \gamma) = (I_p + \alpha_0(\beta'_0 + \theta' \bar{\beta}'_{0\perp})(1 - \Delta_+^{-b_0})) \Lambda_+(\gamma)(C_0 \varepsilon_t + \Delta_+^{b_0} Y_t),$$

and the Gaussian likelihood is

$$L_T(\theta, \gamma) = -\frac{1}{2} \text{tr}\{\Omega_0^{-1} T^{-1} \sum_{t=1}^T \varepsilon_t(\theta, \gamma) \varepsilon_t(\theta, \gamma)'\} = -\frac{1}{2} \text{tr}\{\Omega_0^{-1} M_T(\varepsilon(\theta, \gamma), \varepsilon(\theta, \gamma))\},$$

where  $M_T(a, b) = T^{-1} \sum_{t=1}^T a_t b_t'$ . We will use this simple model to illustrate the role of the processes  $Z_t(b_0)$  and  $D_{b_0} Z_t(b_0)$  and their limits in the analysis of the score functions for  $\gamma$  and  $\theta$  evaluated at  $\lambda_0$ .

The derivative of  $\varepsilon_t(\theta, \gamma)$  with respect to  $\theta$  at  $\lambda = \lambda_0$  in the direction  $\partial\theta \in \mathbb{R}^{(p-r) \times r}$  is denoted  $D_\theta \varepsilon_t|_{\lambda=\lambda_0}(\partial\theta)$  and similarly for  $D_\gamma \varepsilon_t|_{\lambda=\lambda_0}(\partial\gamma)$ ,  $\partial\gamma \in \mathbb{R}^p$ , but with  $\partial\gamma_p = 0$  because  $\gamma_p = 0$ . We find that

$$\begin{aligned} D_\theta \varepsilon_t|_{\lambda=\lambda_0}(\partial\theta) &= \alpha_0(\partial\theta)' \bar{\beta}'_{0\perp} (1 - \Delta_+^{-b_0})(C_0 \varepsilon_t + \Delta_+^{b_0} Y_t) \\ &\simeq -\alpha_0(\partial\theta)' \Delta_+^{-b_0} \xi_t = -T^{b_0-1/2} \alpha_0(\partial\theta)' Z_t(b_0), \\ D_\gamma \varepsilon_t|_{\lambda=\lambda_0}(\partial\gamma) &= (I_p + \alpha_0 \beta'_0 (1 - \Delta_+^{-b_0})) \text{diag}(\partial\gamma) D_\gamma \Lambda_+(\gamma)|_{\gamma=0} (C_0 \varepsilon_t + \Delta_+^{b_0} Y_t) \\ &\simeq -\alpha_0 \beta'_0 \text{diag}(\partial\gamma) D_\gamma \Delta_+^{\gamma-b_0}|_{\gamma=0} \beta_{0\perp} \xi_t = \alpha_0 \beta'_0 \text{diag}(\partial\gamma) \beta_{0\perp} D_{b_0} \Delta_+^{-b_0} \xi_t \\ &= T^{b_0-1/2} (\log T) \alpha_0 \beta'_0 \text{diag}(\partial\gamma) \beta_{0\perp} Z_t^*(b_0), \end{aligned}$$

where  $Z_t$  and  $Z_t^*$  are given in (3) and (4), and where we use “ $\simeq$ ” to indicate that equality holds up to a stationary process that disappears asymptotically when we normalize the nonstationary processes. We identify the score vector  $S_{T,\theta}$  for  $\theta$  from  $D_\theta L_T|_{\lambda=\lambda_0}(\partial\theta) = (\text{vec } \partial\theta)' S_{T,\theta}$ , and similarly for  $\gamma$ . We then find that

$$\begin{aligned} T^{-b_0+1} D_\theta L_T|_{\lambda=\lambda_0}(\partial\theta) &\simeq \text{tr}\{\Omega_0^{-1} \alpha_0(\partial\theta)' T^{1/2} M_T(Z(b_0), \varepsilon)\}, \\ T^{-b_0+1} (\log T)^{-1} D_\gamma L_T|_{\lambda=\lambda_0}(\partial\gamma) &\simeq -\text{tr}\{\Omega_0^{-1} \alpha_0 \beta'_0 \text{diag}(\partial\gamma) \beta_{0\perp} T^{1/2} M_T(Z^*(b_0), \varepsilon)\}, \end{aligned}$$

and, using  $\text{tr}\{A'B\} = (\text{vec } A)' \text{vec } B$ , the scores are

$$\begin{aligned} T^{-b_0+1} S_{T,\theta} &\simeq \text{vec}(T^{1/2} M_T(Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0), \\ T^{-b_0+1} (\log T)^{-1} S_{T,\gamma} &\simeq -B'_0 \text{vec}(T^{1/2} M_T(Z^*(b_0), \varepsilon) \Omega_0^{-1} \alpha_0). \end{aligned}$$

Here, we have defined the  $(p-r)r \times p$  matrix  $B_0 = (\beta'_0 e_1 \otimes \beta'_{0\perp} e_1, \dots, \beta'_0 e_p \otimes \beta'_{0\perp} e_p)$ , with  $e_i$  denoting the  $i$ th unit vector in  $\mathbb{R}^p$ , and used the property that  $\text{tr}\{\beta'_0 \text{diag}(\phi) \beta_{0\perp} M\} = \phi' B'_0 \text{vec } M$  (see Theorem 2 of Johansen and Nielsen, 2021). Thus,  $S_{T,\theta} \in \mathbb{R}^{(p-r)r}$  and  $S_{T,\gamma} \in \mathbb{R}^p$ .

We note that the product moments  $T^{1/2} M_T(Z(b_0), \varepsilon)$  and  $T^{1/2} M_T(Z^*(b_0), \varepsilon)$  converge jointly to their weak limit  $\int_0^1 W(r; b_0) dW'(r)$ , so the scores become

linearly dependent in the limit. We therefore use the relation (7) to eliminate  $Z_t^*(b_0) = Z_t(b_0) + (\log T)^{-1} D_{b_0} Z_t(b_0)$ , and the score for  $\gamma$  becomes

$$T^{-b_0+1} S_{T,\gamma} \simeq -B'_0 \text{vec}(T^{1/2} M_T((\log T)Z(b_0) + D_{b_0} Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0).$$

We can now eliminate the linear dependence in the limit by defining the new parameter

$$\text{vec } \tilde{\theta} = \text{vec } \theta + (\log T) B_0 \gamma \in \mathbb{R}^{(p-r)r}$$

and

$$\tilde{\varepsilon}_t(\text{vec } \tilde{\theta}, \gamma) = \varepsilon_t(\text{vec } \theta, \gamma) = \varepsilon_t(\text{vec } \tilde{\theta} - (\log T) B_0 \gamma, \gamma).$$

Then the scores and their joint limits become

$$\begin{aligned} T^{-b_0+1} S_{T,\tilde{\theta}} &\simeq \text{vec}(T^{1/2} M_T(Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0) \Rightarrow \text{vec} \left( \int_0^1 W(r; b_0) dW'(r) \Omega_0^{-1} \alpha_0 \right), \\ T^{-b_0+1} S_{T,\gamma} &= (\log T) B'_0 \text{vec}(T^{1/2} M_T(Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0) \\ &\quad - B'_0 \text{vec}(T^{1/2} M_T((\log T)Z(b_0) + D_{b_0} Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0) \\ &= -B'_0 \text{vec}(T^{1/2} M_T(D_{b_0} Z(b_0), \varepsilon) \Omega_0^{-1} \alpha_0) \\ &\Rightarrow -B'_0 \text{vec} \left( \int_0^1 D_{b_0} W(r; b_0) dW'(r) \Omega_0^{-1} \alpha_0 \right). \end{aligned}$$

Thus, the introduction of the derivative of the fractional process and its limit allows one to reparameterize the score to find a mixed Gaussian asymptotic distribution, which can then be exploited to conduct inference for some hypotheses in the MFCVAR model. For a detailed analysis, we refer to Johansen and Nielsen (2021).

## 6. CONCLUDING REMARKS

Weak convergence of derivatives of fractional processes is interesting in its own right. However, it is also likely to find application in statistical analysis of inference problems related to multivariate fractional processes.

Hualde (2012) motivated his result (5) with a bivariate regression analysis of so-called “unbalanced cointegration” (see Hualde, 2014), but also anticipated that results like (5) may be useful in the statistical analysis of polynomial cofractionality (see Johansen, 2008; Franchi, 2010).

In Section 5, we presented an application of our results in (11) and Theorem 2 to the asymptotic distribution theory for the maximum likelihood estimators of the fractional parameters in the so-called “multifractional” vector autoregressive model of Johansen and Nielsen (2021). In this setting, the derivative  $D_d Z_{\lfloor Tr \rfloor}(d)$  and its weak limit  $D_d W(r; d)$  play an important role, because they allow avoiding linear dependence in the limit and because the asymptotic distribution is expressed in terms of both  $W(r, d)$  and  $D_d W(r, d)$ .

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