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Centre-valued Index for Toeplitz Operators with Noncommuting Symbols

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Abstract. We formulate and prove a "winding number" index theorem for certain "Toeplitz" operators in the same spirit as Gohberg–Krein, Lesch and others. The "number" is replaced by a self-adjoint operator in a subalgebra $Z \subseteq Z(A)$ of a unital C^*-algebra, A. We assume a faithful Z-valued trace τ on A left invariant under an action $\alpha\colon \mathbf{R} \to \mathrm{Aut}(A)$ leaving Z pointwise fixed. If δ is the infinitesimal generator of α and u is invertible in $\mathrm{dom}(\delta)$, then the "winding operator" of u is $\frac{1}{2\pi i}\tau(\delta(u)u^{-1}) \in Z_{sa}$. By a careful choice of representations we extend (A, Z, τ, α) to a von Neumann setting $(\mathfrak{A}, \mathfrak{Z}, \overline{\tau}, \overline{\alpha})$ where $\mathfrak{A} = A''$ and $\mathfrak{Z} = Z''$. Then $A \subset \mathfrak{A} \subset \mathfrak{A} \rtimes \mathbf{R}$, the von Neumann crossed product, and there is a faithful, dual $\mathfrak{Z}$-trace on $\mathfrak{A} \rtimes \mathbf{R}$. If P is the projection in $\mathfrak{A} \rtimes \mathbf{R}$ corresponding to the non-negative spectrum of the generator of $\mathbf{R}$ inside $\mathfrak{A} \rtimes \mathbf{R}$ and $\widetilde{\pi}\colon A \to \mathfrak{A} \rtimes \mathbf{R}$ is the embedding, then we define $T_u = P\widetilde{\pi}(u)P$ for $u \in A^{-1}$ and show it is Fredholm in an appropriate sense and the $\mathfrak{Z}$-valued index of T_u is the negative of the winding operator. In outline the proof follows that of the scalar case done previously by the authors. The main difficulty is making sense of the constructions with the scalars replaced by $\mathfrak{Z}$ in the von Neumann setting. The construction of the dual $\mathfrak{Z}$-trace on $\mathfrak{A} \rtimes \mathbf{R}$ requires the nontrivial development of a $\mathfrak{Z}$-Hilbert algebra theory. We show that certain of these Fredholm operators fiber as a "section" of Fredholm operators with scalar-valued index and the centre-valued index fibers as a section of the scalar-valued indices.

1 Winding Operator

We consider a unital C^*-algebra A with a unital C^*-subalgebra Z of $Z(A)$, the centre of A. We also assume that there exists a faithful, unital, tracial, conditional expectation $\tau\colon A \to Z$ (a "faithful Z-trace") and a continuous action $\alpha\colon \mathbf{R} \to \mathrm{Aut}(A)$ which leaves τ invariant, *i.e.,* $\tau \circ \alpha_t = \tau$ for all $t \in \mathbf{R}$. That is, our objects of study are 4-tuples (A, Z, τ, α) satisfying these conditions.

Under these hypotheses we show that the "winding number theorem" from [PhR] holds. We will often refer to this as a "winding operator".

Theorem 1.1 *Let (A, Z, τ, α) be a 4-tuple so that A is a unital C^*-algebra and $Z \subseteq Z(A)$ is a unital C^*-subalgebra of the centre of A, $\tau\colon A \to Z$ is a faithful, unital, tracial, conditional expectation, and $\alpha\colon \mathbf{R} \to \mathrm{Aut}(A)$ is a continuous action leaving τ invariant. Let δ be the infinitesimal generator of α. Then $a \mapsto \frac{1}{2\pi i}\tau(\delta(a)a^{-1})\colon \mathrm{dom}(\delta)^{-1} \to Z_{sa}$ is a group homomorphism that is constant on connected components and so extends uniquely to a group homomorphism $A^{-1} \to Z_{sa}$ that is constant on connected components and is 0 on Z^{-1}. We denote this map by $\mathrm{wind}_\alpha(a)$.*

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Proof It is an easy calculation to see that $a \mapsto \tau(\delta(a)a^{-1})\colon \mathrm{dom}(\delta)^{-1} \to (Z,+)$ is a homomorphism. We next calculate that $\alpha_t(z) = z$ for all $z \in Z$ and $t \in \mathbf{R}$.

$$\tau\Big(\big(\alpha_t(z) - z\big)^* \big(\alpha_t(z) - z\big)\Big) = \cdots = \tau(z^* z) - \tau(z^*)z - z^*\tau(z) + \tau(z^* z) = 0.$$

Therefore, $\alpha_t(z) - z = 0$ since τ is faithful. So $Z \subseteq \mathrm{dom}(\delta)$ and $\delta(Z) = \{0\}$. But then for each $z \in Z^{-1}$ we have $\tau(\delta(z)z^{-1}) = 0$.

Now for any $a \in \mathrm{dom}(\delta)$, we have

$$\tau(\delta(a)) = \tau\Big(\lim_{h\to 0} \frac{\alpha_h(a) - a}{h}\Big) = \lim_{h\to 0} \frac{1}{h}\tau(\alpha_h(a) - a) = 0.$$

Hence, by the Leibniz rule, for each $n \geq 1$,

$$\begin{aligned} 0 = \tau\big(\delta(a^n)\big) &= \tau\Big(\sum_{k=0}^{n-1} a^k\delta(a)a^{(n-1)-k}\Big) = \sum_{k=0}^{n-1} \tau(a^k\delta(a)a^{(n-1)-k}) \\ &= \sum_{k=0}^{n-1} \tau(a^{n-1}\delta(a)) = n\tau\big(a^{n-1}\delta(a)\big). \end{aligned}$$

Thus, for each $a \in \mathrm{dom}(\delta)$ and each $k \geq 0$ we have $\tau(\delta(a)a^k) = \tau(a^k\delta(a)) = 0$.

Now if $a \in \mathrm{dom}(\delta)$ and $\|1 - a\| < 1$, then a is invertible and $a^{-1} = \sum_{k=0}^{\infty}(1-a)^k$ which converges in norm. Since $\delta(1) = 0$, we have

$$\begin{aligned} \tau(\delta(a)a^{-1}) = -\tau(\delta(1-a)a^{-1}) &= -\tau\Big(\delta(1-a)\sum_{k=0}^{\infty}(1-a)^k\Big) \\ &= -\sum_{k=0}^{\infty}\tau(\delta(1-a)(1-a)^k) = 0. \end{aligned}$$

To see that the map is constant on connected components, we use the previous paragraph to show that it is locally constant. So we fix $a \in \mathrm{dom}(\delta)^{-1}$ and suppose $b \in \mathrm{dom}(\delta)^{-1}$ where $\|b - a\| < 1/\|a^{-1}\|$. Then $\|ba^{-1} - 1\| \leq \|b - a\|\,\|a^{-1}\| < 1$, so that

$$0 = \tau(\delta(ba^{-1})(ba^{-1})^{-1}) = \tau(\delta(b)b^{-1}) + \tau(\delta(a^{-1})a) = \tau(\delta(b)b^{-1}) - \tau(\delta(a)a^{-1})$$

as required.

Finally, to see that $\tau(\delta(a)a^{-1}) \in iZ_{sa}$, we first observe that $\mathrm{dom}(\delta)$ is a $*$-subalgebra of A, so $a \in \mathrm{dom}(\delta)^{-1}$ implies that $a^*a \in \mathrm{dom}(\delta)^{-1}$. Thus,

$$t \mapsto t1 + (1-t)a^*a$$

defines a path of invertible elements in $\mathrm{dom}(\delta)^{-1}$ connecting 1 to a^*a. Hence,

$$\tau(\delta(a^*a)(a^*a)^{-1}) = \tau(\delta(1)1) = 0.$$

Since the map is a homomorphism, this implies that $\tau(\delta(a^*)(a^*)^{-1}) = -\tau(\delta(a)a^{-1})$. But then

$$[\tau(\delta(a)a^{-1})]^* = \tau\big((a^*)^{-1}\delta(a^*)\big) = \tau(\delta(a^*)(a^*)^{-1}) = -\tau(\delta(a)a^{-1})$$

as required.

Since $\mathrm{dom}(\delta)$ is a dense $*$-subalgebra of A and A^{-1} is open, $\mathrm{dom}(\delta)^{-1}$ is dense in A^{-1} and so the map extends uniquely to A^{-1}. ∎

Definition 1.2 (**Morphism**) For $i = 1, 2$ let $(A_i, Z_i, \tau_i, \alpha^i)$ be two such 4-tuples where A_i is a unital C^*-algebra and Z_i is a unital C^*-subalgebra of the centre of A_i, etc. A morphism from $(A_1, Z_1, \tau_1, \alpha^1)$ to $(A_2, Z_2, \tau_2, \alpha^2)$ is given by a unital $*$-homomorphism $\varphi: A_1 \to A_2$ that maps $Z_1 \to Z_2$ and makes all the appropriate diagrams commute:

$$
\begin{tikzcd}
A_1 \arrow[r, "\varphi"] \arrow[d, "\tau_1"'] & A_2 \arrow[d, "\tau_2"] \\
Z_1 \arrow[r, "\varphi"'] & Z_2
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\varphi"] \arrow[d, "\alpha_t^1"'] & A_2 \arrow[d, "\alpha_t^2"] \\
A_1 \arrow[r, "\varphi"'] & A_2
\end{tikzcd}
$$

Proposition 1.3 *If* $\varphi: A_1 \to A_2$ *is a morphism from* $(A_1, Z_1, \tau_1, \alpha^1)$ *to* $(A_2, Z_2, \tau_2, \alpha^2)$ *and if* $a \in A_1^{-1} \cap (\mathrm{dom}(\delta_1))$, *then* $\varphi(a) \in A_2^{-1} \cap (\mathrm{dom}(\delta_2))$ *and* $\mathrm{wind}_{\alpha^1}(a) \in (Z_1)_{sa}$, *while* $\mathrm{wind}_{\alpha^2}(\varphi(a)) \in (Z_2)_{sa}$ *and also* $\varphi(\mathrm{wind}_{\alpha^1}(a)) = \mathrm{wind}_{\alpha^2}(\varphi(a))$.

Proof We first show that $a \in \mathrm{dom}(\delta_1)$ implies $\varphi(a) \in \mathrm{dom}(\delta_2)$ and that $\varphi(\delta_1(a)) = \delta_2(\varphi(a))$. So if $a \in \mathrm{dom}(\delta_1)$, then

$$\varphi(\delta_1(a)) = \varphi\Big(\lim_{t\to 0} \frac{\alpha_t^1(a) - a}{t}\Big) = \lim_{t\to 0} \varphi\Big(\frac{\alpha_t^1(a) - a}{t}\Big) = \lim_{t\to 0} \frac{\alpha_t^2\big(\varphi(a)\big) - \varphi(a)}{t}.$$

So the right-hand limit exists and defines $\delta_2(\varphi(a))$, that is, $\varphi(\delta_1(a)) = \delta_2\big(\varphi(a)\big)$. Now for $a \in A_1^{-1} \cap (\mathrm{dom}(\delta_1))$,

$$\begin{aligned}
\varphi\big(\mathrm{wind}_{\alpha^1}(a)\big) &= \frac{1}{2\pi i}\varphi\big(\tau_1(\delta_1(a)a^{-1})\big) = \frac{1}{2\pi i}\tau_2\big(\varphi(\delta_1(a)a^{-1})\big) \\
&= \frac{1}{2\pi i}\tau_2\big(\varphi(\delta_1(a))\varphi(a)^{-1}\big) = \frac{1}{2\pi i}\tau_2\big(\delta_2(\varphi(a))\varphi(a)^{-1}\big) \\
&= \mathrm{wind}_{\alpha^2}\big(\varphi(a)\big).
\end{aligned}$$

■

2 Extension to an Enveloping von Neumann Algebra

Key Idea 1 Since the range of our C^*-algebra trace, Z (an abelian C^*-algebra), is no longer restricted to being the scalars, the index of our generalized Toeplitz operators will necessarily take values in an abelian von Neumann algebra, say $\mathfrak{Z}$, containing Z. Unless Z is finite-dimensional (a relatively trivial extension of the scalar-valued trace), we will generally have $Z \neq \mathfrak{Z}$ (if Z is separable but not finite-dimensional we must have $Z \neq \mathfrak{Z}$).

We want our unital C^*-algebra A to be concretely represented on a Hilbert space $\mathcal{H}$ in such a way that the following nontrivial conditions hold. Let $\mathfrak{A} = A''$ and $\mathfrak{Z} = Z''$.

- There exists a necessarily unique faithful, tracial, ultraweakly continuous, conditional expectation, $\overline{\tau}: \mathfrak{A} \to \mathfrak{Z}$ extending τ. We will refer to this as a $\mathfrak{Z}$-trace. Henceforth, we will abbreviate this to uw-continuous.
- The continuous action $\alpha: \mathbf{R} \to \mathrm{Aut}(A)$, which leaves τ invariant extends to an ultraweakly continuous action $\overline{\alpha}: \mathbf{R} \to \mathrm{Aut}(\mathfrak{A})$ that leaves $\overline{\tau}$ invariant.

To achieve this we will assume that Z has a faithful state ω (this is automatically true if Z is separable). We will use the following proposition to define a concrete representation where these conditions obtain. We emphasize that the extension depends on the choice of the faithful state on Z. However, our notation will not show the dependence on this state. Of course, if $Z = \mathbf{C}$, the state is unique. If φ is a morphism from $(A_1, Z_1, \tau_1, \alpha^1)$ to $(A_2, Z_2, \tau_2, \alpha^2)$, we will assume that φ carries the faithful state ω_1 on Z_1 to ω_2 on Z_2, that is, $\omega_1 = \omega_2 \circ \varphi$ restricted to Z_1.

Proposition 2.1 *Let (A, Z, τ, α) be a 4-tuple and let ω be a faithful state on Z. Then $\overline{\omega} := \omega \circ \tau$ is a faithful tracial state on A which is left invariant by the action α. If we let $(\pi, \mathcal{H}, \xi_0)$ be the GNS representation of A afforded by $\overline{\omega}$ with cyclic separating trace vector ξ_0, then there is a continuous unitary representation $\{U_t\}$ of $\mathbf{R}$ on $\mathcal{H}$ so that (π, U) is covariant for α on A. Then $\{U_t\}$ implements a uw-continuous extension of α to $\overline{\alpha}$ acting on $\mathfrak{A} = \pi(A)''$. Moreover, letting $\mathfrak{Z} = \pi(Z)''$, there exists a unique, faithful, unital, uw-continuous $\mathfrak{Z}$-trace $\overline{\tau}: \mathfrak{A} \to \mathfrak{Z}$ extending τ, and $\overline{\alpha}$ leaves $\overline{\tau}$ invariant.*

Proof Denoting the image of $a \in A$ in $\mathcal{H}_{\overline{\omega}} := \mathcal{H}$ by $\widehat{a}$, it is completely standard that $U_t(\widehat{a}) := \widehat{\alpha_t(a)}$ defines a continuous unitary representation of $\mathbf{R}$ on A so that (π, U) is covariant for α. Hence, $\{U_t\}$ implements a uw-continuous extension of α to $\overline{\alpha}$ acting on $\mathfrak{A} = \pi(A)''$. It is also standard that the cyclic and separating vector $\xi_0 = \widehat{1}$ gives an extension of the trace $\overline{\omega}$ to a faithful uw-continuous trace on $\mathfrak{A}$. By an abuse of notation we will drop the notation π for the representation of A and just assume that A acts directly on $\mathcal{H}$. In this way we will also write the extended scalar trace (given by ξ_0) on $\mathfrak{A}$ as $\overline{\omega}$.

With this notation, we invoke [U] to obtain a uw-continuous conditional expectation $E: \mathfrak{A} \to \mathfrak{Z}$ defined by the equation $\overline{\omega}(E(x)y) = \overline{\omega}(xy)$ for $x \in \mathfrak{A}$, $y \in \mathfrak{Z}$. For $x = a \in A$ and $y = z \in Z$, we have

$$\overline{\omega}(\tau(a)z) = \omega(\tau(\tau(a)z)) = \omega(\tau(a)z) = \omega(\tau(az)) = \overline{\omega}(az).$$

Since Z is uw-dense in $\mathfrak{Z}$, we can replace the $z \in Z$ by any $y \in \mathfrak{Z}$ in the previous equation. That is, for $a \in A$ we have $E(a) = \tau(a)$ and so E is just an extension of τ by uw-continuity. We now use the notation $\overline{\tau}$ in place of E, and observe that since τ is tracial, so is $\overline{\tau}$. To see that $\overline{\tau}$ is faithful, suppose $x \in \mathfrak{A}$ and $\overline{\tau}(x^*x) = 0$. Then by the defining equation for $\overline{\tau}$ we have $0 = \overline{\omega}(\overline{\tau}(x^*x)1) = \overline{\omega}(x^*x)$, and since $\overline{\omega}$ is faithful, $x = 0$.

Finally, to see that $\overline{\alpha}$ leaves $\overline{\tau}$ invariant, we let $x \in \mathfrak{A}$ and $t \in \mathbf{R}$. Choose a bounded net $\{a_i\}$ in A that converges to x ultraweakly. Then since $\overline{\alpha}_t$ is spatial, we have $\alpha_t(a_i) = \overline{\alpha}_t(a_i) \to \overline{\alpha}_t(x)$ ultraweakly. Hence,

$$\overline{\tau}(\overline{\alpha}_t(x)) = \lim_i \overline{\tau}(\alpha_t(a_i)) = \lim_i \tau(\alpha_t(a_i)) = \lim_i \tau(a_i) = \lim_i \overline{\tau}(a_i) = \overline{\tau}(x). \qquad \blacksquare$$

Examples (4-**tuples**) **1. Kronecker (scalar trace) Example.** Let $A = C(\mathbf{T}^2)$, the C^*-algebra of continuous functions on the 2-torus, with the usual scalar trace τ_0 given by integration against the Haar measure on $\mathbf{T}^2$. We let $\alpha^\mu: \mathbf{R} \to \mathrm{Aut}(A)$ be the Kronecker flow on A determined by the real number μ (note that μ is not a power, merely a superscript). That is, for $s \in \mathbf{R}$, $f \in A$, and $(z,w) \in \mathbf{T}^2$, we have $(\alpha_s^{\mu}(f))(z,w) = f(e^{-2\pi i s}z, e^{-2\pi i\mu s}w)$.

In terms of the two commuting unitaries that generate $A = C(\mathbf{T}^2)$, namely $U(z,w) = z$ and $V(z,w) = w$, we have $\alpha_s^{\mu}(U) = e^{-2\pi i s}U$ and $\alpha_s^{\mu}(V) = e^{-2\pi i s\mu}V$. Of course, this action leaves our scalar trace τ_0 invariant. In the case where $Z = \mathbf{C}$, the faithful state ω on $Z = \mathbf{C}$ is just the identity mapping and so $\overline{\omega} := \omega \circ \tau_0 = \tau_0$. That is, $\mathcal{H}_{\overline{\omega}} = \mathcal{H}_{\tau_0} = L^2(\mathbf{T}^2)$ with the obvious representation of A on $\mathcal{H}_{\tau_0}$. In this case, $Z = \mathfrak{Z} = \mathbf{C}$ and so $\mathfrak{A} = L^{\infty}(\mathbf{T}^2)$. Clearly τ_0 and α extend to $\overline{\tau}_0$ and $\overline{\alpha}$ as required.

One easily calculates the winding numbers of the generators: $\mathrm{wind}_{\alpha^{\mu}}(U) = -1$ and $\mathrm{wind}_{\alpha^{\mu}}(V) = -\mu$.

1.a. Noncommutative Tori. We quickly observe that the previous construction can be carried over almost verbatim to noncommutative tori. For $\theta \in [0,1)$ let

$$A_\theta = C^*(U,V \mid VU = e^{2\pi i\theta}UV)$$

be the universal C^*-algebra generated by two unitaries U, V satisfying the above relation. For $\theta = 0$, the algebra A_θ is naturally isomorphic to $A = C(\mathbf{T}^2)$ with $U(z,w) = z$ and $V(z,w) = w$. For θ irrational, these algebras are of course the irrational rotation algebras which are simple C^*-algebras. We let $\alpha^{\mu}: \mathbf{R} \to \mathrm{Aut}(A)$ be the flow on A_θ determined by the real number μ, that is, for $s \in \mathbf{R}$ and U, V, the generators of A_θ, we have $\alpha_s^{\mu}(U) = e^{-2\pi i s}U$ and $\alpha_s^{\mu}(V) = e^{-2\pi i s\mu}V$. Since $\alpha_s(U)$ and $\alpha_s(V)$ satisfy the same relation as U and V, this is a well-defined flow on A_θ.

The scalar trace τ_θ on A_θ on the dense subalgebra of finite linear combinations of U^nV^m for m, n in $\mathbf{Z}$ satisfies

$$\tau_\theta(U^nV^m) = \begin{cases} 0 & \text{if } n \neq 0 \text{ or } m \neq 0, \\ 1 & \text{if } n = 0 = m. \end{cases}$$

Again, one easily calculates the winding numbers of the generators:

$$\mathrm{wind}_{\alpha^{\mu}}(U) = -1, \quad \mathrm{wind}_{\alpha^{\mu}}(V) = -\mu.$$

2. Generalized Kronecker and Generalized Noncommutative Tori Examples. We show that any self-adjoint element in any unital commutative C^*-algebra (with a faithful state) can be used as a replacement for the scalar μ in Examples 1 and 1.a to obtain a non-scalar example. Let $Z = C(X)$ be any commutative unital C^*-algebra with a faithful state and let $\eta \in Z_{sa}$ be any self-adjoint element in Z. Let $A = Z \otimes C(\mathbf{T}^2) = C(X, C(\mathbf{T}^2))$ (respectively, $A = Z \otimes A_\theta = C(X, A_\theta)$) and let $\tau: A \to Z$ be given by the "slice-map" $\tau = \mathrm{id}_Z \otimes \tau_\theta$ where τ_θ for $\theta = 0$ is the standard trace on $C(\mathbf{T}^2)$ given by Haar measure (respectively, the usual scalar trace τ_θ on A_θ defined above). Then τ is a faithful, tracial, conditional expectation of A onto Z. In particular, for $f \in A = Z \otimes C(\mathbf{T}^2) \cong C(\mathbf{T}^2, Z)$ we have $\tau(f) = \int_{\mathbf{T}^2} f(z,w)\, d(z,w) \in Z$. In this case we note that for $A = Z \otimes C(\mathbf{T}^2)$, we have $Z(A) = A$ and hence Z is strictly contained in $Z(A)$. On the other hand, for θ irrational, $Z(A) = Z(Z \otimes A_\theta) = Z$ since A_θ is simple. In either case we use the element $\eta \in Z_{sa}$ to define a τ-invariant action $\{\alpha_t^{\eta}\}$ of $\mathbf{R}$ on A: $\alpha_t^{\eta}(f)(x) = \alpha_t^{\eta(x)}(f(x))$, for $f \in A$, $t \in \mathbf{R}$, $x \in X$ (again, η and $\eta(x)$ are not powers, but merely superscripts). It is clear that $(A, Z, \tau, \alpha^{\eta})$ is a 4-tuple.

In both these cases one calculates the following winding operators:

$$\mathrm{wind}_{\alpha^{\eta}}(1 \otimes U) = -1 \otimes 1, \quad \mathrm{wind}_{\alpha^{\eta}}(1 \otimes V) = -\eta \otimes 1.$$

Using the faithful state ω on Z, we define a faithful (tracial) state $\overline{\omega}$ on A via $\overline{\omega} := \omega \circ \tau$. By Proposition 2.1, $\overline{\omega}$ is a faithful (tracial) state on A which is left invariant by α and if $(\pi, \mathcal{H})$ is the GNS representation of A induced by $\overline{\omega}$, then there is a continuous unitary representation $\{U_t\}$ of $\mathbf{R}$ on $\mathcal{H}$ so that (π, U) is covariant for α on A. Also $\{U_t\}$ implements a uw-continuous extension of α to $\overline{\alpha}$ acting on $\mathfrak{A} := \pi(A)''$. Moreover, letting $\mathfrak{Z} := \pi(Z)''$, there exists a unique, faithful, unital, uw-continuous $\mathfrak{Z}$-trace $\overline{\tau}: \mathfrak{A} \to \mathfrak{Z}$ extending τ, and $\overline{\alpha}$ leaves $\overline{\tau}$ invariant.

3. C^*-algebra of the Integer Heisenberg Group

Let A be the C^*-algebra $C^*(H)$ of the integer Heisenberg group H.

$$H = \left\{ \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \middle| \, m, n, p \in \mathbf{Z} \right\}.$$

We view $A = C^*(H)$ as the universal C^*-algebra generated by three unitaries U, V, W satisfying $WU = UW$, $WV = VW$, and $UV = WVU$. Here U, V, W correspond respectively to the three generators of H:

$$u = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.2 *If H is a discrete group with subgroup C, then the map $l^1(H) \to l^1(C)$ defined by $f \mapsto f_{|C}$ extends to a faithful, conditional expectation τ from $C_r{}^*(H) \to C_r{}^*(C)$. If C is the centre of H, then τ is also tracial. Combining τ with the canonical $*$-homomorphism $C^*(H) \to C_r{}^*(H)$, we see that we can also view τ as a trace on $C^*(H)$.*

Proof Let $f \mapsto \pi_H(f)$ and $g \mapsto \pi_C(g)$ denote the left regular representations of $l^1(H)$ and $l^1(C)$ on $l^2(H)$ and $l^2(C)$, respectively. Then for $\eta \in l^2(C) \subseteq l^2(H)$ we have

$$\begin{aligned} \pi_H(f)(\eta)(c) &= \sum_{h \in H} f(ch^{-1})\eta(h) = \sum_{h \in C} f(ch^{-1})\eta(h) \\ &= \sum_{h \in C} f_{|C}(ch^{-1})\eta(h) = \pi_C(f_{|C})(\eta)(c). \end{aligned}$$

In other words, for each $\eta \in l^2(C)$, $\pi_H(f)(\eta) = \pi_C(f_{|C})(\eta)$, so that $\pi_H(f)_{|_{l^2(C)}} = \pi_C(f_{|C})$. We let $E: l^2(H) \to l^2(C)$ denote the canonical projection. Then all $\eta \in l^2(C)$ have the form $\eta = E(\xi)$ for $\xi \in l^2(H)$, and we have

$$\pi_C(f_{|C})E(\xi) = E\pi_C(f_{|C})E(\xi) = E\pi_H(f)E(\xi).$$

We now define $\tau\big(\pi_H(f)\big) = \pi_C(f_{|C})$. To see that τ is bounded in operator norm,

$$\|\pi_C(f_{|C})\| = \|E\pi_H(f)E\| \leq \|\pi_H(f)\|.$$

Thus τ extends by continuity to $\tau: C_r{}^*(H) \to C_r{}^*(C)$. For general $x \in C_r{}^*(H)$ we have $\tau(x) = E\pi_H(x)E$, so that the extended τ is clearly completely positive, onto, and has norm 1, that is, it is a conditional expectation by Tomiyama's theorem.

Now for $f \in l^1(H)$ we have $\tau(\pi_H(f)) = \pi_C(f_{|C})$, so that if C is the centre of H, then in order to see that τ is tracial, it suffices to see that for $f, g \in l^1(H)$, $(f * g)_{|C} = (g * f)_{|C}$. So for $c \in C$ we have

$$(f * g)(c) = \sum_{h \in H} f(ch^{-1})g(h) = \sum_{h \in H} g(h)f(h^{-1}c) = (g * f)(c). \qquad \blacksquare$$

In our example where H is the Heisenberg group, its centre is $C = \{w^n \mid n \in \mathbf{Z}\}$. In our realization of $A = C^*(H)$ as a universal C^*-algebra, the centre of A is $Z = C^*(W)$. Now the dense $*$-subalgebra of A generated by U, V, W has as a basis all elements of the form $W^p V^n U^m$, each of which corresponds uniquely to the group element

$$w^p v^n u^m = \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

in H. In this notation $\tau: A \to Z$ is given by

$$\tau(W^p V^n U^m) = \begin{cases} 0 & \text{if } n \neq 0 \text{ or } m \neq 0, \\ W^p & \text{if } n = 0 = m. \end{cases}$$

In order to define our action $\alpha: \mathbf{R} \to \operatorname{Aut}(A)$, we first fix an element $\eta \in Z_{sa}$. For an explicit example, we arbitrarily choose $\eta = (\mu/3)(W + 1 + W^*)$, where μ is a fixed real number. For this fixed η we define the action α via

$$\alpha_t(U) = e^{-2\pi i t}U, \quad \alpha_t(V) = e^{-2\pi i t\eta}V, \quad \alpha_t(W) = W.$$

So on the basis elements we get

$$\alpha_t(W^p V^n U^m) = e^{-2\pi i n t\eta} e^{-2\pi i m t} W^p V^n U^m = e^{-2\pi i t(n\eta + m)} W^p V^n U^m.$$

One easily checks that for fixed t the operators $U_t := \alpha_t(U)$, $V_t := \alpha_t(V)$, and $W_t := W$ satisfy the same relations as U, V, W, namely,

$$W_t U_t = U_t W_t, \quad W_t V_t = V_t W_t, \quad U_t V_t = W_t V_t U_t.$$

Hence, α_t defines a $*$-representation of H in $A = C^*(H)$ and so extends to a $*$-representation of $C^*(H)$ inside $C^*(H)$. Now W is in the range of this $*$-representation and so $C^*(W)$ is in the range of this $*$-representation. Hence $e^{2\pi i t\eta}$ is in the range of this $*$-representation for any $t \in \mathbf{R}$. Hence $V = e^{2\pi i t\eta} V_t$ is in the range also. Similarly, U is in the range so that $\alpha_t(C^*(H)) = C^*(H)$ since it is dense and closed. Since α_{-t} is the inverse of α_t, α_t is one-to-one and hence an automorphism of $C^*(H)$. One easily checks that $\alpha_{t+s} = \alpha_t \alpha_s$ using the fact that $e^{-2\pi i s\eta}$ is in the centre. The point-norm continuity of $t \mapsto \alpha_t(a)$ is clear.

Thus we have an action $\alpha: \mathbf{R} \to \operatorname{Aut}(A)$ that fixes $Z = C^*(W) = C^*(C)$ and leaves the Z-valued trace τ invariant. That is, $(C^*(H), C^*(C), \tau, \alpha)$ is a 4-tuple. Now the left regular representation of $C^*(C)$ on $l^2(C)$ gives a faithful vector state $\omega(x) = \langle x(\delta_1), \delta_1 \rangle$, which for $x \in l^1(C)$ is just $\omega(x) = x(1)$. Then the state $\overline{\omega}$ on $C^*(H)$ is given for $x \in l^1(H)$ by $\overline{\omega}(x) = (\omega \circ \tau)(x) = \omega(x_{|C}) = x_{|C}(1) = x(1)$. Now if $x, y \in l^1(H)$, then the inner product induced by $\overline{\omega}$ is

$$\langle x, y \rangle_{\overline{\omega}} = \overline{\omega}(x \cdot y^*) = (x \cdot y^*)(1) = \sum_{h \in H} x(1h) y^*(h^{-1}) = \sum_{h \in H} x(h)\overline{y(h)} = \langle x, y \rangle.$$

That is, $\mathcal{H}_{\overline{\omega}} = l^2(H)$ and the representation of $C^*(H)$ on $\mathcal{H}_{\overline{\omega}} = l^2(H)$ is just the left regular representation. So in this case $\mathfrak{A} = W_r^*(H)$, the left regular von Neumann algebra of H.

Now $l^2(H) = \bigoplus_X l^2(C \cdot X)$ over all the cosets $C \cdot X$ of C. Moreover, each coset, $C \cdot (W^p V^n U^m) = C \cdot (V^n U^m)$ is uniquely determined by the pair of integers (n, m), so that $l^2(H) = \bigoplus_{(n,m)} l^2(C \cdot V^n U^m)$. Clearly the left action of C (and hence, of $C^*(C)$) on each coset space is unitarily equivalent to the left regular representation of $C^*(C)$ on $l^2(C)$. Hence, the left action of $C^*(C)$ on $l^2(H)$ is just a countably infinite multiple of the left regular representation of $C^*(C)$ on $l^2(C)$. That is, $\mathfrak{Z} = 1_{\mathbf{Z}^2} \otimes W_r^*(C)$.

Thus the map $\tau: C^*(H) \to C^*(C)$ with both acting on $l^2(H)$ becomes $\tau(x) = 1_{\mathbf{Z}^2} \otimes ExE$ where E is the projection from $l^2(H)$ onto $l^2(C)$. It is clear that this map is weak-operator continuous and so extends by the same formula to a tracial expectation $\overline{\tau}: \mathfrak{A} \to \mathfrak{Z}$. It is also clear that α extends to $\overline{\alpha}$ as needed.

In this example one calculates the following winding operators in $Z = C^*(W)$:

$$\mathrm{wind}_\alpha(U) = -1, \quad \mathrm{wind}_\alpha(V) = -\mu/3(W + 1 + W^*), \quad \mathrm{wind}_\alpha(W) = 0.$$

Examples **(Morphisms) 1. Generalized Kronecker to Kronecker Morphisms.** We let $A_1 = C(X) \otimes C(\mathbf{T}^2)$ and $Z_1 = C(X) \otimes 1$. We let $\tau_1 = \mathrm{id}_{C(X)} \otimes \tau_0$ where $\tau_0: C(\mathbf{T}^2) \to \mathbf{C}$ is given by integration with respect to Haar measure on $\mathbf{T}^2$. We arbitrarily fix an $\eta \in (Z_1)_{sa} = (C(X) \otimes 1)_{sa}$. We also define $\alpha^1: \mathbf{R} \to \mathrm{Aut}(A_1)$ via

$$\alpha_t^1(h)(x, z, w) = h(x, e^{-2\pi i t} z, e^{-2\pi i t \eta(x)} w).$$

As before we let $u \in A_1$ be the unitary $u(x, z, w) = w$.

We let $A_2 = C(\mathbf{T}^2)$ and $Z_2 = \mathbf{C}1$ and $\tau_2 = \tau_0: A_2 \to Z_2$. We arbitrarily fix an $x_0 \in X$ and define the evaluation $*$-homomorphism $\varphi: A_1 \to A_2$ via $\varphi(h)(z, w) = h(x_0, z, w)$. We let $\mu = \eta(x_0)$ and define $\alpha_t^2(h)(z, w) = h(e^{-2\pi i t} z, e^{-2\pi i t \mu} w)$. One easily checks that φ defines a morphism from $(A_1, Z_1, \tau_1, \alpha^1)$ to $(A_2, Z_2, \tau_2, \alpha^2)$ and that $\varphi(u) = v$ where $v(z, w) = w$.

1a. Generalized Noncommutative tori to Kronecker Morphisms. We previously defined $A = C(X) \otimes A_\theta$ and $Z = C(X) \otimes 1$. We also let $\tau_1 = \mathrm{id}_{C(X)} \otimes \tau_\theta$ where $\tau_\theta: A_\theta \to \mathbf{C}$ is defined above. We arbitrarily fixed an $\eta \in (Z)_{sa} = (C(X) \otimes 1)_{sa}$ and then defined $\alpha: \mathbf{R} \to \mathrm{Aut}(A)$ via $\big(\alpha_t(f)\big)(x) = \alpha_t^{\eta(x)}\big(f(x)\big)$ for $f \in A$, $t \in \mathbf{R}$, and $x \in X$. We let $v \in A_1$ be the constant unitary $v(x) = V$.

We now consider A_θ and $Z = \mathbf{C}1$ and $\tau_\theta: A_\theta \to Z$. We arbitrarily fix an $x_0 \in X$ and consider the action of $\mathbf{R}$ on A_θ defined by the real number $\eta(x_0)$, that is, $\alpha^{\eta(x_0)}$. This gives us a 4-tuple, $(A_\theta, \mathbf{C}, \tau_\theta, \alpha^{\eta(x_0)})$. We now have the evaluation $*$-homomorphism $\varphi: A \to A_\theta$ via $\varphi(h) = h(x_0)$. One easily checks that φ defines a morphism from (A, Z, τ_1, α) to $(A_\theta, \mathbf{C}, \tau_\theta, \alpha^{\eta(x_0)})$. Moreover, $\varphi(v) = V$.

2. Heisenberg to Kronecker Morphisms. We let $A_1 = C^*(H)$ and $Z_1 = C^*(W) = C^*(C) \cong C^*(\mathbf{Z}) \cong C(\mathbf{T})$ and recall that

$$\tau_1(W^p V^n U^m) = \begin{cases} 0 & \text{if } n \neq 0 \text{ or } m \neq 0, \\ W^p & \text{if } n = 0 = m, \end{cases}$$

defines a trace $\tau_1: A_1 \to Z_1$. Recall that we (arbitrarily) chose $\theta = (\mu/3)(W+1+W^*) \in (Z_1)_{sa}$ and defined our automorphism group by

$$\alpha_t^1(W^p V^n U^m) = e^{-2\pi i n t\theta} e^{-2\pi i m t} W^p V^n U^m = e^{-2\pi i t(n\theta+m)} W^p V^n U^m.$$

We let $A_2 = C^*(H/C) \cong C^*(\mathbf{Z}^2) \cong C(\mathbf{T}^2)$ where the two isomorphisms are given by

$$\operatorname{Coset}(W^p V^n U^m) = C \cdot (W^p V^n U^m) = C \cdot (V^n U^m) \mapsto (n, m) \mapsto z^n w^m.$$

We let $Z_2 = \mathbf{C}1 \subset A_2$ and define $\tau_2: A_2 \to Z_2 = \mathbf{C}1$ to be the composition of these isomorphisms with the trace on $C(\mathbf{T}^2)$ given by the Haar integral. This clearly implies that

$$\tau_2(C \cdot (V^n U^m)) = \begin{cases} 0 & \text{if } n \neq 0 \text{ or } m \neq 0, \\ 1 & \text{if } n = 0 = m. \end{cases}$$

We now define $\alpha_t^2 \in \operatorname{Aut}(A_2)$ via

$$\begin{aligned} \alpha_t^2((C \cdot V)^n (C \cdot U)^m) &= e^{-2\pi i t n\mu} (C \cdot V)^n e^{-2\pi i t m} (C \cdot U)^m \\ &= e^{-2\pi i t(n\mu+m)} (C \cdot V)^n (C \cdot U)^m. \end{aligned}$$

Clearly, $(A_2, Z_2, \tau_2, \alpha_2)$ is isomorphic to the Kronecker example with scalar μ.

We now define a $*$-homomorphism $\varphi: A_1 = C^*(H) \to A_2 = C^*(H/C)$ as the unique extension of the canonical group homomorphism $H \to H/C$. So

$$\varphi(W^p V^n U^m) = (C \cdot V)^n (C \cdot U)^m.$$

In particular, $\varphi(W^p) = (C{\cdot}1) = 1 \in H/C$. One easily checks that φ defines a morphism from $(A_1, Z_1, \tau_1, \alpha^1)$ to $(A_2, Z_2, \tau_2, \alpha^2)$ and that $\varphi(W^p V^n U^m) = (C \cdot V)^n (C \cdot U)^m$. Hence, $\varphi(\theta) = \varphi((\mu/3)(W^{-1} + 1 + W)) = \mu$ by our choice of θ.

3 Hilbert Algebras Over an Abelian von Neumann Algebra

Key Idea 2 While centre-valued traces are well known (*e.g.*, the Traces Opératorielles of [Dix]) a completely general construction of such traces suitable for use with crossed-products has not, to our knowledge, been attempted before now.

In this section we combine the theory of Hilbert modules [Pa, R] with the theory of Hilbert algebras [Dix] in order to construct centre-valued traces on certain crossed product von Neumann algebras. Although the outline is similar to the usual Hilbert algebra theory, the details are rather subtle. The main difficulties arise because the usual norm completion of these new "Hilbert algebras" is not self-dual in the sense of Paschke [Pa].

Definition 3.1 Let $\mathfrak{B}$ be a von Neumann algebra. A complex vector space $\mathbf{X}$ is a (right) *pre-Hilbert $\mathfrak{B}$-module* if there exists a $\mathfrak{B}$-valued inner product $\langle \cdot, \cdot \rangle$ which is linear in the second co-ordinate satisfying the following.

(a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \leftrightarrow x = 0$ for each $x \in \mathbf{X}$.
(b) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in \mathbf{X}$.
(c) $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in \mathbf{X}$ and $a \in \mathfrak{B}$.
(d) $\operatorname{span}\{\langle x, y \rangle \mid x, y \in \mathbf{X}\}$ is uw-dense in $\mathfrak{B}$.

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Key Idea 3 In the following we consider bounded module mappings between Hilbert modules but we do not assume that these mappings are adjointable. As pointed out by Lance [L], this is too severe a restriction, since for Hilbert modules all such maps arise from inner products. However, most Hilbert modules are not self-dual; self-dual modules Y have the property that $\mathcal{L}(Y)$ is a von Neumann algebra. In the examples that we use later, the Paschke dual $X^\dagger$ of a pre-Hilbert $\mathfrak{B}$-module X is a self-dual module that is usually much larger than X. We need these self-dual modules in order to work in the von Neumann algebra $\mathcal{L}(X^\dagger)$.

Definition 3.2 We follow Paschke [Pa] by defining the *dual* of a pre-Hilbert $\mathfrak{B}$-module $\mathbf{X}$ to be the space $\mathbf{X}^\dagger = \{\theta\colon \mathbf{X} \to \mathfrak{B} \mid \theta \text{ is a bounded } \mathfrak{B}\text{-module map}\}$. In order to make the embedding of $\mathbf{X}$ into $\mathbf{X}^\dagger$ linear, Paschke defines scalar multiplication on $\mathbf{X}^\dagger$ by $(\lambda\theta)(x) := \overline{\lambda}\theta(x)$ for $\lambda \in \mathbf{C}$, $\theta \in \mathbf{X}^\dagger$, and $x \in \mathbf{X}$. Similarly, module multiplication on $\mathbf{X}^\dagger$ is given by $(\theta \cdot a)(x) := (a^*\theta(x))$ for $\theta \in \mathbf{X}^\dagger$, $a \in \mathfrak{B}$, and $x \in \mathbf{X}$.

Therefore, we can identify $\mathbf{X}$ in $\mathbf{X}^\dagger$ via $x \mapsto \widehat{x}$ where $\widehat{x}(y) = \langle x, y\rangle$ for $x, y \in \mathbf{X}$. Since $\mathfrak{B}$ is a von Neumann algebra, Paschke shows how to extend the $\mathfrak{B}$-valued inner product on $\mathbf{X}$ to an inner product on $\mathbf{X}^\dagger$ so that $\mathbf{X}^\dagger$ becomes self-dual [Pa, Theorem 3.2]. This theorem is not trivial.

We recall Paschke's construction [Pa, p. 450]: let $\mathfrak{B}_*$ be the space of ultraweakly continuous linear functionals on $\mathfrak{B}$, that is, the predual of $\mathfrak{B}$. Now for each positive functional ω in $\mathfrak{B}_*$ we have that for $N_\omega = \{x \in \mathbf{X} \mid \omega(\langle x, x\rangle) = 0\}$, the space $\mathbf{X}/N_\omega$ is a pre-Hilbert space with inner product $\langle x + N_\omega, y + N_\omega\rangle_\omega = \omega(\langle x, y\rangle)$. Moreover, for each $\theta \in \mathbf{X}^\dagger$, the mapping $x + N_\omega \mapsto \omega(\theta(x))$ is a well-defined bounded linear functional on $\mathbf{X}/N_\omega$ satisfying $|\omega(\theta(x))| \le \|\omega\|^{1/2}\|\theta\|\,\|x + N_\omega\|_\omega$. Hence, there exists a unique vector θ_ω in $\mathcal{H}_\omega$, the Hilbert space completion of $\mathbf{X}/N_\omega$, with $\omega\big(\theta(x)\big) = \langle\theta_\omega, x + N_\omega\rangle_\omega$ for all $x \in \mathbf{X}$, and $\|\theta_\omega\|_\omega \le \|\omega\|^{1/2}\|\theta\|$. Thus, $\|x\|_\omega := \omega(\langle x, x\rangle)^{1/2}$ is a well-defined seminorm on $\mathbf{X}$ which extends naturally to $\mathbf{X}^\dagger$ via $\|\theta\|_\omega = \langle\theta_\omega, \theta_\omega\rangle_\omega^{1/2}$. Moreover, for all $\omega \in \mathfrak{B}_*^+$, $\theta \in \mathbf{X}^\dagger$, and $x \in \mathbf{X}$ we have

$$\begin{aligned}|\langle\theta_\omega, x + N_\omega\rangle_\omega| &\le \|\theta_\omega\|_\omega\|x + N_\omega\|_\omega\\ &\le \|\omega\|^{1/2}\|\theta\|\,\|\omega\|^{1/2}\|x\| = \|\omega\|\,\|\theta\|\,\|x\|.\end{aligned}$$

We recall [Pa, Proposition 3.8] that $\mathbf{X}^\dagger$ is a dual space with the weak$*$-topology given by the linear functionals $\theta \mapsto \omega(\langle\tau, \theta\rangle)$ for $\omega \in \mathfrak{B}_*$, $\tau \in \mathbf{X}^\dagger$.

Proposition 3.3 *Let $\mathfrak{B}$ be a von Neumann algebra and let $\mathbf{X}$ be a pre-Hilbert $\mathfrak{B}$-module.*

(i) *The unit ball of $\mathbf{X}^\dagger$ is complete in the topology given by the family of seminorms, $\{\|\cdot\|_\omega \mid \omega \in \mathfrak{B}_*^+\}$;*
(ii) *$\mathbf{X}$ is dense in $\mathbf{X}^\dagger$ in this topology; and hence*
(iii) *$\mathbf{X}$ is weak$*$ dense in $\mathbf{X}^\dagger$.*
(iv) *For each $\omega \in \mathfrak{B}_*^+$, $\theta \in \mathbf{X}^\dagger$, and $\epsilon > 0$ there exists an $x \in \mathbf{X}$ with*

$$\|\theta - x\|_\omega^2 = \omega(\langle\theta - x, \theta - x\rangle) < \epsilon^2.$$

Proof (i) Let $\{\theta^\alpha\}$ be a Cauchy net in the unit ball of $\mathbf{X}^\dagger$. Then for a fixed $\omega \in \mathfrak{B}_*^+$, the net $\{(\theta^\alpha)_\omega\}$ is a Cauchy net in the norm $\|\cdot\|_\omega$ on $\mathcal{H}_\omega$ by definition. Hence, there exists an element $\theta_\omega \in \mathcal{H}_\omega$ with $\|(\theta^\alpha)_\omega - \theta_\omega\| \to 0$. Moreover,

$$\|\theta_\omega\| \le \limsup_\alpha \|(\theta^\alpha)_\omega\| \le \|\omega\|^{1/2}\|\theta^\alpha\| \le \|\omega\|^{1/2}.$$

Now for fixed $x \in \mathbf{X}$, $\{\theta^\alpha(x)\}$ is a bounded net in $\mathfrak{B}$. Moreover, for each $\omega \in \mathfrak{B}_*^+$

$$\lim_\alpha \omega\big(\theta^\alpha(x)\big) = \lim_\alpha \langle(\theta^\alpha)_\omega, x + N_\omega\rangle_\omega = \langle\theta_\omega, x + N_\omega\rangle_\omega.$$

Thus for every $\omega \in \mathfrak{B}_*$, the net $\{\omega(\theta^\alpha(x))\}$ converges in $\mathbf{C}$. Clearly, this limit is linear in ω, that is, the bounded net $\{\theta^\alpha(x)\}$ of linear functionals on $\mathfrak{B}_*$ converges pointwise to a linear functional on $\mathfrak{B}_*$ which is therefore bounded by the same bound, $\|x\|$. That is, the pair $(x, \{\theta_\omega \mid \omega \in \mathfrak{B}_*^+\})$ defines an element in $(\mathfrak{B}_*)^* = \mathfrak{B}$ via

$$\omega \mapsto \langle\theta_\omega, x + N_\omega\rangle_\omega.$$

If we call this element $\theta(x)$, then by definition

$$\omega\big(\theta(x)\big) = \langle\theta_\omega, x + N_\omega\rangle_\omega = \lim_\alpha \omega\big(\theta^\alpha(x)\big) \quad \text{and} \quad \|\theta(x)\| \le \|x\|.$$

By this formula, $\theta(x)$ is clearly linear in x, and so $\theta\colon \mathbf{X} \to \mathfrak{B}$ is linear. By construction, $\theta^\alpha(x)$ converges ultraweakly to $\theta(x)$, and since each θ^α is a $\mathfrak{B}$-module map, so is θ. Clearly, $\|\theta\| \le 1$, so θ is in the unit ball of $\mathbf{X}^\dagger$, and θ^α converges to θ. That is, the unit ball of $\mathbf{X}^\dagger$ is complete as claimed.

(ii) To see that $\mathbf{X}$ is dense in $\mathbf{X}^\dagger$, fix $\theta \in \mathbf{X}^\dagger$ and $\epsilon > 0$. Let $\{\omega_1, \omega_2, \dots, \omega_m\}$ be a finite set of functionals in $\mathfrak{B}_*^+$. Given this data, we let $\omega = \omega_1 + \cdots + \omega_m$. Now $\omega \ge \omega_i$ for each $i = 1, 2, \dots, m$ and so by [Pa, Proposition 3.1], the map $x + N_\omega \mapsto x + N_{\omega_i}$ is a well-defined contraction which extends to a contraction $\mathcal{H}_\omega \to \mathcal{H}_{\omega_i}$ carrying θ_ω to θ_{ω_i}. We choose $x \in \mathbf{X}$ so that $\|(x + N_\omega) - \theta_\omega\|_\omega < \epsilon$. Then for each $i = 1, 2, \dots, m$, we have $\|x - \theta\|_{\omega_i} := \|(x + N_{\omega_i}) - \theta_{\omega_i}\|_{\omega_i} \le \|(x + N_\omega) - \theta_\omega\|_\omega < \epsilon$.

(iii) Now fix $\theta \in \mathbf{X}^\dagger$ and let $\epsilon > 0$, $\{\tau_1, \dots, \tau_n\} \subseteq \mathbf{X}^\dagger$, $\{\omega_1, \dots, \omega_m\} \subseteq \mathfrak{B}_*$ define a basic weak$*$-neighbourhood of θ. Since every element of $\mathfrak{B}_*$ is expressible as a linear combination of four elements in $\mathfrak{B}_*^+$, we can assume that $\omega_1, \dots, \omega_m$ are positive. Let $\omega = \omega_1 + \cdots + \omega_m$ and choose $x \in \mathbf{X}$ with

$$\|(x + N_\omega) - \theta_\omega\|_\omega < \frac{\epsilon}{\|\tau_1\| + \cdots + \|\tau_n\|}.$$

Then for each $i = 1, \dots, m$ and $k = 1, \dots, n$, we have

$$\begin{aligned}|\omega_i\langle\tau_k, x - \theta\rangle| &= |\langle\tau_k, x - \theta\rangle_{\omega_i}| \le \|\tau_k\|_{\omega_i}\|x - \theta\|_{\omega_i}\\ &\le \|\tau_k\|_\omega\|x - \theta\|_\omega \le \|\tau_k\|\,\|(x + N_\omega) - \theta_\omega\|_\omega < \epsilon.\end{aligned}$$

(iv) This is just a restatement of the fact that $\mathbf{X}/N_\omega$ is dense in its Hilbert space completion $\mathcal{H}_\omega$ as described in the remarks after Definition 3.2. ■

Remark In the following class of examples we can more or less explicitly calculate $X^\dagger$.

Example 3.4 *Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\xi_n\}$, let $\mathfrak{B}$ be a von Neumann algebra, and let X be the algebraic tensor product $X = \mathcal{H} \otimes \mathfrak{B}$ with the obvious $\mathfrak{B}$-valued inner product. Then X is a pre-Hilbert $\mathfrak{B}$-module and we can identify $X^\dagger$ as*

$$X^\dagger = \Big\{ \sum_n \xi_n \otimes b_n \mid b_n \in \mathfrak{B} \text{ and } \exists M > 0 \text{ with } \Big\| \sum_{n \in F} b_n^* b_n \Big\| \le M, \ \forall \text{ finite } F \Big\}.$$

Such a formal sum defines a bounded $\mathfrak{B}$-module mapping θ on X as follows:

$$\theta\Big(\sum_{k=1}^N \eta_k \otimes a_k \Big) = \sum_{k=1}^N \sum_n \langle \xi_n, \eta_k \rangle b_n^* a_k,$$

where the right-hand side converges in norm.

Proof First let θ denote an arbitrary element in $X^\dagger$. Define $b_n^* := \theta(\xi_n \otimes 1)$. Since θ is also defined on the norm closure of X, we see that θ is defined on each element of the form $\sum_n \xi_n \otimes a_n$ where $\sum_n a_n^* a_n$ converges in norm in $\mathfrak{B}$. In particular, if $\eta \in \mathcal{H}$ so that $\eta = \sum_n \langle \xi_n, \eta \rangle \xi_n$ converges in norm, then $\eta \otimes a = \sum_n \xi_n \otimes \langle \xi_n, \eta \rangle a$ converges in norm, and so

$$\begin{aligned} \theta(\eta \otimes a) &= \sum_n \theta(\xi_n \otimes \langle \xi_n, \eta \rangle a) = \sum_n \langle \xi_n, \eta \rangle \theta(\xi_n \otimes 1) a \\ &= \sum_n \langle \xi_n, \eta \rangle b_n^* a = \sum_n \langle \xi_n, \eta \rangle b_n^* a. \end{aligned}$$

Hence for any element $x = \sum_{k=1}^N \eta_k \otimes a_k \in X$ we have that $x = \sum_{k=1}^N \sum_n \xi_n \otimes \langle \xi_n, \eta_k \rangle a_k$ converges in norm and

$$\theta\Big(\sum_{k=1}^N \eta_k \otimes a_k \Big) = \sum_{k=1}^N \theta(\eta_k \otimes a_k) = \sum_{k=1}^N \sum_n \langle \xi_n, \eta_k \rangle b_n^* a_k,$$

as claimed. To see that the b_n's satisfy the boundedness condition, let F be any finite set of indices. Then

$$\begin{aligned} \Big\| \sum_{n \in F} b_n^* b_n \Big\| &= \Big\| \theta\Big(\sum_{n \in F} \xi_n \otimes b_n \Big) \Big\| \le \|\theta\| \cdot \Big\| \sum_{n \in F} \xi_n \otimes b_n \Big\| \\ &= \|\theta\| \cdot \Big\| \Big\langle \sum_{n \in F} \xi_n \otimes b_n, \sum_{n \in F} \xi_n \otimes b_n \Big\rangle_{\mathfrak{B}} \Big\|^{1/2} = \|\theta\| \cdot \Big\| \sum_{n \in F} b_n^* b_n \Big\|^{1/2}. \end{aligned}$$

That is, $\| \sum_{n \in F} b_n^* b_n \|^{1/2} \le \|\theta\|$ for all finite F, so we can choose $M = \|\theta\|^2$.

On the other hand, if we have such a formal sum $\sum_n \xi_n \otimes b_n$, then we will show that the finite partial sums $\sum_{n \in F} \xi_n \otimes b_n$ form a Cauchy net (in the family of seminorms of Proposition 3.3) in the ball of radius $\sqrt{M}$ in X, and invoke the previous proposition to conclude that they converge pointwise ultraweakly to an element in $X^\dagger$ of norm at most $\sqrt{M}$.

To this end let $\omega \in \mathfrak{B}_*^+$ and let $\epsilon > 0$. Since the finite sums $\{\sum_{n \in F} b_n^* b_n\}_F$ form a bounded increasing net of positive operators in $\mathfrak{B}$, they converge strongly to an element of $\mathfrak{B}$. Hence the net $\{\sum_{n \in F} \omega(b_n^* b_n)\}_F$ converges to a finite nonnegative number. Thus, there exists a large finite set F_0 so that if $F_0 \cap F = \phi$, then $\sum_F \omega(b_n^* b_n) < \epsilon/2$.

Thus if $F_0 \subset F_1$ and $F_0 \subset F_2$, we have

$$\begin{aligned}
&\Big\| \sum_{F_1} \xi_n \otimes b_n - \sum_{F_2} \xi_n \otimes b_n \Big\|_\omega^2 \\
&\quad = \Big\| \sum_{F_1 \sim F_2} \xi_n \otimes b_n - \sum_{F_2 \sim F_1} \xi_n \otimes b_n \Big\|_\omega^2 \\
&\quad = \omega\Big(\Big\langle \Big(\sum_{F_1 \sim F_2} \xi_n \otimes b_n - \sum_{F_2 \sim F_1} \xi_n \otimes b_n \Big), \Big(\sum_{F_1 \sim F_2} \xi_n \otimes b_n - \sum_{F_2 \sim F_1} \xi_n \otimes b_n \Big) \Big\rangle_{\mathfrak{B}} \Big) \\
&\quad = \omega\Big(\sum_{F_1 \sim F_2} b_n^* b_n \Big) + \omega\Big(\sum_{F_2 \sim F_1} b_n^* b_n \Big) < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Hence, the finite sums $\sum_F \xi_n \otimes b_n$ converge to an element $\theta \in X^\dagger$,that is, for each $x \in X$, $\theta(x) = \text{uw-lim}_F \langle \sum_F \xi_n \otimes b_n, x \rangle$. Now for $x = \sum_{k=1}^N \eta_k \otimes a_k \in X$ we have by the first part of the proof that $x = \sum_{k=1}^N \sum_n \xi_n \otimes \langle \xi_n, \eta_k \rangle a_k$ converges in norm. Since θ is bounded, $\theta(x) = \sum_{k=1}^N \sum_n \langle \xi_n, \eta_k \rangle \theta(\xi_n \otimes a_k)$ also converges in norm. But then

$$\theta(\xi_n \otimes a_k) = \underset{F}{\text{uw-lim}} \Big\langle \sum_{m \in F} \xi_m \otimes b_m, \xi_n \otimes a_k \Big\rangle_{\mathfrak{B}} = b_n^* a_k.$$

And so, indeed, $\theta(\sum_{k=1}^N \eta_k \otimes a_k) = \sum_{k=1}^N \sum_n \langle \xi_n, \eta_k \rangle b_n^* a_k$ converges in norm. ■

Key Idea 4 In the definition below of a $\mathfrak{Z}$-Hilbert algebra $\mathcal{A}$, a key idea is the use of the topology given by the seminorms in Proposition 3.3 to replace the norm topology on $\mathcal{H}_\mathcal{A} := \mathcal{A}^\dagger$ when $\mathfrak{Z}$ is not **C**.

Hence, axiom (h) below seems to us the most natural replacement for the usual axiom of the norm-density of $\mathcal{A}^2$ in $\mathcal{A}$. When we come to apply this axiom to the crossed product examples that we construct, we are actually able to show that a stronger condition holds. However, in order to prove that the algebra of bounded elements $\mathcal{A}_b$ also satisfies axiom (h), we need the weaker version below. Moreover, in the converse construction of a $\mathfrak{Z}$-Hilbert algebra from a given $\mathfrak{Z}$-trace one also needs the weaker version of axiom (h) below.

Definition 3.5 Let $\mathfrak{Z}$ be an abelian von Neumann algebra. A complex $*$-algebra $\mathcal{A}$ is called a $\mathfrak{Z}$-*Hilbert algebra* if $\mathcal{A}$ is a right pre-Hilbert $\mathfrak{Z}$-module which satisfies the further four axioms.

(e) $\langle a^*, b^* \rangle = \langle b, a \rangle$ for $a, b \in \mathcal{A}$.
(f) $\langle ab, c \rangle = \langle b, a^* c \rangle$ for $a, b, c \in \mathcal{A}$.
(g) $b \mapsto ab : \mathcal{A} \to \mathcal{A}$ is bounded in the $\mathfrak{Z}$-module norm for each fixed $a \in \mathcal{A}$.
(h) The space $\mathcal{A}^2 = \text{span}\{ab \mid a, b \in \mathcal{A}\}$ is dense in $\mathcal{A}$ in the topology given by the family of seminorms $\{\|\cdot\|_\omega \mid \omega \in \mathfrak{Z}_*^+\}$ defined above.

Remark It is easy to see that if $\mathcal{A}^2$ is norm-dense in $\mathcal{A}$ in the $\mathfrak{Z}$-module norm, $\|a\|^2 = \|\langle a, a \rangle\|$, then axiom (h) is satisfied.

Example 3.6 *Let $\mathfrak{A}$ be a von Neumann algebra and let $\mathfrak{Z}$ be a von Neumann subalgebra of the centre of $\mathfrak{A}$. Suppose $\tau : \mathfrak{A} \to \mathfrak{Z}$ is a faithful, unital, uw-continuous $\mathfrak{Z}$-trace. Then for $a, b \in \mathfrak{A}$ the following inner product makes $\mathfrak{A}$ into a $\mathfrak{Z}$-Hilbert algebra:*

$$\langle a, b \rangle_{\mathfrak{Z}} := \tau(a^* b).$$

Proof The only axioms that are not completely trivial are (c) and (g). Axiom (c) follows from the $\mathfrak{Z}$-linearity of τ, while axiom (g) follows from the calculation

$$\begin{aligned}\|ab\|_{\mathfrak{A}}^2 &= \|\langle ab, ab\rangle_{\mathfrak{Z}}\|_{\mathfrak{Z}} = \|\tau(b^*a^*ab)\|_{\mathfrak{Z}}\\ &\le \|\tau(\|a^*a\|_{op}b^*b)\|_{\mathfrak{Z}} = \|a\|_{op}^2\|b\|_{\mathfrak{A}}^2.\end{aligned}$$

Since τ is unital, it is easy to see that $\|1\|_{\mathfrak{A}} = 1$ and so $\|a\|_{\mathfrak{A}} \le \|a\|_{op}$ for all $a \in \mathfrak{A}$. ∎

Of course, even if $\mathfrak{Z} = \mathbf{C}$, one usually has strict containment $\mathfrak{A} \subset \mathfrak{A}^\dagger := \mathcal{H}_{\mathfrak{A}}$.

Remarks We denote by $\pi(a)$ the operator "left multiplication by a" and note that by axioms (f) and (g), $\pi(a)$ is adjointable with adjoint $\pi(a^*)$ and hence $\pi(a)$ is a $\mathfrak{Z}$-module mapping. That is, $a(bz) = (ab)z\ for\ a, b \in \mathcal{A}\ , z \in \mathfrak{Z}$.

We denote by $\pi'(a)$ the operator "right multiplication by a" and note that by axioms (e), (f), and (g), $\pi'(a)$ is also bounded and adjointable with adjoint $\pi'(a^*)$ and therefore is also a $\mathfrak{Z}$-module mapping. That is, $(bz)a = (ba)z$ for $a, b \in \mathcal{A}$, $z \in \mathfrak{Z}$. A little playing with the axioms and using the fact that $\mathfrak{Z}$ is abelian yields the further useful identity, $(az)^* = a^*z^*$ for $a \in \mathcal{A}$, $z \in \mathfrak{Z}$.

Whenever $\mathcal{A}$ is a $\mathfrak{Z}$-Hilbert algebra, we will use the suggestive notation $\mathcal{H}_{\mathcal{A}}$ in place of $\mathcal{A}^\dagger$ for the Paschke dual of $\mathcal{A}$. That is,

$$\mathcal{H}_{\mathcal{A}} = \mathcal{A}^\dagger = \{\theta : \mathcal{A} \to \mathfrak{Z} \mid \theta \text{ is a bounded } \mathfrak{Z}\text{-module map}\}.$$

By [Pa, Theorem 3.2], $\mathcal{H}_{\mathcal{A}}$ is a self-dual Hilbert $\mathfrak{Z}$-module. For $\xi \in \mathcal{H}_{\mathcal{A}}$ and $a \in \mathcal{A}$ we have $\xi(a) = \langle \xi, \widehat{a}\rangle$ where $\widehat{a} \in \mathcal{H}_{\mathcal{A}}$ is given by $\widehat{a}(b) = \langle a, b\rangle$ for $b \in \mathcal{A}$. We identify a with $\widehat{a} \in \mathcal{H}_{\mathcal{A}}$ so that $\mathcal{A} \subseteq \mathcal{H}_{\mathcal{A}}$ and so, of course, $\mathcal{A}^- \subseteq \mathcal{H}_{\mathcal{A}}$. By [Pa, Corollary 3.7] each $\pi(a)$ (respectively, $\pi'(a)$) extends uniquely to an element of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ which we will also denote by $\pi(a)$ (respectively, $\pi'(a)$) and moreover, the map

$$\mathcal{A} \stackrel{\pi}{\to} \mathcal{L}(\mathcal{H}_{\mathcal{A}})$$

is a $*$-monomorphism. Similarly, the map

$$\mathcal{A} \stackrel{\pi'}{\to} \mathcal{L}(\mathcal{H}_{\mathcal{A}})$$

is a $*$-anti-monomorphism.

We note that with this notation, axiom (h) ensures that $\mathcal{A}^2$ is weak$*$-dense in $\mathcal{H}_{\mathcal{A}}$ by Proposition 3.3 (iii).

Proposition 3.7 *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra where $\mathfrak{Z}$ is an abelian von Neumann algebra. For $z \in \mathfrak{Z}$ and $\xi \in \mathcal{H}_{\mathcal{A}}$ the mapping $\xi \mapsto z\cdot\xi := \xi z$ embeds $\mathfrak{Z}$ into $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. With this embedding we have $\mathfrak{Z} = Z(\mathcal{L}(\mathcal{H}_{\mathcal{A}}))$, the centre of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. Moreover, $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a Type I von Neumann algebra.*

Proof It is easy to check that this mapping embeds $\mathfrak{Z}$ into $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ and since each $T \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is $\mathfrak{Z}$-linear, we have that $\mathfrak{Z} \hookrightarrow Z(\mathcal{L}(\mathcal{H}_{\mathcal{A}}))$. Now by [R, Corollary 7.10], $\mathfrak{Z}$ and $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ are Morita equivalent in the sense of [R] and so by [R, Theorem 8.11], $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a Type I von Neumann algebra.

Now by the construction of [R, Corollary 7.10], $\mathcal{H}_{\mathcal{A}}$ becomes a left Hilbert $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$-module with the inner product $\langle \xi, \eta\rangle_{\mathcal{L}(\mathcal{H}_{\mathcal{A}})}(\mu) = \xi\langle \eta, \mu\rangle_{\mathfrak{Z}}$ for $\xi, \eta, \mu \in \mathcal{H}_{\mathcal{A}}$. That is,

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$\langle \xi, \eta \rangle_{\mathcal{L}(\mathcal{H}_{\mathcal{A}})}$ is the "finite-rank" operator $\xi \otimes \overline{\eta}$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. Then for $T \in Z\big(\mathcal{L}(\mathcal{H}_{\mathcal{A}})\big)$,

$$\begin{aligned}\langle T\xi, \eta \rangle_{\mathcal{L}(\mathcal{H}_{\mathcal{A}})} &= (T\xi) \otimes \overline{\eta} = T(\xi \otimes \overline{\eta}) \\ &= (\xi \otimes \overline{\eta})T = \xi \otimes \overline{T^*\eta} = \langle \xi, T^*\eta \rangle_{\mathcal{L}(\mathcal{H}_{\mathcal{A}})}.\end{aligned}$$

Thus, such a T is adjointable and clearly $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$-linear. By [R, Corollary 7.10], T must be of the form $T\xi = \xi z = z \cdot \xi$ for some $z \in \mathfrak{Z}$, that is, $\mathfrak{Z} = Z(\mathcal{L}(\mathcal{H}_{\mathcal{A}}))$. ∎

Key Idea 5 The fact that $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a Type I von Neumann algebra with centre $\mathfrak{Z}$ is one key idea which makes the theory of $\mathfrak{Z}$-Hilbert algebras possible. That is, if $\mathfrak{R}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ which contains $\mathfrak{Z}$, then $\mathfrak{R}$ is uw-closed if and only if $\mathfrak{R} = \mathfrak{R}''$ where $'$ denotes commutant within $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. This follows from [Dix, complément 13, III.7] and allows us to use commutation (pure algebra) to determine inclusion or equality of certain algebras.

4 Commutation Theorem for $\mathfrak{Z}$-Hilbert Algebras

Throughout this section $\mathfrak{Z}$ is an abelian von Neumann algebra and $\mathcal{A}$ is a $\mathfrak{Z}$-Hilbert algebra with Paschke dual $\mathcal{H}_{\mathcal{A}}$. Given the machinery we have developed for $\mathfrak{Z}$-Hilbert algebras, the proof of the commutation theorem below follows the outline of the classical case quite closely.

Lemma 4.1 *If T is a nonzero operator in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, then there exists $a \in \mathcal{A}$ with $T\pi(a) \neq 0$.*

Proof If $T(\mathcal{A}^2) = \{0\}$, then for all $\xi \in \mathcal{H}_{\mathcal{A}}$, $\langle T^*\xi, ab \rangle = \langle \xi, T(ab) \rangle = 0$. Hence, for each positive $\omega \in \mathfrak{Z}_*$ we have $0 = \omega(\langle ab, T^*\xi \rangle) = \langle ab, T^*\xi \rangle_\omega$. Then by Definition 3.5 (h) and Proposition 3.3 (b) we must have $T^*\xi = 0$ for all $\xi \in \mathcal{H}_{\mathcal{A}}$, that is, $T^* = 0$ and hence $T = 0$. Therefore, there exists $a, b \in \mathcal{A}$ with $0 \neq T(ab) = T(\pi(a)b) = (T\pi(a))(b)$. So $T\pi(a) \neq 0$. ∎

Since $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a von Neumann algebra, it has a natural ultraweak (uw) topology. This is the topology we refer to in the following lemma.

Lemma 4.2 *With the standing assumptions of this section, we have*

(i) $(\pi(\mathcal{A}))^{-\mathrm{uw}} = (\pi(\mathcal{A}))''$,
(ii) $\mathfrak{Z} \subseteq (\pi(\mathcal{A}))^{-\mathrm{uw}}$.

Proof Since $\mathfrak{Z}$ is the centre of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, by Proposition 3.7 we see that

$$(\pi(\mathcal{A}))' = [\mathrm{alg}\{\pi(\mathcal{A}), \mathfrak{Z}\}]'.$$

Moreover, since $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is Type I with centre $\mathfrak{Z}$ and $\mathfrak{Z} \subseteq \mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})$, we have by [Dix, complément 13, III.7] that $[\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]'' = [\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}$. Hence,

$$(\pi(\mathcal{A}))'' = [\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]'' = [\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}. \tag{4.1}$$

Now $\pi(\mathcal{A})$ is a $*$-ideal in the $*$-algebra $\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})$ so that $(\pi(\mathcal{A}))^{-\mathrm{uw}}$ is a $*$-ideal in $[\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}$ so that there exists a central projection E in

$$[\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}$$

with $(\pi(\mathcal{A}))^{-uw} = E[\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}$. If $E \neq 1$, then $1 - E \neq 0$ but $(1 - E)\pi(\mathcal{A}) = \{0\}$, contradicting the previous lemma. Hence,

$$(4.2) \qquad (\pi(\mathcal{A}))^{-uw} = [\mathrm{alg}(\pi(\mathcal{A}), \mathfrak{Z})]^{-\mathrm{uw}}.$$

Equations (4.1) and (4.2) imply part (i). Part (ii) follows since $\mathfrak{Z}$ is contained in any commutant. ■

Lemma 4.3 *The map $*$ extends to a conjugate-linear isometry of $\mathcal{H}_{\mathcal{A}}$ (also denoted by $*$) by defining $\xi^*(a) := (\xi(a^*))^*$ for $\xi \in \mathcal{H}_{\mathcal{A}}$ and $a \in \mathcal{A}$. This extension satisfies*

$$\langle \xi, \eta \rangle^* = \langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle,$$

for all $\xi, \eta \in \mathcal{H}_{\mathcal{A}}$.

Proof It is easy to see that ξ^* is a bounded $\mathfrak{Z}$-module map and that $\|\xi^*\| \leq \|\xi\|$. Since $\xi^{**} = \xi$, we see that $*$ is isometric on $\mathcal{H}_{\mathcal{A}}$. By axioms (b) and (e) we have for $a, b \in \mathcal{A}$,

$$(\widehat{b})^*(a) = (\widehat{b}(a^*))^* = \langle b, a^* \rangle^* = \langle a^*, b \rangle = \langle b^*, a \rangle = \widehat{b^*}(a),$$

so that this $*$ really is an extension from $\mathcal{A}$ to $\mathcal{H}_{\mathcal{A}}$. Moreover, using the definition of module multiplication given in Definition 3.2, it is easy to check that $(\xi z)^* = \xi^* z^*$ for all $z \in \mathfrak{Z}$ and $\xi \in \mathcal{H}_{\mathcal{A}}$.

We observe that $\mathfrak{Z}$ is a self-dual Hilbert $\mathfrak{Z}$-module with the inner product $\langle z_1, z_2 \rangle = z_1^* z_2$. For if $\theta: \mathfrak{Z} \to \mathfrak{Z}$ is a bounded $\mathfrak{Z}$-module map, then $\theta(z) = \theta(1)z = \langle \theta(1)^*, z \rangle$.

Now if $\xi \in \mathcal{H}_{\mathcal{A}}$, then by [Pa, Proposition 3.6], ξ extends uniquely to a bounded $\mathfrak{Z}$-module mapping: $\mathcal{H}_{\mathcal{A}} \to \mathfrak{Z}$. But using the first paragraph of the proof one checks that $\eta \mapsto \langle \xi, \eta \rangle$ and $\eta \mapsto \langle \xi^*, \eta^* \rangle^*$ are two such extensions. Hence, $\langle \xi, \eta \rangle = \langle \xi^*, \eta^* \rangle^*$ as claimed.

The equality $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ follows from axiom (b) since $\mathcal{H}_{\mathcal{A}}$ is a (self-dual) Hilbert $\mathfrak{Z}$-module by [Pa, Theorem 3.2]. ■

Definition 4.4 The isometry $\eta \mapsto \eta^*: \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}}$ of the previous lemma will be denoted by J, that is, $J(\eta) = \eta^*$ for all $\eta \in \mathcal{H}$.

Remarks The unique extension of [Pa, Proposition 3.6] used in the previous proof will be used several more times in this paper under the name "unique extension property."

Lemma 4.5 *With the standing assumptions of this section, we have the following.*

(i) $\mathfrak{Z} \subseteq (\pi'(\mathcal{A}))^{-\mathrm{uw}} = (\pi'(\mathcal{A}))''$.
(ii) $\pi(\mathcal{A}) \subseteq (\pi'(\mathcal{A}))'$.
(iii) $\pi'(\mathcal{A}) \subseteq (\pi(\mathcal{A}))'$.

Proof (i) This is the same proof as Lemma 4.2.

(ii) and (iii) By the unique extension property, it suffices to see that $\pi'(a)\pi(b) = \pi(b)\pi'(a)$ on the space $\mathcal{A} \subseteq \mathcal{H}_{\mathcal{A}}$. This is trivial to check. ■

4.1 Bounded Elements in $\mathcal{H}_{\mathcal{A}}$

Let $\xi \in \mathcal{H}_{\mathcal{A}}$ and suppose that the map $a \mapsto \pi'(a)\xi: \mathcal{A} \to \mathcal{H}_{\mathcal{A}}$ is bounded. We note that by the remarks following Example 3.6, $\pi(az) = \pi(a)z = z\pi(a)$ and $\pi'(az) = \pi'(a)z = z\pi'(a)$, for all $a \in \mathcal{A}$ and $z \in \mathfrak{Z}$. Therefore,

$$(az) \mapsto \pi'(az)\xi = z\pi'(a)\xi = (\pi'(a)\xi)z,$$

so that this bounded map is also $\mathfrak{Z}$-linear. Hence, by the unique extension property this map extends uniquely to a bounded module mapping $\mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}}$ which we denote by $\pi(\xi)$, that is, $\pi(\xi)a = \pi'(a)\xi$ for all $a \in \mathcal{A}$. By [Pa, Proposition 3.4] $\pi(\xi)$ is adjointable and $\pi(\xi) \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})$. Such an element $\xi \in \mathcal{H}_{\mathcal{A}}$ is called "left-bounded" and the set of all such elements is denoted $\mathcal{A}_l$. Clearly, $\mathcal{A} \subseteq \mathcal{A}_l$.

Similarly, we let $\mathcal{A}_r = \{\eta \in \mathcal{H}_{\mathcal{A}} \,|\, \pi'(\eta) \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})\}$ where, of course, $\pi'(\eta)a = \pi(a)\eta$ for all $a \in \mathcal{A}$.

Proposition 4.6 *With the standing assumptions of this section:*

(i) $\pi(\mathcal{A}_l) \subseteq (\pi'(\mathcal{A}))'$ *and similarly* $\pi'(\mathcal{A}_r) \subseteq (\pi(\mathcal{A}))'$,

(ii) $\pi(\mathcal{A}_l)$ *is a left ideal in* $(\pi'(\mathcal{A}))'$ *and* $T\pi(\xi) = \pi(T\xi)$ *for* $\xi \in \mathcal{A}_l$ *and* $T \in (\pi'(\mathcal{A}))'$. *In particular,* $\pi(\eta)\pi(\xi) = \pi(\pi(\eta)\xi)$ *for* $\eta, \xi \in \mathcal{A}_l$. *Similarly,* $\pi'(\mathcal{A}_r)$ *is a left ideal in* $(\pi(\mathcal{A}))'$, *etc.*

(iii) $\mathcal{A}_l$ *is an associative algebra with the multiplication* $\xi\eta = \pi(\xi)\eta$ *and* $\pi: \mathcal{A}_l \to \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ *is a monomorphism. Similarly,* $\mathcal{A}_r$ *is an associative algebra with the multiplication* $\xi\eta = \pi'(\eta)\xi$, *and* π' *is an anti-monomorphism.*

(iv) $\mathcal{A}_l$ *is invariant under* $*$ *and* $\pi(\xi^*) = \pi(\xi)^*$ *so that* $\pi(\mathcal{A}_l)$ *is a* $*$*-ideal in* $(\pi'(\mathcal{A}))'$ *and* π *is a* $*$*-monomorphism. A similar statement holds for* $\mathcal{A}_r$.

Proof (i) By the unique extension property, it suffices to check that if $\xi \in \mathcal{A}_l$ and $b \in \mathcal{A}$, then $\pi(\xi)\pi'(b) = \pi'(b)\pi(\xi)$ on the space $\mathcal{A}$. To this end let $a \in \mathcal{A}$. Then

$$(\pi(\xi)\pi'(b))(a) = \pi(\xi)(ab) = \pi'(ab)(\xi) = \pi'(b)\pi'(a)(\xi) = \pi'(b)\pi(\xi)(a),$$

as required.

(ii) If $\xi \in \mathcal{A}_l$, $T \in (\pi'(\mathcal{A}))'$, and $a \in \mathcal{A}$, then $\pi(T\xi)a = \pi'(a)T\xi = T\pi'(a)\xi = T\pi(\xi)a$, that is, $T\xi \in \mathcal{A}_l$ and $\pi(T\xi) = T\pi(\xi)$ by the unique extension property.

(iii) By (ii), $\xi\eta := \pi(\xi)\eta$ is in $\mathcal{A}_l$ if $\xi, \eta \in \mathcal{A}_l$. Moreover, by (2) $\pi(\xi\eta) = \pi(\xi)\pi(\eta)$. Since $\pi: \mathcal{A}_l \to \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is clearly linear, it suffices to see that π is also one-to-one. But if $\pi(\xi) = 0$, then for all $a, b \in \mathcal{A}$ we have $0 = \langle \pi(\xi)a, b\rangle_\omega = \langle \pi'(a)\xi, b\rangle_\omega = \langle \xi, ba^*\rangle_\omega$ for all positive $\omega \in \mathfrak{Z}_*$, that is, $\xi = 0$ by axiom (h) and Proposition 3.3.

(iv) Let $\xi \in \mathcal{A}_l$ and let $a, b \in \mathcal{A}$. Using Lemma 4.3 and the fact that $\mathcal{H}_{\mathcal{A}}$ is a Hilbert $\mathfrak{Z}$-module, we get the following calculation:

$$\begin{aligned}\langle \pi(\xi)^* a, b\rangle &= \langle b, \pi(\xi)^* a\rangle^* = \langle \pi(\xi)b, a\rangle^* = \langle \pi'(b)\xi, a\rangle^* \\ &= \langle \xi, ab^*\rangle^* = \langle \xi^*, ba^*\rangle = \langle \xi^*, \pi'(a^*)b\rangle = \langle \pi'(a)\xi^*, b\rangle \\ &= \langle \pi(\xi^*)a, b\rangle.\end{aligned}$$

Thus, as module maps $\pi(\xi)^* a$ and $\pi(\xi^*)a$ agree for all $b \in \mathcal{A}$ and so $\pi(\xi)^* a = \pi(\xi^*)a$ for all $a \in \mathcal{A}$, that is, ξ^* is left-bounded and $\pi(\xi^*) = \pi(\xi)^*$. Moreover, for $\xi, \eta \in \mathcal{A}_l$

$$\pi\big((\xi\eta)^*\big) = [\pi(\xi\eta)]^* = [\pi(\xi)\pi(\eta)]^* = \pi(\eta)^*\pi(\xi)^* = \pi(\eta^*)\pi(\xi^*) = \pi(\eta^*\xi^*)$$

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and so $(\xi\eta)^* = \eta^*\xi^*$ as π is one-to-one. ∎

Corollary 4.7 *With the standing assumptions of this section, we have*

(i) $(\pi(\mathcal{A}_l))'' = \pi(\mathcal{A}_l)^{-\mathrm{uw}} = (\pi'(\mathcal{A}))'$,
(ii) $(\pi'(\mathcal{A}_r))'' = \pi'(\mathcal{A}_r)^{-\mathrm{uw}} = (\pi(\mathcal{A}))'$.

Proof (i) By Proposition 4.6, $\pi(\mathcal{A}_l)^{-\mathrm{uw}}$ is a $*$-ideal in $(\pi'(\mathcal{A}))'$. But by Lemma 4.2, $1 \in \mathfrak{Z} \subseteq \pi(\mathcal{A})^{-uw} \subseteq \pi(\mathcal{A}_l)^{-\mathrm{uw}}$ and so $\pi(\mathcal{A}_l)^{-\mathrm{uw}} = (\pi'(\mathcal{A}))'$. Now since $\mathfrak{Z} \subseteq (\pi(\mathcal{A}_l))^{-\mathrm{uw}}$, we have [Dix, complément 13, III.7] that $(\pi(\mathcal{A}_l)^{-\mathrm{uw}})'' = \pi(\mathcal{A}_l)^{-\mathrm{uw}}$. But since commutants are always ultraweakly closed,

$$\big(\pi(\mathcal{A}_l)\big)'' = \big(\pi(\mathcal{A}_l)''\big)^{-\mathrm{uw}} \supseteq \big(\pi(\mathcal{A}_l)\big)^{-\mathrm{uw}} = (\pi(\mathcal{A}_l)^{-\mathrm{uw}})'' \supseteq (\pi(\mathcal{A}_l))''.$$

The proof of (ii) is similar. ∎

Proposition 4.8 *With the standing assumptions of this section, we have $\mathcal{A}_l = \mathcal{A}_r$ and*

(i) $\pi'(\xi)a = [\pi(\xi^*)a^*]^*$ *for* $\xi \in \mathcal{A}_l$, $a \in \mathcal{A}$.
(ii) $\pi(\xi)a = [\pi'(\xi^*)a^*]^*$ *for* $\xi \in \mathcal{A}_r$, $a \in \mathcal{A}$.

Proof (i) Let $\xi \in \mathcal{A}_l$. Then for $a, b \in \mathcal{A}$

$$\begin{aligned}\langle \pi'(\xi)a, b\rangle &= \langle \pi(a)\xi, b\rangle = \langle \xi, a^*b\rangle \\ &= \langle \xi^*, b^*a\rangle^* = \langle \pi'(a^*)\xi^*, b^*\rangle^* = \langle \pi(\xi^*)a^*, b^*\rangle^* \\ &= \langle [\pi(\xi^*)a^*]^*, b\rangle.\end{aligned}$$

Therefore, $\xi \in \mathcal{A}_r$ so that $\mathcal{A}_l \subseteq \mathcal{A}_r$ and (i) holds. Similarly, $\mathcal{A}_r \subseteq \mathcal{A}_l$ and (ii) holds. ∎

Corollary 4.9 *For all $\xi \in \mathcal{A}_l = \mathcal{A}_r$ and $\eta \in \mathcal{H}_{\mathcal{A}}$*

(i) $\pi'(\xi)\eta = [\pi(\xi^*)\eta^*]^*$,
(ii) $\pi(\xi)\eta = [\pi'(\xi^*)\eta^*]^*$.

Proof (i) Recall $J\colon \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}}$ is the conjugate-linear isometry $J\eta = \eta^*$. As noted in the proof of Lemma 4.3, $J(\eta z) = (J\eta)z^*$ for $z \in \mathfrak{Z}$. Now by Proposition 4.8 (i), we see that for $\xi \in \mathcal{A}_l = \mathcal{A}_r$, $\pi'(\xi)$ and $J\pi(\xi^*)J$ agree on $\mathcal{A}$. Since both of these maps are bounded $\mathfrak{Z}$-module maps they agree on $\mathcal{H}_{\mathcal{A}}$ by uniqueness. This proves (i); the proof of (ii) is similar. ∎

Proposition 4.10 *Let $\xi, \eta \in \mathcal{A}_l = \mathcal{A}_r$.*

(i) $\pi(\xi)\eta = \pi'(\eta)\xi$, *so that the two multiplications of Proposition 4.6 agree.*
(ii) $\pi(\xi)\pi'(\eta) = \pi'(\eta)\pi(\xi)$.

Proof (i) Fix $a \in \mathcal{A}$. Then

$$\begin{aligned}\langle \pi(\xi)\eta, a\rangle &= \langle (\pi(\xi)\eta)^*, a^*\rangle^* = \langle \pi'(\xi^*)\eta^*, a^*\rangle^* = \langle \eta^*, \pi'(\xi)a^*\rangle^* = \langle \eta^*, \pi(a^*)\xi\rangle^* \\ &= \langle \pi(a)\eta^*, \xi\rangle^* = \langle \pi'(\eta^*)a, \xi\rangle^* = \langle a, \pi'(\eta)\xi\rangle^* = \langle \pi'(\eta)\xi, a\rangle\end{aligned}$$

so that (i) holds.

(ii) Again fix $a \in \mathcal{A}$. Then

$$\begin{aligned}\pi(\xi)\pi'(\eta)a &= \pi(\xi)\pi(a)\eta = \pi(\pi(\xi)a)\eta \qquad \text{by Proposition 4.6 (ii)}\\ &= \pi'(\eta)(\pi(\xi)a) = \pi'(\eta)\pi(\xi)a.\end{aligned}$$ ∎

Notation Since $\mathcal{A}_l = \mathcal{A}_r$ (even as $*$-algebras), we now use the notation $\mathcal{A}_b$ to denote the $*$-algebra of *bounded* elements in $\mathcal{H}_{\mathcal{A}}$.

Theorem 4.11 (Commutation Theorem) *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra over the abelian von Neumann algebra $\mathfrak{Z}$. Then*

(i) $\pi(\mathcal{A})^{-\mathrm{uw}} = (\pi(\mathcal{A}))'' = (\pi(\mathcal{A}_b))'' = \pi(\mathcal{A}_b)^{-\mathrm{uw}} = (\pi'(\mathcal{A}_b))' = (\pi'(\mathcal{A}))'$,

(ii) $\pi'(\mathcal{A})^{-\mathrm{uw}} = (\pi'(\mathcal{A}))'' = (\pi'(\mathcal{A}_b))'' = \pi'(\mathcal{A}_b)^{-\mathrm{uw}} = (\pi(\mathcal{A}_b))' = (\pi(\mathcal{A}))'$.

Proof (i) By Corollary 4.7 (i) we have

$$\big(\pi(\mathcal{A}_b)\big)^{-\mathrm{uw}} = \big(\pi(\mathcal{A}_b)\big)'' = \big(\pi'(\mathcal{A})\big)' \supseteq \big(\pi'(\mathcal{A}_b)\big)'.$$

However, by Corollary 4.9 (ii) we have $(\pi(\mathcal{A}_b))'' \subseteq (\pi'(\mathcal{A}_b))''' = (\pi'(\mathcal{A}_b))'$. Hence,

$$\big(\pi(\mathcal{A}_b)\big)^{-\mathrm{uw}} = \big(\pi(\mathcal{A})\big)'' = \big(\pi'(\mathcal{A})\big)' = \big(\pi'(\mathcal{A}_b)\big)'.$$

On the other hand, by Corollary 4.7 (ii) $(\pi(\mathcal{A}))'' = (\pi'(\mathcal{A}_b))''' = (\pi'(\mathcal{A}_b))'$. Since $\pi(\mathcal{A})^{-\mathrm{uw}} = (\pi(\mathcal{A}))''$, by Lemma 4.2, we are done.

The proof of (ii) is similar. ∎

Definition 4.12 We define the *left von Neumann algebra* of $\mathcal{A}$ to be

$$\mathcal{U}(\mathcal{A}) := (\pi(\mathcal{A}))''.$$

We define the *right von Neumann algebra* of $\mathcal{A}$ to be $\mathcal{V}(\mathcal{A}) := (\pi'(\mathcal{A}))''$.

Corollary 4.13 *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra over the abelian von Neumann algebra $\mathfrak{Z}$. Then for all $\xi, \eta \in \mathcal{A}_b$, with J as in Definition 4.4*

(i) *$J\pi(\xi)J = \pi'(J\xi)$ and $J\pi'(\xi)J = \pi(J\xi)$,*

(ii) *$J\mathcal{U}(\mathcal{A})J = \mathcal{V}(\mathcal{A})$ and $J\mathcal{V}(\mathcal{A})J = \mathcal{U}(\mathcal{A})$.*

Proof Item (i) is just Corollary 4.7.

To see item (ii), let $T \in \mathcal{U}(\mathcal{A}) = (\pi'(\mathcal{A}_b))'$. Then for $\xi \in \mathcal{A}_b$ and $\eta \in \mathcal{H}_{\mathcal{A}}$ we get

$$\begin{aligned}JTJ\pi(\xi)\eta &= JTJ\pi(\xi)J\eta^* = JT\pi'(J\xi)\eta^* = J\pi'(J\xi)T\eta^*\\ &= J\pi'(J\xi)JJTJ\eta = \pi(\xi)JTJ\eta.\end{aligned}$$

Therefore, $J\mathcal{U}(\mathcal{A})J \subseteq (\pi(\mathcal{A}_b))' = \mathcal{V}(\mathcal{A})$. Similarly, $J\mathcal{V}(\mathcal{A})J \subseteq \mathcal{U}(\mathcal{A})$. Since $J^2 = 1$, we are done. ∎

Remarks At this point we could show that $\mathcal{A}_b$ is a $\mathfrak{Z}$-Hilbert algebra satisfying $\mathcal{H}_{\mathcal{A}_b} = \mathcal{H}_{\mathcal{A}}$, $\mathcal{U}(\mathcal{A}_b) = \mathcal{U}(\mathcal{A})$, and $\mathcal{V}(\mathcal{A}_b) = \mathcal{V}(\mathcal{A})$. Since we do not appear to need this now, we defer the statement and proof to Proposition 6.4.

5 Centre-valued Traces

With the same hypotheses and notation of the previous section we show how to construct a natural $\mathfrak{Z}$-valued trace on the von Neumann algebra, $\mathcal{U}(\mathcal{A})$. We first remind the reader of Paschke's results that both $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ are dual spaces, and that since $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a von Neumann algebra, its weak$*$-topology must also be its uw-topology, since pre-duals for von Neumann algebras are unique.

Key Idea 6 The problem of convergence is one of our main headaches. The topology of Proposition 3.3 (closely related to a topology introduced by Paschke [Pa]) and [Pa, Proposition 3.10] are exactly what is needed to prove the following result, which is used several times in the remainder of this paper.

Proposition 5.1 *If $\mathcal{A}$ is a pre-Hilbert $\mathfrak{Z}$-module (not necessarily a $\mathfrak{Z}$-Hilbert algebra) with Paschke dual $\mathcal{H}_{\mathcal{A}}$, then*

(i) *a bounded net $\{\xi_\alpha\}$ in $\mathcal{H}_{\mathcal{A}}$ converges weak$*$ to $\xi \in \mathcal{H}_{\mathcal{A}}$ if and only if*

$$\langle \eta, \xi_\alpha \rangle \to \langle \eta, \xi \rangle$$

ultraweakly in $\mathfrak{Z}$ for all $\eta \in \mathcal{H}_{\mathcal{A}}$,

(ii) *a net $\{T_\alpha\}$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ converges ultraweakly to $T \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ if and only if*

$$\langle T_\alpha \xi, \eta \rangle \to \langle T\xi, \eta \rangle$$

ultraweakly in $\mathfrak{Z}$ for all $\xi, \eta \in \mathcal{H}_{\mathcal{A}}$,

(iii) *a bounded net $\{T_\alpha\}$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ converges ultraweakly to $T \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ if and only if $\langle T_\alpha a, b \rangle \to \langle Ta, b \rangle$ ultraweakly in $\mathfrak{Z}$ for all $a, b \in \mathcal{A}$.*

Proof (i) is just [Pa, Remark 3.9] and works for any self-dual Hilbert module over a von Neumann algebra.

(ii) follows immediately from the definition of the weak$*$-topology on $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ in Remark 3.9 and the proof of Proposition 3.10 of [Pa]. This result also holds for any self-dual Hilbert module over a von Neumann algebra.

(iii) follows from item (ii) and the usual $\epsilon/3$-argument using Proposition 3.3 (iv). ∎

Since $\pi(\mathcal{A}_b^2)$ is going to be the domain of definition of our $\mathfrak{Z}$-valued trace on $\mathcal{U}(\mathcal{A})$, we need a condition on an operator $T \in \mathcal{U}(\mathcal{A})$ (involving $\mathfrak{Z}$-valued inner products) to be an element of $\pi(\mathcal{A}_b)$.

Remark In Example 3.6 where our $\mathfrak{Z}$-Hilbert algebra is itself a von Neumann algebra $\mathfrak{A}$ with $\mathfrak{Z} \subseteq Z(\mathfrak{A})$ and a faithful, tracial, uw-continuous $\mathfrak{Z}$-trace $\tau: \mathfrak{A} \to \mathfrak{Z}$, one can use Proposition 5.1 (iii) to show that $\pi(\mathfrak{A}) = (\pi(\mathfrak{A}))''$, as expected.

Proposition 5.2 *If $T \in \mathcal{U}(\mathcal{A})$, then $T \in \pi(\mathcal{A}_b)$ if and only if*

$$\{\langle T\xi, T\xi \rangle \mid \xi \in \mathcal{A}_b \text{ and } \|\pi(\xi)\| \le 1\}$$

is bounded above in $\mathfrak{Z}_+$. In this case, $T = \pi(\eta)$ where $z = \langle \eta, \eta \rangle$ and z is the supremum of this set in $\mathfrak{Z}_+$.

Proof ($\Leftarrow$) Let z be an upper bound for this set in $\mathfrak{Z}_+$. Let $\{\pi(\xi_\alpha)\}$ be a net in $\pi(\mathcal{A}_b)$ converging ultraweakly to 1 and norm bounded by 1. Then

$$\|T\xi_\alpha\| = \|\langle T\xi_\alpha, T\xi_\alpha\rangle\|^{1/2} \le \|z\|^{1/2},$$

so that $\{T\xi_\alpha\}$ is a bounded net in the dual space $\mathcal{H}_\mathcal{A}$ and so we can assume that it converges weak$*$ to some $\eta \in \mathcal{H}_\mathcal{A}$, that is,

$$T\xi_\alpha \overset{w^*}{\to} \eta \quad \text{and} \quad \pi(T\xi_\alpha) = T\pi(\xi_\alpha) \overset{\text{uw}}{\to} T.$$

By Proposition 5.1 we see that for all $a \in \mathcal{A}$ and all $\mu \in \mathcal{H}_\mathcal{A}$

$$\begin{aligned}\langle Ta, \mu\rangle &= \lim_\alpha \langle \pi(T\xi_\alpha)a, \mu\rangle = \lim_\alpha \langle \pi'(a)T\xi_\alpha, \mu\rangle = \lim_\alpha \langle T\xi_\alpha, \pi'(a^*)\mu\rangle \\ &= \langle \eta, \pi'(a^*)\mu\rangle = \langle \pi(\eta)a, \mu\rangle.\end{aligned}$$

So $Ta = \pi(\eta)a$ for all $a \in \mathcal{A}$ and hence $T = \pi(\eta)$ where $\eta \in \mathcal{A}_b$.

($\Rightarrow$) On the other hand, if $T = \pi(\eta)$ for some $\eta \in \mathcal{A}_b$, then for all $\xi \in \mathcal{A}_b$ with $\|\pi(\xi)\| \le 1$ we get by x [Pa, Remark 3.9].

$$\begin{aligned}\langle T\xi, T\xi\rangle &= \langle \eta\xi, \eta\xi\rangle = \langle \xi^*\eta^*, \xi^*\eta^*\rangle \\ &= \langle \pi(\xi\xi^*)\eta^*, \eta^*\rangle \le \|\pi(\xi\xi^*)\|\langle \eta, \eta\rangle \le \langle \eta, \eta\rangle \in \mathfrak{Z}.\end{aligned}$$

Now since $\mathfrak{Z}$ is abelian, the supremum of any finite set of self-adjoint elements exists and so the supremum of the bounded set $\{\langle T\xi, T\xi\rangle \mid \xi \in \mathcal{A}_b \text{ and } \|\pi(\xi)\| \le 1\}$ can be written as the limit of a bounded increasing net of elements in $\mathfrak{Z}_+$, which exists (in $\mathfrak{Z}_+$) by Vigier's Theorem. We let z_0 be this supremum. Then if $T = \pi(\eta)$ for $\eta \in \mathcal{A}_b$, we see by the second part of the above argument that $z_0 \le \langle \eta, \eta\rangle$.

On the other hand, If we choose the net $\{\xi_\alpha\}$ as in the first part of the above argument to also satisfy $\xi_\alpha^* = \xi_\alpha$, then

$$\begin{aligned}\langle T\xi_\alpha, T\xi_\alpha\rangle &= \langle \eta\xi_\alpha, \eta\xi_\alpha\rangle = \langle \xi_\alpha\eta^*, \xi_\alpha\eta^*\rangle \\ &= \langle \pi(\xi_\alpha)^2\eta^*, \eta^*\rangle \xrightarrow{\text{uw}} \langle \eta^*, \eta^*\rangle = \langle \eta, \eta\rangle.\end{aligned}$$

That is, $\langle \eta, \eta\rangle \ge z_0$ and we are done. ∎

Lemma 5.3 *Let $\mathcal{I} = \pi(\mathcal{A}_b)^2 := \operatorname{span}\{\pi(\xi)\pi(\eta) \mid \xi, \eta \in \mathcal{A}_b\}$. Then $\mathcal{I}$ is a uw dense $*$-ideal in $\mathcal{U}(\mathcal{A})$ and $\mathcal{I}_+ = \{\pi(\xi^*)\pi(\xi) \mid \xi \in \mathcal{A}_b\}$.*

Proof It follows from Proposition 4.6 and Theorem 4.11 that $\mathcal{I}$ is a uw dense $*$-ideal in $\mathcal{U}(\mathcal{A})$. Let $\mathcal{I}_0 = \{\pi(\xi^*)\pi(\xi) \mid \xi \in \mathcal{A}_b\}$. We verify that $\mathcal{I}_0$ satisfies the conditions (i)–(iii) of Lemme 1 of [Dix, I.1.6].

(i) $\mathcal{I}_0$ is unitarily invariant in $\mathcal{U}(\mathcal{A})$ since $\pi(\mathcal{A}_b)$ is an ideal in $\mathcal{U}(\mathcal{A})$.

(ii) Let $\eta \in \mathcal{A}_b$ and let $T \in \mathcal{U}(\mathcal{A})_+$ with $0 \le T \le \pi(\eta^*)\pi(\eta)$. Then for each $\xi \in \mathcal{A}_b$ with $\|\pi(\xi)\| \le 1$ we get:

$$\begin{aligned}\langle T^{1/2}\xi, T^{1/2}\xi\rangle &= \langle T\xi, \xi\rangle \le \langle \pi(\eta^*)\pi(\eta)\xi, \xi\rangle \\ &= \langle \eta\xi, \eta\xi\rangle = \langle \xi^*\eta^*, \xi^*\eta^*\rangle \le \|\pi(\xi^*)\|^2\langle \eta^*, \eta^*\rangle \le \langle \eta, \eta\rangle.\end{aligned}$$

By Proposition 5.2, $T^{1/2} = \pi(\mu)$ for some $\mu \in \mathcal{A}_b$. That is, $T = \pi(\mu^*)\pi(\mu) \in \mathcal{I}_0$.

(iii) If $S = \pi(\eta^*\eta)$ and $T = \pi(\mu^*\mu)$ are in $\mathcal{I}_0$, then for all $\xi \in \mathcal{A}_b$ with $\|\pi(\xi)\| \le 1$ we have

$$\begin{aligned}\langle (S+T)^{1/2}\xi, (S+T)^{1/2}\xi\rangle &= \langle S\xi,\xi\rangle + \langle T\xi,\xi\rangle \\ &= \langle \pi(\eta^*\eta)\xi,\xi\rangle + \langle \pi(\mu^*\mu)\xi,\xi\rangle \le \cdots \le \langle \eta,\eta\rangle + \langle \mu,\mu\rangle.\end{aligned}$$

Again by Proposition 5.2, $(S+T)^{1/2} = \pi(\gamma)$ for some $\gamma \in \mathcal{A}_b$, and so $S+T = \pi(\gamma^*\gamma) \in \mathcal{I}_0$. Hence, $\mathcal{I}_0 = \mathcal{J}_+$ the positive part of an ideal $\mathcal{J}$ with $\mathcal{J} = \operatorname{span}\mathcal{I}_0$. Clearly, $\mathcal{J} \subseteq \mathcal{I}$. On the other hand, if $\xi,\eta \in \mathcal{A}_b$, then $\pi(\xi)\pi(\eta^*) = \frac{1}{4}\sum_{k=0}^{3} i^k\pi(\xi+i^k\eta)\pi((\xi+i^k\eta)^*)$ is in $\mathcal{J}$. Thus, $\mathcal{I} \subseteq \mathcal{J}$, and so they are equal, that is, $\{\pi(\xi^*)\pi(\xi) \mid \xi \in \mathcal{A}_b\} = \mathcal{I}_0 = \mathcal{J}_+ = \mathcal{I}_+$. ∎

Corollary 5.4 *With the above hypotheses,*

$$\mathcal{I} := \operatorname{span}\{\pi(\xi)\pi(\eta) \mid \xi,\eta \in \mathcal{A}_b\} = \{\pi(\xi)\pi(\eta) \mid \xi,\eta \in \mathcal{A}_b\}.$$

Proof Let $T \in \mathcal{I}$ and $T = V|T|$ be the polar decomposition of T in $\mathcal{U}(\mathcal{A})$. Then $|T| = V^*T \in \mathcal{I}_+$. Hence, $T = V|T| = V\pi(\xi)\pi(\xi^*) = \pi(V\xi)\pi(\xi^*)$ by Proposition 4.6 (ii). ∎

Remarks At this point we can define a "trace" on the ideal $\mathcal{I}$ in the usual way,

$$\tau(\pi(\xi\eta)) := \langle \xi^*,\eta\rangle,$$

as in the following theorem. However, in order to connect this up with Dixmier's "trace opératorielle" [Dix] which includes unbounded operators affiliated with $\mathfrak{Z}$ in its range (and also includes a notion of normal) we are forced to work a little harder.

Theorem 5.5 *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra over the abelian von Neumann algebra $\mathfrak{Z}$. Let $\mathcal{I} = \pi(\mathcal{A}_b^2)$ be the canonical uw-dense $*$-ideal in $\mathcal{U}(\mathcal{A}) = (\pi(\mathcal{A}))''$, the left von Neumann algebra of $\mathcal{A}$. Then $\tau\colon \mathcal{I} \to \mathfrak{Z}$ defined by $\tau(\pi(\xi\eta)) = \langle \xi^*,\eta\rangle$ is a well-defined positive $\mathfrak{Z}$-linear mapping which is*

(i) *faithful,* i.e., *$\tau(T) = 0$ and $T \ge 0 \Rightarrow T = 0$,*
(ii) *tracial,* i.e., *$\tau(TS) = \tau(ST)$ for $T \in \mathcal{U}(\mathcal{A})$ and $S \in \mathcal{I}$.*

Proof To see that τ is well defined, fix a net $\{\xi_\alpha\}$ in $\mathcal{A}_b$ with $\pi'(\xi_\alpha) \to 1$ ultraweakly. Let $T = \pi(\xi\eta) \in \mathcal{I}$. Then the element $\xi\eta \in \mathcal{A}_b^2$ is unique since π is one-to-one (of course, its representation as a product is not unique). Now

$$\tau(T) = \langle \xi^*,\eta\rangle = \operatorname{uw-lim}_{\alpha}\langle \pi'(\xi_\alpha)\xi^*,\eta\rangle = \operatorname{uw-lim}_{\alpha}\langle \xi_\alpha,\xi\eta\rangle,$$

that is, $\tau(T)$ is uniquely determined by T. Thus, $\tau(T)$ is well defined and $\mathfrak{Z}$-linear.

If $T \in \mathcal{I}_+$, then $T = \pi(\xi^*\xi)$ by Lemma 5.3 and $\tau(T) = \langle \xi,\xi\rangle \ge 0$ so that τ is positive. Clearly, $\tau(T) = 0 \Rightarrow \xi = 0 \Rightarrow \pi(\xi) = 0 \Rightarrow T = 0$, that is, τ is faithful.

To see that τ is tracial, let $S = \pi(\xi\eta) \in \mathcal{I}$ and let $T \in \mathcal{U}(\mathcal{A})$. Then

$$\begin{aligned}\tau(TS) &= \tau(T\pi(\xi)\pi(\eta)) = \tau(\pi(T\xi)\pi(\eta)) = \langle (T\xi)^*,\eta\rangle = \langle T\xi,\eta^*\rangle^* \\ &= \langle \xi, T^*(\eta^*)\rangle^* = \langle \xi^*, (T^*(\eta^*))^*\rangle = \tau(\pi(\xi)\pi(T^*(\eta^*))^*) \\ &= \tau(\pi(\xi)[T^*\pi(\eta^*)]^*) = \tau(\pi(\xi)\pi(\eta)T) = \tau(ST).\end{aligned}$$

∎

6 Traces Opératorielles

We recall here J. Dixmier's definition of a "$\mathfrak{Z}$-trace" [Dix]. We begin by paraphrasing (and translating) Dixmier's discussion of the formal set-up.

Let $\mathfrak{A}$ be a von Neumann algebra and let $\mathfrak{Z}$ be a von Neumann subalgebra of the centre of $\mathfrak{A}$. In this section we fix a locally compact Hausdorff space X, a positive measure ν on X, and an isomorphism of $L^\infty(X,\nu)$ with $\mathfrak{Z}$ (see [Dix, théorème 1 of I.7]). Then $\mathfrak{Z}_+$ is embedded in the set $\widehat{\mathfrak{Z}}_+$ of nonnegative measurable functions on X which are not necessarily finite-valued. Of course, we identify functions in $\widehat{\mathfrak{Z}}_+$ which are equal ν-almost everywhere. As mentioned before, any bounded increasing net in $\mathfrak{Z}_+$ has a supremum in $\mathfrak{Z}_+$. It is clear that the same thing holds for the set $\widehat{\mathfrak{Z}}_+$.

Definition 6.1 With the above notation, we define a $\mathfrak{Z}$-trace on $\mathfrak{A}_+$ to be a mapping $\phi\colon\mathfrak{A}_+\to\widehat{\mathfrak{Z}}_+$ that satisfies the following.

(i) If $S,T\in\mathfrak{A}_+$, then $\phi(S+T)=\phi(S)+\phi(T)$.
(ii) If $S\in\mathfrak{A}_+$ and $T\in\mathfrak{Z}_+$, then $\phi(TS)=T\phi(S)$.
(iii) If $S\in\mathfrak{A}_+$ and U is a unitary in $\mathfrak{A}$, then $\phi(USU^*)=\phi(S)$.

We call ϕ *faithful* if $S\in\mathfrak{A}_+$ and $\phi(S)=0\Rightarrow S=0$. We call ϕ *finite* if $\phi(S)\in\mathfrak{Z}_+$ for all $S\in\mathfrak{A}_+$. We call ϕ *semifinite* if for each nonzero $S\in\mathfrak{A}_+$ there exists a nonzero $T\in\mathfrak{A}_+$ with $T\le S$ and $\phi(T)\in\mathfrak{Z}_+$. We call ϕ *normal* if for every bounded increasing net $\{S_\alpha\}$ in $\mathfrak{A}_+$ with supremum $S\in\mathfrak{A}_+$, $\phi(S)$ is the supremum of the increasing net $\{\phi(S_\alpha)\}$ in $\widehat{\mathfrak{Z}}_+$.

We now show that if $\mathcal{A}$ is a $\mathfrak{Z}$-Hilbert algebra, then there is a natural $\mathfrak{Z}$-trace on the von Neumann algebra $\mathcal{U}(\mathcal{A})$ constructed in the usual way.

Theorem 6.2 ([Dix, Théorème 1, I.6.2]) *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra over the abelian von Neumann algebra $\mathfrak{Z}$ and let $\tau\colon\mathcal{I}=\pi(\mathcal{A}_b^2)\to\mathfrak{Z}$ be the tracial mapping defined in Theorem 5.5. Then τ restricted to $\mathcal{I}_+$ extends to a mapping $\overline{\tau}\colon\mathcal{U}(\mathcal{A})_+\to\widehat{\mathfrak{Z}}_+$ via $\overline{\tau}(T)=\sup\{\tau(S)\mid S\in\mathcal{I}_+,\ S\le T\}$. This extension is a faithful, normal, semifinite $\mathfrak{Z}$-trace in the sense of Dixmier and moreover, $\{T\in\mathcal{U}(\mathcal{A})_+\mid\overline{\tau}(T)\in\mathfrak{Z}_+\}=\mathcal{I}_+$. Clearly, $\overline{\tau}$ is the unique normal extension of τ.*

Proof This proof is similar in outline to Théorème 1, I.6.2 of [Dix]. However, there are many complications (some subtle) in this degree of generality. At least it is clear that $\overline{\tau}$ extends τ.

(i) $\overline{\tau}$ is additive. Trivially we have for $T_1,T_2\in\mathcal{U}(\mathcal{A})_+$, $\overline{\tau}(T_1)+\overline{\tau}(T_2)\le\overline{\tau}(T_1+T_2)$. On the other hand, let $T=T_1+T_2$ for $T_1,T_2\in\mathcal{U}(\mathcal{A})$. Then by [Dix, p. 86], $T_1^{1/2}=AT^{1/2}$ and $T_2^{1/2}=BT^{1/2}$ for $A,B\in\mathcal{U}(\mathcal{A})$ and $E=A^*A+B^*B$ is the range projection of T. Now if $0\le S\le T$ with $S\in U(\mathcal{A})_+$, then

$$ASA^*\le ATA^*=(AT^{1/2})(AT^{1/2})^*=T_1^{1/2}T_1^{1/2}=T_1,$$

and similarly, $BSB^*\le T_2$. Since $\mathcal{I}$ is an ideal, ASA^* and BSB^* are in $\mathcal{I}_+$. Thus, since $ES=S$,

$$\tau(S)=\tau(ES)=\tau(A^*AS)+\tau(B^*BS)=\tau(ASA^*)+\tau(BSB^*)\le\overline{\tau}(T_1)+\overline{\tau}(T_2).$$

Taking the supremum over all such S yields the other inequality:

$$\overline{\tau}(T) \leq \overline{\tau}(T_1) + \overline{\tau}(T_2).$$

(ii) $\overline{\tau}$ is $\mathfrak{Z}_+$-linear. Unlike the scalar case this is not completely trivial. If E is a projection in $\mathfrak{Z}_+$ and $T \in \mathcal{U}(\mathcal{A})_+$, then one easily checks that

$$(S \in \mathcal{I}_+ \text{ and } S \leq ET) \Longleftrightarrow (S = ER \text{ for } R \in \mathcal{I}_+ \text{ with } R \leq T).$$

Applying the definition of $\overline{\tau}$, we get $\overline{\tau}(ET) = E\overline{\tau}(T)$.

Now if $z_0 \in \mathfrak{Z}_+$ and if there exists $z_1 \in \mathfrak{Z}_+$ with $z_1 z_0 = E$, the range projection of z_0, then again one shows that

$$(S \in \mathcal{I}_+ \text{ and } S \leq z_0 T) \Longleftrightarrow (S = z_0 R \text{ for } R \in \mathcal{I}_+ \text{ with } R \leq T).$$

Hence, $\overline{\tau}(z_0 T) = z_0 \overline{\tau}(T)$ if z_0 is bounded away from 0 on its range projection.

Now for an arbitrary $z_0 \in \mathfrak{Z}_+$ and $T \in \mathcal{U}(\mathcal{A})_+$ we work pointwise on X where we have identified $\mathfrak{Z} = L^\infty(X.\nu)$. So fix $x \in X$. There are two cases. If $z_0(x) = 0$, then $[z_0\overline{\tau}(T)](x) = z_0(x)\overline{\tau}(T)(x) = 0$. On the other hand, if $S \leq z_0 T$ and $S \in \mathcal{I}_+$, then $S = ES$ where E, the range projection of z_0, satisfies $E(x) = 0$. Then

$$\tau(S)(x) = \tau(ES)(x) = \big(E\tau(S)\big)(x) = E(x)\tau(S)(x) = 0.$$

Taking the supremum over such S we get $\overline{\tau}(z_0 T)(x) = 0$, that is, if $z_0(x) = 0$, then $\overline{\tau}(z_0 T)(x) = [z_0 \overline{\tau}(T)](x) = 0$.

In the second case $z_0(x) > 0$, so we can write $z_0 = z_1 + z_2$ in $\mathfrak{Z}_+$ where z_1 is bounded away from 0 on its support (which contains x) and $z_2(x) = 0$. Then

$$\begin{aligned}\overline{\tau}(z_0 T)(x) &= [\overline{\tau}(z_1 T) + \overline{\tau}(z_2 T)](x) = [z_1\overline{\tau}(T) + \overline{\tau}(z_2 T)](x)\\ &= z_1(x)\overline{\tau}(T)(x) + \overline{\tau}(z_2 T)(x) = z_0(x)\overline{\tau}(T)(x) + 0 = [z_0\overline{\tau}(T)](x).\end{aligned}$$

Hence, $\overline{\tau}(z_0 T) = z_0 \overline{\tau}(T)$.

(iii) $\overline{\tau}$ is unitarily invariant. This follows easily from Theorem 5.5 (ii).

(iv) $\overline{\tau}$ is faithful. If $\overline{\tau}(T) = 0$, then the only $S \in \mathcal{I}_+$ with $S \leq T$ is $S = 0$. However, if $\{\pi(\xi_\alpha)\}$ is a net in $\pi(\mathcal{A}_b)$ converging ultraweakly to 1 and having norm ≤ 1, then

$$0 \leq T^{1/2}\pi(\xi_\alpha\xi_\alpha^*)T^{1/2} \leq T.$$

But $T^{1/2}\pi(\xi_\alpha\xi_\alpha^*)T^{1/2}$ is in $\mathcal{I}_+$ and converges ultraweakly to T. Hence, $T = 0$.

(v) $\overline{\tau}$ is semifinite. This is the same argument as in (iv).

(vi) $\{T \in \mathcal{U}(\mathcal{A})_+ \mid \overline{\tau}(T) \in \mathfrak{Z}_+\} = \mathcal{I}_+$. Clearly, $\mathcal{I}_+$ is contained in this set. So suppose $\overline{\tau}(T) = z \in \mathfrak{Z}_+$. We apply Proposition 5.2: let $\xi \in \mathcal{A}_b$ satisfy $\|\pi(\xi)\| \leq 1$. Then

$$\pi\big[(T^{1/2}(\xi))(T^{1/2}(\xi))^*\big] = T^{1/2}\pi(\xi\xi^*)T^{1/2} \leq T$$

and so $\tau\big(\pi\big[(T^{1/2}(\xi))(T^{1/2}(\xi))^*\big]\big) \leq \overline{\tau}(T) = z$. But

$$\tau\big(\pi\big[(T^{1/2}(\xi))(T^{1/2}(\xi))^*\big]\big) = \langle (T^{1/2}(\xi))^*, (T^{1/2}(\xi))^*\rangle = \langle T^{1/2}(\xi), T^{1/2}(\xi)\rangle.$$

Therefore, by Proposition 5.2, $T^{1/2} = \pi(\eta)$ for some $\eta \in \mathcal{A}_b$ and so $T = \pi(\eta^*\eta) \in \mathcal{I}_+$.

(vii) $\overline{\tau}$ is normal. We first show that $\overline{\tau}$ satisfies the normality condition when the relevant operators are all in $\mathcal{I}_+$. Suppose that $\{\pi(\xi_\alpha^*\xi_\alpha)\}$ is an increasing net in $\mathcal{I}_+$ with least upper bound $\pi(\xi^*\xi)$ also in $\mathcal{I}_+$. Now for any $\eta \in \mathcal{A}_b$ we have by the polar

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decomposition theorem that $|\pi(\eta)| = V\pi(\eta) = \pi(V\eta)$ and that $V\eta \in \mathcal{A}_b$. Hence, for any $\eta \in \mathcal{A}_b$,

$$\pi(\eta^*\eta) = |\pi(\eta)|^2 = \pi((V\eta)^2), \quad \text{and} \quad \pi(V\eta) \geq 0.$$

Thus we can assume that ξ_α and ξ are self-adjoint and that $\pi(\xi_\alpha) \geq 0$ and $\pi(\xi) \geq 0$. Then $\pi(\xi_\alpha) = (\pi(\xi_\alpha^*\xi_\alpha))^{1/2}$ and $\pi(\xi) = (\pi(\xi^*\xi))^{1/2}$.

Now $\pi(\xi_\alpha^2) \to \pi(\xi^2)$ in the strong operator topology by Vigier's Theorem and by the proof of Théorème 1 of I.6.2 of [Dix] we also have $\pi(\xi_\alpha) \to \pi(\xi)$ in the strong operator topology. As the square root function is operator monotone, this implies that $\pi(\xi) = \sup_\alpha \pi(\xi_\alpha)$.

It easily follows that $\|\xi_\alpha\| \leq \|\xi\|$ for all α. Since $\mathcal{H}_\mathcal{A}$ is a dual space, we can find a subnet $\{\xi_\beta\}$ which converges weak∗ to some $\zeta \in \mathcal{H}_\mathcal{A}$. To see that $\zeta = \xi$, let $\lambda, \mu \in \mathcal{A}_b$. Then by Proposition 5.1

$$\langle \zeta, \lambda\mu \rangle = \lim_\beta \langle \xi_\beta, \lambda\mu \rangle = \lim_\beta \langle \pi(\xi_\beta)\mu^*, \lambda \rangle = \langle \pi(\xi)\mu^*, \lambda \rangle = \langle \xi, \lambda\mu \rangle.$$

Thus, ζ and ξ define the same $\mathfrak{Z}$-valued mapping on $\mathcal{A}_b^2 \supseteq \mathcal{A}^2$ and therefore the same mapping on $\mathcal{A}$, that is, $\zeta = \xi$.

Now since τ is positive, we have $\tau(\pi(\xi^*\xi)) \geq \sup_\alpha \tau(\pi(\xi_\alpha^*\xi_\alpha))$. On the other hand, by Kaplansky's Cauchy–Schwarz inequality [K] (which holds since $\mathfrak{Z}$ is abelian) we have $|\langle \xi_\beta, \xi \rangle| \leq \langle \xi_\beta, \xi_\beta \rangle^{1/2} \langle \xi, \xi \rangle^{1/2}$ for all β. Since ξ and ξ_β are self-adjoint, it is seen that $\langle \xi_\beta, \xi \rangle$ is also self-adjoint and so, in fact, $\langle \xi_\beta, \xi \rangle \leq \langle \xi_\beta, \xi_\beta \rangle^{1/2} \langle \xi, \xi \rangle^{1/2}$ for all β. Hence,

$$\begin{aligned}\langle \xi, \xi \rangle &= \text{uw-}\lim_\beta \langle \xi_\beta, \xi \rangle \leq \sup_\beta \langle \xi_\beta, \xi_\beta \rangle^{1/2} \langle \xi, \xi \rangle^{1/2} \\ &\leq (\sup_\alpha \langle \xi_\alpha, \xi_\alpha \rangle^{1/2}) \langle \xi, \xi \rangle^{1/2}.\end{aligned}$$

Since $\mathfrak{Z}$ is abelian, this implies that $\langle \xi, \xi \rangle^{1/2} \leq \sup_\alpha \langle \xi_\alpha, \xi_\alpha \rangle^{1/2}$ and so

$$\langle \xi, \xi \rangle \leq \sup_\alpha \langle \xi_\alpha, \xi_\alpha \rangle.$$

That is, $\tau(\pi(\xi^*\xi)) \leq \sup_\alpha \tau(\pi(\xi_\alpha^*\xi_\alpha))$, and so they are equal.

Now we let $\{T_\alpha\}$ be an increasing net in $\mathcal{U}(\mathcal{A})_+$ with supremum $T \in \mathcal{U}(\mathcal{A})_+$. We define $f = \sup_\alpha(\overline{\tau}(T_\alpha))$, in $\widehat{\mathfrak{Z}}_+$. Let $E = \{x \in X \mid f(x) = +\infty\}$. Since $\overline{\tau}(T_\alpha) \leq \overline{\tau}(T)$ for all α, we have $f \leq \overline{\tau}(T)$. Hence f agrees with $\overline{\tau}(T)$ on the measurable set E. The complement of E is the countable union of the measurable sets

$$E_N := \{x \in X \mid f(x) \leq N\},$$

so it suffices to see that f agrees with $\overline{\tau}(T)$ (almost everywhere) on each E_N. To this end, let z_N be the characteristic function of E_N. Clearly, $z_N \in \mathfrak{Z}_+$ and $z_N T = \sup_\alpha z_N T_\alpha$ in $\mathcal{U}(\mathcal{A})_+$. Now for each α,

$$\overline{\tau}(z_N T_\alpha) = z_N \overline{\tau}(T_\alpha) \leq z_N f \leq N z_N \in \mathfrak{Z}_+.$$

So by an earlier part of the proof, there exists $\xi_\alpha = \xi_\alpha^* \in \mathcal{A}_b$ with $z_N T_\alpha = \pi(\xi_\alpha^* \xi_\alpha)$ and $\langle \xi_\alpha, \xi_\alpha \rangle \le N z_N$. Now for each $\eta \in \mathcal{A}_b$ with $\|\pi(\eta)\| \le 1$ we have

$$\begin{aligned}\langle z_N T^{1/2}\eta, z_N T^{1/2}\eta\rangle = \langle z_N T\eta, \eta\rangle &= \lim_\alpha \langle z_N T_\alpha \eta, \eta\rangle = \lim_\alpha \langle \xi_\alpha \eta, \xi_\alpha \eta\rangle \\ &= \lim_\alpha \langle \eta^* \xi_\alpha, \eta^* \xi_\alpha\rangle = \lim_\alpha \langle \pi(\eta\eta^*)\xi_\alpha, \xi_\alpha\rangle \le \sup_\alpha \langle \xi_\alpha, \xi_\alpha\rangle \le N z_N.\end{aligned}$$

Therefore, by Proposition 5.2 there exists a $\zeta \in \mathcal{A}_b$ with $z_N T^{1/2} = \pi(\zeta)$. Moreover,

$$\sup_\alpha \pi(\xi_\alpha^* \xi_\alpha) = \sup_\alpha z_N T_\alpha = z_N T = \pi(\zeta^* \zeta).$$

Hence by the first part of the proof of normality of $\overline{\tau}$,

$$\overline{\tau}(z_N T) = \overline{\tau}\big(\pi(\zeta^*\zeta)\big) = \sup_\alpha \overline{\tau}\big(\pi(\xi_\alpha^*\xi_\alpha)\big) = \sup_\alpha \overline{\tau}(z_N T_\alpha).$$

That is, for $x \in E_N$ we have

$$\begin{aligned} f(x) = (z_N f)(x) &= (z_N \sup_\alpha \overline{\tau}(T_\alpha))(x) \\ &= (\sup_\alpha \overline{\tau}(z_N T_\alpha))(x) = (\overline{\tau}(z_N T))(x) \\ &= (z_N \overline{\tau}(T))(x) = \overline{\tau}(T)(x)\end{aligned}$$

as required. ∎

Remarks In the above setting we want to observe that $\mathcal{A}_b$ is also a $\mathfrak{Z}$-Hilbert algebra and that $\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A}_b)$, etc. It turns out that the only subtle point is the fact that $\mathcal{H}_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}_b}$.

Lemma 6.3 *Suppose* $\mathbf{X} \subseteq \mathbf{Y} \subseteq \mathbf{X}^\dagger$ *as pre-Hilbert* $\mathfrak{B}$*-modules where* $\mathfrak{B}$ *is a von Neumann algebra. Then, in fact,* $\mathbf{X}^\dagger = \mathbf{Y}^\dagger$.

Proof If $\theta \in \mathbf{X}^\dagger$, then $y \mapsto \langle \theta, y\rangle_{\mathbf{X}^\dagger}: \mathbf{Y} \to \mathfrak{B}$ is a bounded $\mathfrak{B}$-module map and so there is a unique $\widetilde{\theta} \in \mathbf{Y}^\dagger$ so that

$$\langle \widetilde{\theta}, \widehat{y}\rangle_{\mathbf{Y}^\dagger} = \langle \theta, y\rangle_{\mathbf{X}^\dagger} \quad \text{for all } y \in \mathbf{Y}. \tag{6.1}$$

That is, $\theta \mapsto \widetilde{\theta}$ embeds $\mathbf{X}^\dagger$ in $\mathbf{Y}^\dagger$. We first show that this embedding preserves inner products.

Now given $\eta \in \mathbf{X}^\dagger$, $\theta \mapsto \langle \widetilde{\eta}, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger}: \mathbf{X}^\dagger \to \mathfrak{B}$ is an element of $\mathbf{X}^{\dagger\dagger} = \mathbf{X}^\dagger$ and so there exists a unique $\gamma \in \mathbf{X}^\dagger$ so that

$$\langle \gamma, \theta\rangle_{\mathbf{X}^\dagger} = \langle \widetilde{\eta}, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger} \quad \text{for all } \theta \in \mathbf{X}^\dagger. \tag{6.2}$$

In particular, for all $x \in \mathbf{X}$ we get $\langle \gamma, x\rangle_{\mathbf{X}^\dagger} = \langle \widetilde{\eta}, \widehat{x}\rangle_{\mathbf{Y}^\dagger} = \langle \eta, x\rangle_{\mathbf{X}^\dagger}$ by equation (6.1). Hence, $\gamma = \eta$ and equation (6.2) becomes $\langle \eta, \theta\rangle_{\mathbf{X}^\dagger} = \langle \widetilde{\eta}, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger}$ for all $\eta, \theta \in \mathbf{X}^\dagger$. That is, $\mathbf{X}^\dagger$ is a pre-Hilbert $\mathfrak{B}$-submodule of $\mathbf{Y}^\dagger$ and we have $\mathbf{Y} \subseteq \mathbf{X}^\dagger \subseteq \mathbf{Y}^\dagger$ as pre-Hilbert $\mathfrak{B}$-modules.

Now for each $\mu \in \mathbf{Y}^\dagger$ the map $\theta \mapsto \langle \mu, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger}: \mathbf{X}^\dagger \to \mathfrak{B}$ defines a unique element $\check{\mu} \in \mathbf{X}^\dagger$ satisfying $\langle \mu, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger} = \langle \check{\mu}, \theta\rangle_{\mathbf{X}^\dagger} = \langle \widetilde{\check{\mu}}, \widetilde{\theta}\rangle_{\mathbf{Y}^\dagger}$ for all $\theta \in \mathbf{X}^\dagger$. But since $\mathbf{Y} \subseteq \mathbf{X}^\dagger$ we must have $\mu = \widetilde{\check{\mu}}$, that is, $\sim: \mathbf{X}^\dagger \to \mathbf{Y}^\dagger$ is onto. ∎

Proposition 6.4 *Let $\mathcal{A}$ be a $\mathfrak{Z}$-Hilbert algebra over the abelian von Neumann algebra $\mathfrak{Z}$. Then $\mathcal{A}_b$ is also a $\mathfrak{Z}$-Hilbert algebra and*

(i) $\mathcal{H}_{\mathcal{A}_b} = \mathcal{H}_{\mathcal{A}}$,
(ii) $\mathcal{U}(\mathcal{A}_b) = \mathcal{U}(\mathcal{A})$ *and* $\mathcal{V}(\mathcal{A}_b) = \mathcal{V}(\mathcal{A})$,
(iii) $(\mathcal{A}_b)_b = \mathcal{A}_b$.

Proof Since $\mathfrak{Z} \subseteq \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ and $\pi(\mathcal{A}_b)$ is a left ideal in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, we see that $\mathcal{A}_b$ is a pre-Hilbert $\mathfrak{Z}$-submodule of $\mathcal{H}_{\mathcal{A}}$ containing $\mathcal{A}$. Hence, by Lemma 6.3 $\mathcal{H}_{\mathcal{A}_b} = \mathcal{H}_{\mathcal{A}}$. Thus, axioms (a)–(d) are automatically satisfied.

That $\mathcal{A}_b$ is a $*$-algebra follows from Proposition 4.6. Now axiom (e) follows from Lemma 4.3. Axiom (f) follows from Proposition 4.6 (iv) since $\pi(\xi^*) = \pi(\xi)^*$ for $\xi \in \mathcal{A}_b$. Axiom (g) follows from the definition of $\mathcal{A}_b$ and Proposition 4.6 (iii).

To see axiom (h), we first note that $\mathcal{A}^2 \subseteq \mathcal{A}_b^2 \subseteq \mathcal{A}_b \subseteq \mathcal{H}_{\mathcal{A}_b} = \mathcal{H}_{\mathcal{A}}$. Since $\mathcal{A}^2$ is dense in $\mathcal{A}$ by definition and $\mathcal{A}$ is dense in $\mathcal{H}_{\mathcal{A}}$ by Proposition 3.3, it follows that $\mathcal{A}_b^2$ is dense in $\mathcal{H}_{\mathcal{A}_b}$ and hence in $\mathcal{A}_b$. Thus, $\mathcal{A}_b$ is also a $\mathfrak{Z}$-Hilbert algebra and (ii) and (iii) follow easily. ■

7 $\mathfrak{Z}$-Hilbert Algebras From $\mathfrak{Z}$-Traces

Here we suppose that ϕ is a faithful, normal, semifinite $\mathfrak{Z}$-trace (in Dixmier's sense) on the von Neumann algebra $\mathfrak{A}$ where $\mathfrak{Z}$ is a von Neumann subalgebra of the centre of $\mathfrak{A}$. We abuse notation and also let ϕ denote the unique linear extension of the original ϕ from $\mathcal{I}_+ = \{x \in \mathfrak{A} \mid \phi(x) \in \mathfrak{Z}_+\}$ to the ideal $\mathcal{I} = \operatorname{span} \mathcal{I}_+$, defined in [Dix, Proposition 1 of III.4.1]. Then by [Dix, I.1.6] the space $\mathcal{A} = \{x \in \mathfrak{A} \mid \phi(x^*x) \in \mathfrak{Z}_+\}$ is an ideal in $\mathfrak{A}$ with $\mathcal{A}^2 = \mathcal{I}$.

Proposition 7.1 *With the above hypotheses, the ideal $\mathcal{A} = \{x \in \mathfrak{A} \mid \phi(x^*x) \in \mathfrak{Z}_+\}$ is a $\mathfrak{Z}$-Hilbert algebra, with the $\mathfrak{Z}$-valued inner product $\langle x, y\rangle = \phi(x^*y)$.*

Proof Since $\mathcal{A}$ is an ideal in $\mathfrak{A}$, it is certainly a right $\mathfrak{Z}$-module. Axiom (a) is just the statement that ϕ is faithful. Axiom (b) follows since the extended ϕ is clearly self-adjoint. Axiom (c) follows as the original ϕ is $\mathfrak{Z}_+$-linear.

To see that Axiom (d) holds requires a little thought. First, it is clear that

$$\operatorname{span}\big(\phi(\mathcal{A}^2)\big)$$

is an ideal in $\mathfrak{Z}$. Therefore, its uw-closure is an ideal in $\mathfrak{Z}$ of the form $E\mathfrak{Z}$ for some projection $E \in \mathfrak{Z}$. If $(1-E) \neq 0$, then since ϕ is semifinite, there exists $x \in \mathfrak{A}_+$ with $0 \neq x \leq (1-E)$ and $\phi(x) \in \mathfrak{Z}_+$ so that $x^{1/2} \in \mathcal{A}$. But then

$$0 \neq \phi(x) = \phi\big((1-E)x\big) = (1-E)\phi(x)$$

lies in $E\mathfrak{Z}$, a contradiction. Hence $E = 1$ and the span of the inner products is uw-dense in $\mathfrak{Z}$.

Axiom (e) follows from the tracial property of Proposition 1 of III.4.1 of [Dix]. Axiom (f) is trivial, and Axiom (g) is proved as in Example 3.6.

To see axiom (h) we first show that $\mathcal{A}$ is uw-dense in $\mathfrak{A}$. Now the ultraweak closure of $\mathcal{A}$ is a uw closed ideal in $\mathfrak{A}$ and so has the form $F\mathfrak{A}$ for some projection F in $Z(\mathfrak{A})$.

If $(1-F)\neq 0$, then, since ϕ is semifinite, there exists $y\in\mathfrak{A}_+$ with $0\neq y\leq(1-F)$ and $\phi(y)\in\mathfrak{Z}_+$ so that $y^{1/2}\in\mathcal{A}$. But then $y\in\mathcal{A}$ and so $y\leq F$, a contradiction as $y\neq 0$. Thus $F=1$ and $\mathcal{A}$ is uw-dense in $\mathfrak{A}$.

Now given $\omega\geq 0$ in the predual of $\mathfrak{Z}$, we have that $\phi_\omega := \omega\circ\phi$ is a normal, semifinite trace on $\mathfrak{A}$ by Proposition 2 of III.4.3 of [Dix]. Moreover, the GNS Hilbert space of the normal representation π_ω of $\mathfrak{A}$ induced by ϕ_ω is the same as the Hilbert space $\mathcal{H}_\omega$ of Section 3. For $a,b\in\mathcal{A}$ we have $\pi_\omega(a)(b+N_\omega)=ab+N_\omega$. Since π_ω is normal, $\pi_\omega(\mathcal{A})$ is uw-dense in $\pi_\omega(\mathfrak{A})$. Therefore, it is also s.o.-dense and hence given any $b\in\mathcal{A}$ and $\epsilon>0$, there exists $a\in\mathcal{A}$ with $\|\pi_\omega(a)(b+N_\omega)-(b+N_\omega)\|_\omega<\epsilon$. That is, $\|ab-b\|_\omega<\epsilon$ and axiom (h) is satisfied. ∎

In this setting, each $x\in\mathfrak{A}$ defines an operator $\widetilde{x}$ on the ideal

$$\mathcal{A}=\{a\in\mathfrak{A}\mid\phi(a^*a)\in\mathfrak{Z}_+\}$$

via $\widetilde{x}(a)=xa$. Clearly, $\widetilde{x}$ is $\mathfrak{Z}$-linear and it is easy to check that $\widetilde{x}$ is a bounded $\mathfrak{Z}$-module map on $\mathcal{A}$ and therefore extends uniquely to a bounded module map on $\mathcal{H}_{\mathcal{A}}$ also denoted by $\widetilde{x}$. As left multiplications commute with right multiplications, we see that $\widetilde{x}\in(\pi'(\mathcal{A}))'=\mathcal{U}(\mathcal{A})$ by the Commutation Theorem, 4.11.

Lemma 7.2 *Let $\mathcal{A}$ be a uw-dense $*$-ideal in the von Neumann algebra $\mathfrak{A}$. Then each $T\in\mathfrak{A}_+$ is the increasing limit of a net in $\mathcal{A}_+$.*

Proof It follows from the proof of Theorem 1.4.2 of [Ped] that $\{a\in\mathcal{A}_+\mid\|a\|<1\}$ is an increasing net in the usual ordering of positive elements and hence converges in $\mathfrak{A}_+$ by Vigier's Theorem. By the Kaplansky Density Theorem there is a subnet of this one converging ultraweakly to the identity in $\mathfrak{A}$, and therefore this net converges ultraweakly to $1\in\mathfrak{A}$.

Thus, if $T\in\mathfrak{A}_+$, the net $\{T^{1/2}aT^{1/2}\mid a\in\mathcal{A}_+ \text{ and } \|a\|<1\}$ is an increasing net in $\mathcal{A}_+$ converging ultraweakly to T. ∎

Theorem 7.3 *Let ϕ be a faithful normal semifinite $\mathfrak{Z}$-trace on the von Neumann algebra $\mathfrak{A}$, where $\mathfrak{Z}$ is a von Neumann subalgebra of the centre of $\mathfrak{A}$. Let*

$$\mathcal{A}=\{a\in\mathfrak{A}\mid\phi(a^*a)\in\mathfrak{Z}_+\}$$

be the corresponding $\mathfrak{Z}$-Hilbert algebra. Then the mapping $x\mapsto\widetilde{x}:\mathfrak{A}\to\mathcal{U}(\mathcal{A})$ is an isomorphism of von Neumann algebras.

Proof It is clear the the mapping is a $*$-homomorphism. Since $\mathcal{A}$ is uw-dense in $\mathfrak{A}$, the mapping is also one-to-one. Hence, it suffices to see that the mapping is onto $\mathcal{U}(\mathcal{A})$. So let $T\in\mathcal{U}(\mathcal{A})_+$. Since $\pi(\mathcal{A})$ is a uw-dense $*$-ideal in $\mathcal{U}(\mathcal{A})$, there is a net $\{b_\alpha\}$ in $\mathcal{A}_+$ with $\pi(b_\alpha)$ increasing to T in $\mathcal{U}(\mathcal{A})\subseteq\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. Since, $\{b_\alpha\}$ is an increasing net in $\mathcal{A}_+\subseteq\mathfrak{A}_+$ bounded by $\|T\|$, it converges to an element $x\in\mathfrak{A}_+$. To see that $\widetilde{x}=T$ it suffices to see that $\omega(\langle Ta,c\rangle)=\omega(\langle xa,c\rangle)$ for all $a,c\in\mathcal{A}$ and $\omega\geq 0$ in $\mathfrak{Z}_*$.

Now since $\omega\circ\phi$ is a normal scalar trace on $\mathfrak{A}$ by [Dix, Proposition 2 of III.4.3] and since $ca^*\in\mathcal{A}^2=\mathcal{I}$ is contained in the ideal of definition of this normal scalar

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trace, the map $y \mapsto \omega \circ \phi(yca^*): \mathfrak{A} \to \mathbf{C}$ is a normal (and so uw-continuous) linear functional on $\mathfrak{A}$. Hence,

$$\omega(\langle xa, c\rangle) = \omega\big(\phi(a^*xc)\big) = \omega\big(\phi(xca^*)\big) = \lim_\alpha \omega(\phi(b_\alpha ca^*)) = \lim_\alpha \omega(\langle \pi(b_\alpha)a, c\rangle).$$

But by Proposition 5.1 (ii) this last term equals $\omega(\langle Ta, c\rangle)$ since $\pi(b_\alpha) \overset{\mathrm{uw}}{\to} T$. ∎

8 The $\mathfrak{Z}$-Trace on the Crossed Product von Neumann Algebra

Let (A, Z, τ, α) be a 4-tuple as in Section 1. We also assume that Z has a faithful state ω to apply Proposition 2.1 so that $\overline{\omega} = \omega \circ \tau$ is a faithful tracial state on A and representing A on the GNS Hilbert space $\mathcal{H}_{\overline{\omega}}$ we obtain $\mathfrak{A} = A''$ and $\mathfrak{Z} = Z''$ and a $\mathfrak{Z}$-trace $\overline{\tau}: \mathfrak{A} \to \mathfrak{Z}$ extending τ and an extension of α to an ultraweakly continuous action $\overline{\alpha}: \mathbf{R} \to \mathrm{Aut}(\mathfrak{A})$ that leaves $\overline{\tau}$ invariant.

Remark The following construction of the $\mathfrak{Z}$-trace on the crossed-product algebra works in much greater generality. The action of $\mathbf{R}$ on A leaving τ invariant can be replaced by an action of a unimodular locally compact group G on A leaving τ invariant. We leave the minor modifications to the interested reader. All the results up to the end of Section 8.5 work in this generality.

We let $A_{\mathfrak{Z}}$ denote the C^*-subalgebra of $\mathfrak{A}$ generated by A and $\mathfrak{Z}$. Clearly,

$$A_{\mathfrak{Z}} = \Big\{ \sum_{i=1}^n a_i z_i \,|\, a_i \in A, z_i \in \mathfrak{Z} \Big\}^{-\|\cdot\|}.$$

It is clear that

- $A_{\mathfrak{Z}}$ contains A and $\mathfrak{Z}$ and is therefore ultraweakly dense in $\mathfrak{A}$,
- $\overline{\tau}: A_{\mathfrak{Z}} \to \mathfrak{Z}$ is a faithful, unital $\mathfrak{Z}$-trace,
- $\overline{\alpha}: \mathbf{R} \to \mathrm{Aut}(A_{\mathfrak{Z}})$ is a *norm*-continuous action on $A_{\mathfrak{Z}}$ leaving $\overline{\tau}$ invariant and leaving $\mathfrak{Z}$ pointwise fixed.

Key Idea 7 The introduction of this hybrid algebra $A_{\mathfrak{Z}}$ allows us to treat $\mathfrak{Z}$ as scalars and use *norm*-continuity in most of our calculations. This permits the use of C^*-algebra crossed products and is a considerable simplification. We note also that one cannot simply use the space of norm-continuous functions $C_c(\mathbf{R}, \mathfrak{A})$ below since $\overline{\alpha}$-twisting the multiplication might take us out of the realm of norm-continuity. However, as a vector space (and pre-Hilbert $\mathfrak{Z}$-module), $C_c(\mathbf{R}, \mathfrak{A})$ will have its uses.

With this set-up and notation, we have the following definition.

Definition 8.1 $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}})$, the space of norm-continuous compactly supported functions from $\mathbf{R}$ to $A_{\mathfrak{Z}}$. We require norm-continuity so that $\mathcal{A}$ becomes a $*$-algebra with the usual $\overline{\alpha}$-twisted multiplication $x \cdot y(s) = \int x(t)\overline{\alpha}_t(y(s-t))\,dt$ and involution $x^*(s) = \overline{\alpha}_s((x(-s))^*)$.

Moreover, $\mathcal{A}$ becomes a (right) pre-Hilbert $\mathfrak{Z}$-module with the inner product

$$\langle x, y\rangle = \int \overline{\tau}(x(s)^* y(s))\, ds$$

and $\mathfrak{Z}$-action $(xz)(s) = x(s)z$.

Axioms (a)–(c) are routine calculations. To see axiom (d), we observe that the set of inner products $\{\langle x, y\rangle \mid x, y \in \mathcal{A}\}$ is exactly equal to $\mathfrak{Z}$. It comes as no surprise that $\mathcal{A}$ is, in fact, a $\mathfrak{Z}$-Hilbert algebra.

Remark We will also have occasion to use the completion of $\mathcal{A}$ in the vector-valued Banach L^2 norm $\|x\|_2 = (\int \|x(s)\|^2\, ds)^{1/2}$. We define this completion to be $L^2(\mathbf{R}, A_{\mathfrak{Z}})$ and observe that since $\|x\|_{\mathcal{A}} \le \|x\|_2$, we have a natural inclusion

$$L^2(\mathbf{R}, A_{\mathfrak{Z}}) \hookrightarrow \mathcal{A}^{-\|\cdot\|_{\mathcal{A}}} \subset \mathcal{H}_{\mathcal{A}}.$$

Proposition 8.2 *With the above inner product and $\mathfrak{Z}$-action, the $*$-algebra $\mathcal{A}$ is a $\mathfrak{Z}$-Hilbert algebra.*

Proof Axioms (e) and (f) are routine calculations. Since $\mathcal{A}$ contains all the scalar-valued functions in $C_c(\mathbf{R})$, it is easy to see that $\mathcal{A}^2$ is dense in $\mathcal{A}$ in the vector-valued L^2 norm.

Since $\|x\|_{\mathcal{A}} \le \|x\|_2$, $\mathcal{A}^2$ is dense in $\mathcal{A}$ in the $\mathfrak{Z}$-Hilbert algebra norm and so axiom (h) is satisfied by the remark after Definition 3.5.

Axiom (g) requires a little more thought. We will show that the left regular representation of the $*$-algebra $\mathcal{A}$ on the pre-Hilbert $\mathfrak{Z}$-module $\mathcal{A}$ is the integrated form of a covariant pair of representations $(\pi_{\mathcal{A}}, U)$ of the system $(A_{\mathfrak{Z}}, \mathbf{R}, \overline{\alpha})$ inside the von Neumann algebra, $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. To this end we represent $A_{\mathfrak{Z}}$ on the $\mathfrak{Z}$-module $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}})$ via $[\pi_{\mathcal{A}}(a)x](s) = ax(s)$ for $a \in A_{\mathfrak{Z}}, x \in \mathcal{A}, s \in \mathbf{R}$. Similarly, we represent $\mathbf{R}$ on $\mathcal{A}$ via $[U_t(x)](s) = \overline{\alpha}_t(x(s-t))$ for $t, s \in \mathbf{R}, x \in \mathcal{A}$.

One easily checks that these are representations as bounded, adjointable $\mathfrak{Z}$-module mappings. Now for fixed $x \in \mathcal{A}$ the map $t \mapsto U_t(x)$ is $\|\cdot\|_2$-norm continuous and so $\|\cdot\|_{\mathcal{A}}$-norm continuous. By Proposition 5.1 (iii) this easily implies that

$$t \mapsto U_t\colon \mathbf{R} \to \mathcal{L}(\mathcal{H}_{\mathcal{A}})$$

is an ultraweakly continuous representation. Moreover, the following are easily verified:

- $\|\pi_{\mathcal{A}}(a)\| \le \|a\|$ for $a \in A_{\mathfrak{Z}}$.
- $\langle U_t(x), U_t(y)\rangle = \langle x, y\rangle$ for $t \in \mathbf{R}, x, y \in \mathcal{A}$.
- $\pi_{\mathcal{A}}(a)^* = \pi_{\mathcal{A}}(a^*)$ and $U_t^* = U_{-t}$ for $a \in A_{\mathfrak{Z}}, t \in \mathbf{R}$.
- $U_t \pi_{\mathcal{A}}(a) U_t^* = \pi_{\mathcal{A}}(\overline{\alpha}_t(a))$ for $t \in \mathbf{R}$ and $a \in A_{\mathfrak{Z}}$. This is the covariance condition.

Combining this covariant pair of representations of the system, $(A_{\mathfrak{Z}}, \mathbf{R}, \overline{\alpha})$ in $\mathcal{L}(\mathcal{A})$ with the $*$-monomorphism embedding $\mathcal{L}(\mathcal{A}) \hookrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ (by [Pa, Corollary 3.7]) we obtain a representation $\pi_{\mathcal{A}} \times U$ of the C^*-algebra $A_{\mathfrak{Z}} \rtimes \mathbf{R}$ in the von Neumann algebra $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. One then easily checks that for $x \in \mathcal{A} \subset A_{\mathfrak{Z}} \rtimes \mathbf{R}$ and $y \in \mathcal{A} \subset \mathcal{H}_{\mathcal{A}}$ that $[(\pi_{\mathcal{A}} \times U)(x)(y)](s) = \int x(t)\overline{\alpha}_t(y(s-t))\, dt = (x \cdot y)(s)$. That is, left-multiplication by x on the $\mathfrak{Z}$-module $\mathcal{A}$ is bounded in the $\mathfrak{Z}$-module norm and axiom (g) is satisfied. ∎

Lemma 8.3 *If $\mathcal{A} = C_c(\mathbf{R}, A_3)$ as above, then the following hold.*

(i) *The norm-decreasing embedding $(\mathcal{A}, \|\cdot\|_2) \to (\mathcal{H}_{\mathcal{A}}, \|\cdot\|_3)$ extends by continuity to a norm-decreasing embedding of $L^2(\mathbf{R}, A_3)$ into $\mathcal{H}_{\mathcal{A}}$. Moreover, $L^2(\mathbf{R}, A_3)$ is a $\mathfrak{Z}$-module and the $\mathfrak{Z}$-valued inner product on $\mathcal{H}_{\mathcal{A}}$ restricts to $L^2(\mathbf{R}, A_3)$ so that it is, in fact, a pre-Hilbert $\mathfrak{Z}$-module.*

(ii) *If $x \in L^2(\mathbf{R}, A_3) \subseteq \mathcal{H}_{\mathcal{A}}$ and $y \in \mathcal{A}$, then in the $\mathfrak{Z}$-Hilbert algebra notation, the element $\pi(x)y := \pi'(y)x \in \mathcal{H}_{\mathcal{A}}$ is identical to the element $x \cdot y \in L^2(\mathbf{R}, A_3)$ given by the twisted convolution $(x \cdot y)(s) = \int x(t)\overline{\alpha}_t(y(s-t))\,dt$.*

(iii) *If $x, y \in L^2(\mathbf{R}, A_3)$ and if $\pi(x)$ and $\pi(y)$ are bounded, then the operator*

$$\pi(x)^*\pi(y)$$

is in the ideal of definition of the $\mathfrak{Z}$-trace, σ on $\mathcal{U}(\mathcal{A})$, and

$$\sigma[\pi(x)^*\pi(y)] = \langle x, y\rangle = \int \overline{\tau}(x(t)^* y(t))\,dt.$$

Proof The first statement of (i) follows trivially from the inequality $\|x\|_{\mathcal{A}} \le \|x\|_2$.

To see the second statement of (i), suppose $\{x_n\}$ is a sequence in $\mathcal{A}$ which is Cauchy in the $\|\cdot\|_2$ norm and that $z \in \mathfrak{Z}$. Then $\|x_n z - x_m z\|_2 \le \|x_n - x_m\|_2\|z\| \to 0$, so that $L^2(\mathbf{R}, A_3)$ is a $\mathfrak{Z}$-module. Similarly, if $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{A}$ which are Cauchy in the $\|\cdot\|_2$ norm, then by the Cauchy–Schwarz inequality

$$\begin{aligned}\|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle\| &= \|\langle x_n - x_m, y_n\rangle - \langle x_m, y_m - y_n\rangle\|\\ &\le \|x_n - x_m\|_{\mathcal{A}}\|y_n\|_{\mathcal{A}} + \|x_m\|_{\mathcal{A}}\|y_m - y_n\|_{\mathcal{A}}\\ &\le \|x_n - x_m\|_2\|y_n\|_2 + \|x_m\|_2\|y_m - y_n\|_2.\end{aligned}$$

Therefore, the $\mathfrak{Z}$-valued inner product on $\mathcal{H}_{\mathcal{A}}$ restricts to a $\mathfrak{Z}$-valued inner product on $L^2(\mathbf{R}, A_3)$.

To see (ii), let $\{x_n\}$ be a sequence in $\mathcal{A}$ with $\|x_n - x\|_2 \to 0$. Then

$$\|x_n \cdot y - x \cdot y\|_{\mathcal{A}} \le \|x_n \cdot y - x \cdot y\|_2 \le \|x_n - x\|_2\|y\|_1 \to 0.$$

On the other hand, since x_n and y are both in $\mathcal{A}$, we have that $\pi'(y)x_n = x_n \cdot y$ by definition, and so

$$\begin{aligned}\|x_n \cdot y - \pi(x)y\|_{\mathcal{A}} = \|\pi'(y)x_n - \pi'(y)x\|_{\mathcal{A}} &\le \|\pi'(y)\|\,\|x_n - x\|_{\mathcal{A}}\\ &\le \|\pi'(y)\|\,\|x_n - x\|_2 \to 0.\end{aligned}$$

So $\pi(x)y = x \cdot y$.

(iii) follows from from the definition of the trace (Theorem 5.5) and (i) ■

Lemma 8.4 *The representation $\pi_{\mathcal{A}}\colon A_3 \to \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ extends to an ultraweakly continuous representation (also denoted $\pi_{\mathcal{A}}$) of $\mathfrak{A}$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$.*

Proof We first observe that the space of norm-continuous functions, $C_c(\mathbf{R}, \mathfrak{A}) \subset \mathcal{H}_{\mathcal{A}}$ in a natural way. That is, if $x \in C_c(\mathbf{R}, \mathfrak{A})$, then for $y \in \mathcal{A}$ the map

$$y \mapsto \int \overline{\tau}\big((x(t))^* y(t)\big)\,dt$$

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is a bounded $\mathfrak{Z}$-module mapping from $\mathcal{A}$ to $\mathfrak{Z}$ and so defines a unique element in $\mathcal{H}_{\mathcal{A}}$. If we abuse notation and denote this element in $\mathcal{H}_{\mathcal{A}}$ by x, then we get the formula

$$\langle x, y\rangle = \int \overline{\tau}((x(t))^* y(t))\, dt.$$

Clearly, $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}}) \subset C_c(\mathbf{R}, \mathfrak{A}) \subset \mathcal{H}_{\mathcal{A}}$. The extension of $\pi_{\mathcal{A}}$ to $\mathfrak{A}$ is now obvious: $[\pi_{\mathcal{A}}(a)x](s) = ax(s)$ for $a \in \mathfrak{A}, x \in C_c(\mathbf{R}, \mathfrak{A}), s \in \mathbf{R}$. It is easy to check that this is a well-defined extension to $\mathfrak{A}$ as $\mathfrak{Z}$-module mappings on the $\mathfrak{Z}$-submodule $C_c(\mathbf{R}, \mathfrak{A}) \subset \mathcal{H}_{\mathcal{A}}$. These $\pi_{\mathcal{A}}(a)$ extend uniquely to $\mathfrak{Z}$-module mappings on $\mathcal{H}_{\mathcal{A}}$ since $\mathcal{H}_{\mathcal{A}}$ is also the Paschke dual of $C_c(\mathbf{R}, \mathfrak{A})$ by Lemma 6.3.

To see that $\pi_{\mathcal{A}}: \mathfrak{A} \to \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is normal, it suffices to see that $\pi_{\mathcal{A}}(\mathfrak{A})$ is ultraweakly closed in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ by [Dix, Corollaire I.4.1]. To this end, it suffices to see that the unit ball in $\pi_{\mathcal{A}}(\mathfrak{A})$ is ultraweakly closed. So let $\{a_n\}$ be a net in $\mathfrak{A}$ with $\|a_n\| = \|\pi_{\mathcal{A}}(a_n)\| \leq 1$ and $\pi_{\mathcal{A}}(a_n) \to T$ ultraweakly in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. Since the unit ball in $\mathfrak{A}$ is ultraweakly compact, we can assume (by choosing a subnet if necessary) that there is an $a \in \mathfrak{A}$ such that $a_n \to a$ ultraweakly. By Proposition 5.1 (iii), we have for all $x, y \in C_c(\mathbf{R}, \mathfrak{A})$

$$\langle x, \pi_{\mathcal{A}}(a_n)y\rangle \to \langle x, Ty\rangle \text{ ultraweakly in } \mathfrak{Z}.$$

On the other hand, if $x = cf$ and $y = bg$ for $c, b \in \mathfrak{A}$ and $f, g \in C_c(\mathbf{R})$, then one easily calculates that $\langle x, \pi_{\mathcal{A}}(a_n)y\rangle = \overline{\tau}(a_n bc^*) \int \overline{f}(t)g(t)\, dt$ which converges ultraweakly in $\mathfrak{Z}$ to $\langle x, \pi_{\mathcal{A}}(a)y\rangle$. Thus, for all such x, y we have

$$\langle x, \pi_{\mathcal{A}}(a)y\rangle = \langle x, Ty\rangle.$$

Clearly, the same equation holds for all finite linear combinations of such x and y. Since such combinations are $\|\cdot\|_2$-dense in $C_c(\mathbf{R}, \mathfrak{A})$ (and so $\|\cdot\|_{\mathfrak{Z}}$-dense) we have the equation holding for all $x, y \in C_c(\mathbf{R}, \mathfrak{A})$. Hence, for all $y \in C_c(\mathbf{R}, \mathfrak{A})$ we have

$$\pi_{\mathcal{A}}(a)y = Ty.$$

Since $\pi_{\mathcal{A}}(a)$ leaves the pre-Hilbert $\mathfrak{Z}$-module $C_c(\mathbf{R}, \mathfrak{A})$ invariant, Proposition 3.6 of [Pa] implies that $T = \pi_{\mathcal{A}}(a)$ as required. ∎

Key Idea 8 Now the natural embedding of the $\mathfrak{Z}$-module, $L^2(\mathbf{R}) \otimes_{\text{alg}} A_{\mathfrak{Z}}$ into $L^2(\mathbf{R}, A_{\mathfrak{Z}})$ induces an embedding $L^2(\mathbf{R}, A_{\mathfrak{Z}}) \hookrightarrow L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}}$ where the latter is defined to be the completion of the algebraic tensor product in the pre-Hilbert $\mathfrak{Z}$-module norm [L]. Thus we get a series of inclusions of pre-Hilbert $\mathfrak{Z}$-modules each of which is strict unless A is finite-dimensional:

$$L^2(\mathbf{R}) \otimes_{\text{alg}} A_{\mathfrak{Z}} \subset L^2(\mathbf{R}, A_{\mathfrak{Z}}) \subset L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}} \subset \mathcal{H}_{\mathcal{A}}.$$

One could insert another (generally strict) series of containments,

$$L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}} \subset L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} \mathfrak{A} \subset \mathcal{H}_{\mathcal{A}},$$

or even the diagram of containments,

$$\begin{array}{ccccc} C_c(\mathbf{R}, A_{\mathfrak{Z}}) & = & \mathcal{A} & = & C_c(\mathbf{R}, A_{\mathfrak{Z}}) \\ \cup & & & & \cap \\ C_c(\mathbf{R}) \otimes_{\text{alg}} A_{\mathfrak{Z}} & \subset & L^2(\mathbf{R}) \otimes_{\text{alg}} A_{\mathfrak{Z}} & \subset & L^2(\mathbf{R}, A_{\mathfrak{Z}}). \end{array}$$

In general, one might be able to realize $\mathcal{H}_{\mathcal{A}}$ as some sort of collection of measurable L^2-functions from $\mathbf{R}$ into the $\mathfrak{Z}$-module $\mathcal{H}_{A_{\mathfrak{Z}}} = \mathcal{H}_{\mathfrak{A}}$; however, this does not seem

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particularly useful, so we refrain from exploring this idea further. The important point is that each of these $\mathfrak{Z}$-modules has the *same* Paschke dual $\mathcal{H}_{\mathcal{A}}$ and so we can define operators in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ by defining bounded adjointable $\mathfrak{Z}$-module mappings on any one of them by [Pa, Corollary 3.7]. Of course, any one such operator may or may not leave the other $\mathfrak{Z}$-modules invariant.

Proposition 8.5 *Let $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}})$.*

(i) *For $x \in \mathcal{A}$ we have $\pi(x) = (\pi_{\mathcal{A}} \times U)(x) = \int \pi_{\mathcal{A}}(x(t))U_t\,dt$, where the integral converges in the norm of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$.*

(ii) $\mathcal{U}(\mathcal{A}) = [(\pi_{\mathcal{A}} \times U)(A_{\mathfrak{Z}} \rtimes \mathbf{R})]'' = [\pi_{\mathcal{A}}(\mathfrak{A}) \cup \{U_t\}_{t\in\mathbf{R}}]''$.

(iii) $\mathcal{U}(\mathcal{A}) = [(\pi_{\mathcal{A}} \times U)(A \rtimes \mathbf{R})]''$.

Proof To see (i) we note that in the proof of Lemma 8.3(ii) it was shown that for $x, y \in \mathcal{A}$, $\pi(x)y = (\pi_{\mathcal{A}} \times U)(x)y$. By [Pa, Proposition 3.6], this implies that $\pi(x) = (\pi_{\mathcal{A}} \times U)(x)$ as elements of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. The second equality in (i) is true for any crossed product when x is a compactly supported continuous function from the group into the C^*-algebra. To see (ii) we first note that by (i),

$$
\begin{aligned}
(\pi_{\mathcal{A}} \times U)(A_{\mathfrak{Z}} \rtimes \mathbf{R}) &= (\pi_{\mathcal{A}} \times U)(C_c(\mathbf{R}, A_{\mathfrak{Z}}))^{-\|\cdot\|} \\
&= (\pi_{\mathcal{A}} \times U)(\mathcal{A})^{-\|\cdot\|} \\
&= \pi(\mathcal{A})^{-\|\cdot\|}.
\end{aligned}
$$

Hence, $\mathcal{U}(\mathcal{A}) = [\pi(\mathcal{A})]'' = [\pi(\mathcal{A})^{-\|\cdot\|}]'' = [(\pi_{\mathcal{A}} \times U)(A_{\mathfrak{Z}} \rtimes \mathbf{R})]''$. Now by the Commutation Theorem 4.11 $\mathcal{U}(\mathcal{A}) = (\pi'(\mathcal{A}))'$ and it is an easy calculation that $\pi_{\mathcal{A}}(A_{\mathfrak{Z}}) \subset (\pi'(\mathcal{A}))'$. Since the representation $\pi_{\mathcal{A}}$ is ultraweakly continuous on $\mathfrak{A}$ and $A_{\mathfrak{Z}}$ is ultraweakly dense in $\mathfrak{A}$, we see that

$$\pi_{\mathcal{A}}(\mathfrak{A}) = \pi_{\mathcal{A}}(A_{\mathfrak{Z}})^{-\mathrm{uw}} \subset \big(\pi'(\mathcal{A})\big)' = \mathcal{U}(\mathcal{A}).$$

It is a straightforward calculation (since the operators U_t leave $\mathcal{A}$ invariant) that

$$\{U_t\}_{t\in\mathbf{R}} \subset \big(\pi'(\mathcal{A})\big)' = \mathcal{U}(\mathcal{A}).$$

Thus, $[\pi_{\mathcal{A}}(\mathfrak{A}) \cup \{U_t\}_{t\in\mathbf{R}}]'' \subset \mathcal{U}(\mathcal{A})$.

On the other hand, if $T \in [\pi_{\mathcal{A}}(\mathfrak{A}) \cup \{U_t\}_{t\in\mathbf{R}}]'$, then $T \in [\pi_{\mathcal{A}}(A_{\mathfrak{Z}}) \cup \{U_t\}_{t\in\mathbf{R}}]'$ and by the full force of (i), we see that $T \in (\pi(\mathcal{A}))' = \mathcal{U}(\mathcal{A})'$ by Theorem 4.11. That is,

$$[\pi_{\mathcal{A}}(\mathfrak{A}) \cup \{U_t\}_{t\in\mathbf{R}}]' \subset \mathcal{U}(\mathcal{A})' \quad \text{or} \quad [\pi_{\mathcal{A}}(\mathfrak{A}) \cup \{U_t\}_{t\in\mathbf{R}}]'' \supset \mathcal{U}(\mathcal{A})$$

as required.

To see (iii), we observe that since A is ultraweakly dense in $\mathfrak{A}$, Lemma 8.4 implies that $\pi_{\mathcal{A}}(\mathfrak{A}) = \pi_{\mathcal{A}}(A)'' \subset [(\pi_{\mathcal{A}} \times U)(A \rtimes \mathbf{R})]''$. Since

$$\{U_t\}_{t\in\mathbf{R}} \subset [(\pi_{\mathcal{A}} \times U)(A \rtimes \mathbf{R})]'',$$

we have by (ii) that $\mathcal{U}(\mathcal{A}) \subset [(\pi_{\mathcal{A}} \times U)(A \rtimes \mathbf{R})]''$. The other containment is trivial. ∎

Definition 8.6 (The Induced Representation) Now there is another representation of $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}})$ (and hence $A_{\mathfrak{Z}} \rtimes \mathbf{R}$) on $\mathcal{H}_{\mathcal{A}}$ that is unitarily equivalent to $\pi = \pi_{\mathcal{A}} \times U$. In the remainder of the paper we will use the standard notation for this representation, namely Ind (see below). Later when we define the notion of index, we will use the notation Index to avoid confusion. To define the representation Ind we first define a single unitary $V \in \mathcal{L}(\mathcal{H}_{\mathcal{A}})$ via $(V\xi)(t) = \overline{\alpha}_t^{-1}(\xi(t))$ for $\xi \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$. One easily checks that V is a bounded, adjointable $\mathfrak{Z}$-module mapping on the $\mathfrak{Z}$-module $L^2(\mathbf{R}, A_{\mathfrak{Z}})$ and therefore on $\mathcal{H}_{L^2(\mathbf{R},A_{\mathfrak{Z}})} = \mathcal{H}_{\mathcal{A}}$ by the previous remarks. One easily checks that for $a \in A_{\mathfrak{Z}}$, $t \in \mathbf{R}$, and $\xi \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$

$$V\pi_{\mathcal{A}}(a)V^* = \widetilde{\pi}(a) \quad \text{and} \quad VU_tV^* = \lambda_t,$$

where

$$(\widetilde{\pi}(a)\xi)(s) = \overline{\alpha}_s^{-1}(a)\xi(s) \quad \text{and} \quad (\lambda_t\xi)(s) = \xi(s-t).$$

Another straightforward calculation shows that for $x, \xi \in \mathcal{A}$

$$(V\pi(x)V^*\xi)(s) = \int \overline{\alpha}_s^{-1}(x(t))\xi(s-t)\,dt,$$

and that this formula easily extends to $\xi \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$.

Now if $x \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$, $\pi(x)$ is bounded and $\xi \in \mathcal{A}$, then using the formula of Lemma 8.3 (ii) one easily calculates that we obtain the same formula, namely

$$(V\pi(x)V^*\xi)(s) = \int \overline{\alpha}_s^{-1}(x(t))\xi(s-t)\,dt.$$

Since this representation of $A_{\mathfrak{Z}} \rtimes \mathbf{R}$, $x \mapsto V\pi(x)V^*$ is induced from the left multiplication of $A_{\mathfrak{Z}}$ on itself via the action of $\mathbf{R}$ on $A_{\mathfrak{Z}}$, we denote it by $\mathrm{Ind}(x)$, that is, $\mathrm{Ind}(x) := V\pi(x)V^*$. Now the von Neumann algebra, $\mathcal{U}(\mathcal{A})$ contains the representations $\pi_{\mathcal{A}}$ of $A_{\mathfrak{Z}}$ and U of $\mathbf{R}$, which integrate to give the representation $\pi = \pi_{\mathcal{A}} \times U$ of $\mathcal{A}$ (and hence of $A_{\mathfrak{Z}} \rtimes \mathbf{R}$) in $\mathcal{U}(\mathcal{A})$. We define the von Neumann algebra $\mathcal{M} = V\mathcal{U}(\mathcal{A})V^*$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ which also has centre $\mathfrak{Z}$ and is unitarily equivalent to $\mathcal{U}(\mathcal{A})$ but for which the machinery of $\mathfrak{Z}$-Hilbert algebras is not directly applicable. So $\mathcal{M}$ is generated by the representations, $\widetilde{\pi}(\cdot) := V\pi_{\mathcal{A}}(\cdot)V^*$ of $A_{\mathfrak{Z}}$ and $\lambda_{(\cdot)} := VU_{(\cdot)}V^*$ of $\mathbf{R}$. The integrated representation $\widetilde{\pi} \times \lambda$ is, of course, Ind. The trace on $\mathcal{M}$ is denoted by $\widehat{\tau}$ and is defined on $\mathcal{M}^{\widehat{\tau}} := V\mathcal{U}(\mathcal{A})^{\sigma}V^*$ via $\widehat{\tau}(T) := \sigma(V^*TV)$. It follows from Lemma 8.3 (iii) that if $x, y \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$ and if $\pi(x)$ and $\pi(y)$ are bounded, then the operator $\mathrm{Ind}(x)^*\,\mathrm{Ind}(y)$ is in the ideal of definition of the $\mathfrak{Z}$-trace, $\widehat{\tau}$ on $\mathcal{M}$ and

$$\widehat{\tau}[\mathrm{Ind}(x)^*\,\mathrm{Ind}(y)] = \widehat{\tau}[V\pi(x)^*\pi(y)V^*] = \langle x, y\rangle = \int \overline{\tau}(x(t)^*y(t))\,dt.$$

Definition 8.7 (The Hilbert Transform) The Hilbert transform $H_{\mathbf{R}}$ on $L^2(\mathbf{R})$ is defined for $\xi \in L^2(\mathbf{R})$ by $H_{\mathbf{R}}(\xi) = (\widehat{\xi}\,\mathrm{sgn})^{\vee}$, where $\widehat{\ }, {}^{\vee}$ are the usual Fourier transform and inverse transform and sgn is the usual signum function on $\mathbf{R}$.

Then $H_{\mathbf{R}}$ is a self-adjoint unitary, so that $H_{\mathbf{R}}^2 = 1$ and $P_{\mathbf{R}} := \frac{1}{2}(H_{\mathbf{R}} + 1)$ is the projection onto the Hardy space, $\mathcal{H}^2(\mathbf{R})$. By [L], $H := H_{\mathbf{R}} \otimes 1$ and $P := P_{\mathbf{R}} \otimes 1$ define bounded adjointable $\mathfrak{Z}$-module maps on $L^2(\mathbf{R}) \otimes_{\mathrm{alg}} A_{\mathfrak{Z}}$ (and therefore on $\mathcal{H}_{\mathcal{A}}$) with the same properties. That is, $H^2 = 1$ and $P = \frac{1}{2}(H+1)$ satisfies $P = P^* = P^2$.

In the lemma below, we identify $L^2(\mathbf{R})$ with $L^2(\mathbf{R}) \cdot 1_A$ inside $L^2(\mathbf{R}, A_{\mathfrak{Z}})$.

Lemma 8.8 *The operators H and P are in $\mathcal{M}$. In fact, if for $\epsilon > 0$ we define the function f_ϵ in $L^2(\mathbf{R}) \subset L^2(\mathbf{R}, A_{\mathfrak{Z}}) \subset \mathcal{H}_{\mathcal{A}}$ via $f_\epsilon(t) = \frac{1}{\pi i t}$ for $|t| \geq \epsilon$, then the $\pi(f_\epsilon)$ (technically,*

$\pi(f_\epsilon \cdot 1_A)$) are uniformly bounded, and as $\epsilon \to 0$, $\operatorname{Ind}(f_\epsilon) = V\pi(f_\epsilon)V^* \to H$ strongly on $L^2(\mathbf{R}) \otimes \overline{A}_{\mathfrak{Z}}$, so $\operatorname{Ind}(f_\epsilon) = V\pi(f_\epsilon)V^* \to H$ ultraweakly on $\mathcal{H}_{\mathcal{A}}$.

Proof It follows from [DM] that left convolution by the functions f_ϵ, $\lambda(f_\epsilon)$ are uniformly bounded on $L^2(\mathbf{R})$ and converge strongly to $H_{\mathbf{R}}$. It is trivial then that $\lambda(f_\epsilon)\otimes 1$ converges strongly to $H_{\mathbf{R}}\otimes 1$ on $L^2(\mathbf{R})\otimes_{\mathrm{alg}} A_{\mathfrak{Z}}$. Since these operators are all uniformly bounded, adjointable $\mathfrak{Z}$-module maps by [L], we see by the usual $\delta/3$-argument that their extensions to the completion $L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}}$ satisfy $\lambda(f_\epsilon) \otimes 1 \to H_{\mathbf{R}} \otimes 1 = H$ strongly on $L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}}$. It now follows from Lemma 5.1 (iii) (with $L^2(\mathbf{R}) \otimes_{\mathfrak{Z}} A_{\mathfrak{Z}}$ in place of $\mathcal{A}$) and Key Idea 8 that $\lambda(f_\epsilon) \otimes 1 \to H$ ultraweakly on $\mathcal{H}_{L^2(\mathbf{R})\otimes_{\mathfrak{Z}} A_{\mathfrak{Z}}} = \mathcal{H}_{\mathcal{A}}$. It remains to see that $\lambda(f_\epsilon) \otimes 1 = \operatorname{Ind}(f_\epsilon)$ on $\mathcal{H}_{\mathcal{A}}$. Now the former is initially defined on $L^2(\mathbf{R}) \otimes_{\mathrm{alg}} A_{\mathfrak{Z}}$ while the latter is initially defined on $V(\mathcal{A}) = \mathcal{A}$. Since they are both defined on the common dense domain $C_c(\mathbf{R}) \otimes A_{\mathfrak{Z}}$, it suffices to check equality there. This is a trivial calculation. ■

Remark It follows from Lemma 8.8 that for $\xi \in \mathcal{A}$

$$H(\xi) = \operatorname{norm}\lim_{\epsilon\to 0} V\pi(f_\epsilon)V^*\xi.$$

And since $V\pi(f_\epsilon)V^*\xi(s) = \int f_\epsilon(t)\xi(s-t)\,dt = \int_{|t|\geq 0} \frac{1}{\pi i t}\xi(s-t)\,dt$ for $s \in \mathbf{R}$, we can formally write $(H\xi)(s) = \int \frac{1}{\pi i t}\xi(s-t)\,dt$ for $\xi \in \mathcal{A}$ and $s \in \mathbf{R}$, where we understand the integral to be the principal-value integral converging in the norm of $\mathcal{H}_{\mathcal{A}}$.

9 The Index Theorem

We quickly recap for the benefit of the reader what we have done so far. We begin with a unital C^*-algebra A and a unital C^*-subalgebra Z of the centre of A. We assume that we have a faithful, unital Z-trace τ and a continuous action $\alpha\colon \mathbf{R} \to \operatorname{Aut}(A)$ leaving τ and hence Z invariant. In short, the 4-tuple (A, Z, τ, α) is our object of study. As *Standing Assumptions*, we will assume that we have a concrete $*$-representation of A on a Hilbert space $\mathcal{H}$ which carries a *faithful*, unital uw-continuous $\mathfrak{Z}$-trace $\overline{\tau}\colon \mathfrak{A} \to \mathfrak{Z}$ extending τ, where, as before $\mathfrak{A}$ and $\mathfrak{Z}$ denoted respectively the ultraweak closures of A and Z on $\mathcal{H}$. Since A is concretely represented on this Hilbert space, we do not carry a special notation for this representation. Moreover there is an ultraweakly continuous action $\overline{\alpha}\colon \mathbf{R} \to \operatorname{Aut}(\mathfrak{A})$ extending α and leaving $\overline{\tau}$ and $\mathfrak{Z}$ invariant. If Z has a faithful state ω, then the GNS representation of the state $\overline{\omega} = \omega \circ \tau$ gives us a representation of A satisfying the Standing Assumptions by Proposition 2.1.

We defined $A_{\mathfrak{Z}}$ to be the C^*-subalgebra of $\mathfrak{A}$ generated by A and $\mathfrak{Z}$, so that $\overline{\alpha}$ restricts to a norm-continuous action of $\mathbf{R}$ on $A_{\mathfrak{Z}}$ and $\overline{\tau}$ restricts to a faithful, unital $\mathfrak{Z}$-trace on $A_{\mathfrak{Z}}$. We defined $\mathcal{A} = C_c(\mathbf{R}, A_{\mathfrak{Z}})$ to be a $*$-algebra with the usual $\overline{\alpha}$-twisted convolution multiplication. There is a natural (right) pre-Hilbert $\mathfrak{Z}$-module structure on $\mathcal{A}$ making it into a $\mathfrak{Z}$-Hilbert algebra as defined in Section 3. We defined $\mathcal{H}_{\mathcal{A}}$ to be the Paschke dual of all bounded $\mathfrak{Z}$-module mappings from $\mathcal{A}$ to $\mathfrak{Z}$ (*i.e.*, all $\mathfrak{Z}$-linear "$\mathfrak{Z}$-valued functionals" on $\mathcal{A}$). Then $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a Type I von Neumann algebra with centre $\mathfrak{Z}$. The point of this set-up is that the von Neumann subalgebra $\mathcal{U}(\mathcal{A})$ of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ generated by the left multiplications $\pi(x)$ of $\mathcal{A}$ on $\mathcal{H}_{\mathcal{A}}$ contains $\mathfrak{Z}$ in its centre and has a faithful, normal, semifinite $\mathfrak{Z}$-trace σ defined on the two-sided ideal

$\mathcal{U}(\mathcal{A})^\sigma = \pi(\mathcal{A}_b^2)$ via $\sigma(\pi(\xi\eta)) = \langle \xi^*, \eta\rangle$, for $\xi, \eta \in \mathcal{A}_b$ the (full) $\mathfrak{Z}$-Hilbert algebra of (left) bounded elements in $\mathcal{H}_{\mathcal{A}}$.

At this point we look at a von Neumann algebra $\mathcal{M} = V\mathcal{U}(\mathcal{A})V^*$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ which also contains $\mathfrak{Z}$ in its centre. Then $\mathcal{M}$ is generated by representations $\widetilde{\pi}(\cdot) := V\pi_{\mathcal{A}}(\cdot)V^*$ of $A_{\mathfrak{Z}}$ and $\lambda_{(\cdot)} := VU_{(\cdot)}V^*$ of $\mathbf{R}$. The integrated representation $\widetilde{\pi}\times\lambda$ is denoted by Ind. The canonical trace on $\mathcal{M}$ is denoted by $\widehat{\tau}$ and has domain of definition $\mathcal{M}^{\widehat{\tau}} = \{S \in \mathcal{M} \mid S = V\pi(\xi\eta)V^* \text{ some } \xi, \eta \in \mathcal{A}_b\}$. And for $S = V\pi(\xi\eta)V^*$, $\widehat{\tau}(S) = \langle \xi^*, \eta\rangle$. In particular, if $x, y \in L^2(\mathbf{R}, A_{\mathfrak{Z}})$ with $\pi(x)$ and $\pi(y)$ bounded, then the operator $\mathrm{Ind}(x)^*\,\mathrm{Ind}(y)$ is in the ideal of definition of the $\mathfrak{Z}$-trace, $\widehat{\tau}$ on $\mathcal{M}$, and

$$\widehat{\tau}[\mathrm{Ind}(x)^*\,\mathrm{Ind}(y)] = \int \overline{\tau}(x(t)^*y(t))\,dt.$$

Definition 9.1 We consider the semifinite von Neumann algebra, $\mathcal{N} := P\mathcal{M}P$ with the faithful, normal, semifinite $\mathfrak{Z}$-trace obtained by restricting $\widehat{\tau}$. For $a \in A$ we define the Toeplitz operator $T_a := P\widetilde{\pi}(a)P \in \mathcal{N}$.

We recall from Section 1 that δ is the infinitesimal generator of α on A and that

$$a \mapsto \frac{1}{2\pi i}\tau(\delta(a)a^{-1})\colon \mathrm{dom}(\delta)^{-1} \to Z_{sa}$$

is a group homomorphism which is constant on connected components and so extends uniquely to a group homomorphism $A^{-1} \to Z_{sa}$ which is constant on connected components and is 0 on Z^{-1}. With this convention and all the above notation, we state our index theorem. Much of the work that we have done so far is to make sense of the statement of the following theorem and to make sense of the index calculations of [CMX, PhR] in this generality. It is interesting that the conclusions of the theorem are insensitive to the choice of a suitable representation of A satisfying the standing assumptions. In particular, if the representation is chosen using Proposition 2.1, the conclusions of the theorem are insensitive to the choice of a faithful state on Z.

Theorem 9.2 *Let A be a unital C^*-algebra and let $Z \subseteq Z(A)$ be a unital C^*-subalgebra of the centre of A. Let $\tau\colon A \to Z$ be a faithful, unital Z-trace that is invariant under a continuous action α of $\mathbf{R}$. Then for any $a \in A^{-1} \cap \mathrm{dom}(\delta)$, the Toeplitz operator T_a is Fredholm relative to the trace $\widehat{\tau}$ on $\mathcal{N} = P(\mathrm{Ind}(A \rtimes \mathbf{R})'')P$, and*

$$\widehat{\tau}\text{-}\,\mathrm{Index}(T_a) = \frac{-1}{2\pi i}\tau\big(\,\delta(a)a^{-1}\big).$$

We follow the second proof of [CMX, Section 25.2] (*cf.* [PhR, Section 3]). Now relative to the decomposition $1 = P + (1-P)$, we see that

$$\widetilde{\pi}(a) = \begin{bmatrix} T_a & B \\ C & D \end{bmatrix},$$

where $B = P\widetilde{\pi}(a)(1-P) = P[P, \widetilde{\pi}(a)] = \frac{1}{2}P[H, \widetilde{\pi}(a)]$, and similarly,

$$C = \frac{1}{2}[H, \widetilde{\pi}(a)]P.$$

Thus, we are led to calculate the general commutator $[H, \widetilde{\pi}(a)]$ for $a \in \mathrm{dom}(\delta)$.

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Lemma 9.3 *For any $a \in \mathrm{dom}(\delta)$, $[H, \widetilde{\pi}(a)]$ belongs to $\mathcal{M}_2^{\widehat{\tau}}$. In fact, $[H, \widetilde{\pi}(a)] = \mathrm{Ind}(x)$, where $x \in C_0(\mathbf{R}, A_3) \cap L^2(\mathbf{R}, A_3)$ is given by $x(t) = \frac{\alpha_t(a) - a}{\pi i t}$.*

Proof Now $\mathrm{Ind}(f_\epsilon)$ converges strongly on $\mathcal{A}$ to H, so we easily compute for $\xi \in \mathcal{A}$, $[\mathrm{Ind}(f_\epsilon), \widetilde{\pi}(a)]\xi = \mathrm{Ind}(x_\epsilon)\xi$, where

$$x_\epsilon(t) = \begin{cases} \frac{\alpha_t(a) - a}{\pi i t} & |t| \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

So the $\mathrm{Ind}(x_\epsilon)$ are uniformly bounded operators that converge pointwise on $\mathcal{A}$ to $[H, \widetilde{\pi}(a)]$. Now since $x(t) \to (\pi i)^{-1}\delta(a)$ as $t \to 0$ and

$$\|x(t)\|^2 \leq \frac{4\|a\|^2}{\pi^2 t^2},$$

we see that $x \in C_0(\mathbf{R}, A_3) \cap L^2(\mathbf{R}, A_3)$. One easily calculates that for $\xi \in \mathcal{A}$

$$\|\mathrm{Ind}(x)\xi - \mathrm{Ind}(x_\epsilon)\xi\|_3 \leq \|\mathrm{Ind}(x)\xi - \mathrm{Ind}(x_\epsilon)\xi\|_2 \to 0,$$

and so $\mathrm{Ind}(x)$ and $[H, \widetilde{\pi}(a)]$ agree on $\mathcal{A}$. That is, by the discussion in Proposition 8.5 $\pi(x) = V^* \mathrm{Ind}(x) V$ is left bounded and $\mathrm{Ind}(x) = [H, \widetilde{\pi}(a)]$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. ∎

We want to use the $\mathfrak{Z}$-trace version of Hörmander's formula (Theorem A3 and Corollary A4 in the Appendix) to calculate the $\widehat{\tau}$-index of the Toeplitz operator T_a as $\widehat{\tau}([T_a, T_{a^{-1}}])$. So we are led to examine such commutators in the hopes that they are in fact trace-class (they are).

Corollary 9.4 *If $a, b \in \mathrm{dom}(\delta)$, we have $T_a T_b - T_{ab} \in \mathcal{M}^{\widehat{\tau}} \cap \mathcal{N} = \mathcal{N}^{\widehat{\tau}}$. In particular, if $b = a^{-1}$, then T_a and T_b are $\widehat{\tau}$-Fredholm operators in $\mathcal{N}$. In general, if $ab = ba$, then $[T_a, T_b] \in \mathcal{N}^{\widehat{\tau}}$.*

Proof We easily calculate (see [PhR, Corollary 3.3])

$$T_a T_b - T_{ab} = P\widetilde{\pi}(a)(P - 1)\widetilde{\pi}(b)P = \cdots = \frac{1}{4}P[H, \widetilde{\pi}(a)][H, \widetilde{\pi}(b)]P \tag{9.1}$$

which is in $\mathcal{M}^{\widehat{\tau}} \cap P\mathcal{M}P = \mathcal{N}^{\widehat{\tau}}$. If $ab = ba$, then

$$[T_a, T_b] = (T_a T_b - T_{ab}) + (T_{ba} - T_b T_a) \in \mathcal{N}^{\widehat{\tau}}.$$ ∎

Discussion In the case that $a, b \in \mathrm{dom}(\delta)$ commute we have by equation (9.1) and a small calculation:

$$[T_a, T_b] = P\widetilde{\pi}(a)(P - 1)\widetilde{\pi}(b)P - P\widetilde{\pi}(b)(P - 1)\widetilde{\pi}(a)P \tag{9.2}$$

$$= \cdots = \frac{1}{2}P(\widetilde{\pi}(a)H\widetilde{\pi}(b) - \widetilde{\pi}(b)H\widetilde{\pi}(a))P, \tag{9.3}$$

and both of these terms are trace-class. Applying the trace to equation (9.3) we get:

$$\widehat{\tau}([T_a, T_b]) = \frac{1}{2}\widehat{\tau}\Big(P\big(\widetilde{\pi}(a)H\widetilde{\pi}(b) - \widetilde{\pi}(b)H\widetilde{\pi}(a)\big)P\Big).$$

On the other hand, applying the trace to equation (9.2), using the cyclic property of the trace and a little calculation (see [PhR]) we get

$$\widetilde{\tau}([T_a, T_b]) = \frac{1}{2}\widetilde{\tau}\Big(\big(1-P\big)\big(\widetilde{\pi}(a)H\widetilde{\pi}(b) - \widetilde{\pi}(b)H\widetilde{\pi}(a)\big)(1-P)\Big). \tag{9.4}$$

Defining $T := \widetilde{\pi}(a)H\widetilde{\pi}(b) - \widetilde{\pi}(b)H\widetilde{\pi}(a)$, and averaging equations (9.3) and (9.4), we get

$$\widetilde{\tau}([T_a, T_b]) = \frac{1}{4}\widetilde{\tau}(PTP + (1-P)T(1-P)),$$

and both of these terms are trace-class. Unfortunately, T itself is not usually trace-class. However, T is in $\mathcal{M}_2^{\widetilde{\tau}}$ by the following lemma.

Lemma 9.5 (*cf.* [PhR, Lemma 3.4]) *Suppose $a, b \in \operatorname{dom}(\delta)$ and $ab = ba$. Then*

$$T = \widetilde{\pi}(a)H\widetilde{\pi}(b) - \widetilde{\pi}(b)H\widetilde{\pi}(a)$$

belongs to $\mathcal{M}_2^{\widetilde{\tau}}$; in fact, it has the form $\operatorname{Ind}(y)$ where y is the function in $C_0(\mathbf{R}, A_3) \cap L^2(\mathbf{R}, A_3)$ given by $y(t) = (\pi i t)^{-1}(a\alpha_t(b) - b\alpha_t(a))$.

Proof It is straightforward to verify that we can also write

$$T = [H, \widetilde{\pi}(b)]\widetilde{\pi}(a) - [H, \widetilde{\pi}(a)]\widetilde{\pi}(b).$$

Then by Lemma 9.3 we see that $T = \operatorname{Ind}(y)$ where

$$y(t) = \frac{(\alpha_t(b) - b)\alpha_t(a)}{\pi i t} - \frac{(\alpha_t(a) - a)\alpha_t(b)}{\pi i t} = \frac{a\alpha_t(b) - b\alpha_t(a)}{\pi i t}.$$

Since $y(t) \to (\pi i)^{-1}(\delta(b)a - \delta(a)b)$ in the norm of A as $t \to 0$, y is a continuous A-valued function. As $\|y(t)\| \le 2\|a\|\,\|b\|/\pi t$ for $t \neq 0$, we also see that $y \in L^2(\mathbf{R}, A_3)$. ∎

Remark In Lemma 9.5 $y(0) = (\pi i)^{-1}(\delta(b)a - \delta(a)b) = -2(\pi i)^{-1}\delta(a)b$. Combining this with equation (7) of the previous discussion would yield the desired formula, $\widetilde{\tau}([T_a, T_b]) = \frac{-1}{2\pi i}\widetilde{\tau}(\delta(a)b)$, assuming that the operator T is trace-class. Since T is generally not trace-class, we need an approximate identity argument.

Lemma 9.6 *If $S \in \mathcal{M}^{\widetilde{\tau}}$ and $\{f_n\}$ is a sequence of functions in $C_c(\mathbf{R})^+ \subset C_c(\mathbf{R}, A_3)$ each having integral* 1 *and symmetric supports about* 0 *shrinking to* 0*, then*

$$\widetilde{\tau}(S) = \operatorname*{uw\text{-}lim}_{n\to\infty} \widetilde{\tau}(\operatorname{Ind}(f_n)S).$$

Proof As in the proof of Lemma 8.7, we see that the operators $\mathrm{Ind}(f_n) = V\pi(f_n)V^*$ are uniformly bounded on $\mathcal{H}_{\mathcal{A}}$ by 1 and converge strongly to 1 on $L^2(\mathbf{R})\otimes\overline{A_3}$. In particular, for all $x, y \in \mathcal{A}$ we have by Paschke's Cauchy–Schwarz inequality [Pa, Proposition. 2.3]:

$$\begin{aligned}\widehat{\tau}[\mathrm{Ind}(x)\,\mathrm{Ind}(y)] = \langle x^*, y\rangle = \langle y^*, x\rangle &= \operatorname{norm}\lim_{n\to\infty}\langle y^*, \pi(f_n)x\rangle\\ &= \operatorname{norm}\lim_{n\to\infty}\langle (f_n x)^*, y\rangle = \operatorname{norm}\lim_{n\to\infty}\widehat{\tau}[\mathrm{Ind}(f_n x)\,\mathrm{Ind}(y)]\\ &= \operatorname{norm}\lim_{n\to\infty}\widehat{\tau}[\mathrm{Ind}(f_n)\,\mathrm{Ind}(x)\,\mathrm{Ind}(y)].\end{aligned}$$

Now by Lemma 5.1 (iii) we see that for all $\xi, \eta \in \mathcal{A}_b$

$$\widehat{\tau}[\mathrm{Ind}(\xi)\,\mathrm{Ind}(\eta)] = \operatorname*{uw-lim}_{n\to\infty}\widehat{\tau}[\mathrm{Ind}(f_n)\,\mathrm{Ind}(\xi)\,\mathrm{Ind}(\eta)].$$

Since every $S \in \mathcal{M}^{\widehat{\tau}}$ has the form $S = \mathrm{Ind}(\xi)\,\mathrm{Ind}(\eta)$ for some $\xi, \eta \in \mathcal{A}_b$, we are done. ∎

Proposition 9.7 *If $a, b \in \mathrm{dom}(\delta)$ and $ab = ba$, then $[T_a, T_b] \in \mathcal{N}^{\widehat{\tau}}$ and*

$$\widehat{\tau}[T_a, T_b] = \frac{-1}{2\pi i}\tau(\delta(a)b).$$

Proof Let $\{f_n\}$ be as in the previous lemma. Then by equation (7) of the Discussion, the previous two lemmas, and the fact that $\mathrm{Ind}(f_n)P = P\,\mathrm{Ind}(f_n)$ we get

$$\begin{aligned}\widehat{\tau}([T_a, T_b]) &= \frac{1}{4}\widehat{\tau}\big(PTP + (1-P)T(1-P)\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(\mathrm{Ind}(f_n)(PTP + (1-P)T(1-P))\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(\mathrm{Ind}(f_n)PTP + \mathrm{Ind}(f_n)(1-P)T(1-P)\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(P\,\mathrm{Ind}(f_n)TP + (1-P)\,\mathrm{Ind}(f_n)T(1-P)\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(P\,\mathrm{Ind}(f_n)T + (1-P)\,\mathrm{Ind}(f_n)T\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(\mathrm{Ind}(f_n)T\big)\\ &= \operatorname{uw-lim}\frac{1}{4}\widehat{\tau}\big(\mathrm{Ind}(f_n)\,\mathrm{Ind}(y)\big)\\ &= \operatorname{uw-lim}\frac{1}{4\pi i}\int f_n(t)\tau\Big(\frac{\alpha_t(b)-b}{t}a - \frac{\alpha_t(a)-a}{t}b\Big)\,dt.\end{aligned}$$

In fact, this last limit is easily seen to converge in norm so that using $\tau(\delta(ab)) = 0$ we get

$$\widehat{\tau}([T_a, T_b]) = \frac{1}{4\pi i}\tau\big(\delta(b)a - \delta(a)b\big) = \frac{-1}{2\pi i}\tau\big(\delta(a)b\big).$$ ∎

Proof of Theorem 9.2 Recall that relative to the decomposition $1 = P + (1-P)$ we have

$$\widetilde{\pi}(a) = \begin{bmatrix} T_a & B \\ C & D\end{bmatrix},$$

where $B = P\widetilde{\pi}(a)(1-P) = P[P,\widetilde{\pi}(a)] = \frac{1}{2}P[H,\widetilde{\pi}(a)] \in \mathcal{M}_2^{\widehat{\tau}}$, and

$$C = \frac{1}{2}[H,\widetilde{\pi}(a)]P \in \mathcal{M}_2^{\widehat{\tau}}.$$

By Corollary A4 of the Appendix and the previous proposition we have

$$\widehat{\tau}\text{-}\operatorname{Index}(T_a) = \widehat{\tau}([T_a, T_{a^{-1}}]) = \frac{-1}{2\pi i}\tau(\delta(a)a^{-1}).$$

This completes the proof of Theorem 9.2. ■

Corollary 9.8 *Suppose $\varphi: A_1 \to A_2$ defines a morphism from $(A_1, Z_1, \tau_1, \alpha^1)$ to $(A_2, Z_2, \tau_2, \alpha^2)$ and $a \in A_1^{-1} \cap (\operatorname{dom}(\delta_1))$. Then $\varphi(a) \in A_2^{-1} \cap (\operatorname{dom}(\delta_2))$ and $\widehat{\tau}_1$–$\operatorname{Index}(T_a) \in (Z_1)_{sa}$ while $\widehat{\tau}_2$–$\operatorname{Index}(T_{\varphi(a)}) \in (Z_2)_{sa}$ and also $\varphi(\widehat{\tau}_1$–$\operatorname{Index}(T_a)) = \widehat{\tau}_2 - \operatorname{Index}(T_{\varphi(a)})$.*

Proof This follows immediately from Proposition 1.3 and Theorem 9.2. ■

10 Examples

10.1 Kronecker (Scalar Trace) Example

Recall that $A = C(\mathbf{T}^2)$, the C^*-algebra of continuous functions on the 2-torus, with the usual scalar trace τ given by the Haar measure on $\mathbf{T}^2$ and $\alpha: \mathbf{R} \to \operatorname{Aut}(A)$ is the Kronecker flow on A determined by the real number μ. That is, for $s \in \mathbf{R}$, $f \in A$, and $(z,w) \in \mathbf{T}^2$ we have $(\alpha_s f)(z,w) = f(e^{-2\pi is}z,\, e^{-2\pi i\mu s}w)$. In this case, $Z = \mathfrak{Z} = \mathbf{C}$ and so $A_{\mathfrak{Z}} = A$. Hence our $\mathfrak{Z}$-Hilbert algebra $\mathcal{A} = C_c(\mathbf{R}, A)$ is just a Hilbert algebra in the ordinary sense and $\mathcal{H}_{\mathcal{A}} = L^2(\mathbf{R}, L^2(\mathbf{T}^2))$. Now denoting $\mathcal{H} = L^2(\mathbf{T}^2)$, we have that the C^*-crossed product $A \rtimes_\alpha \mathbf{R}$ is represented on $L^2(\mathbf{R},\mathcal{H})$ by the induced representation of Definition 8.6 as follows: for $s, t \in \mathbf{R}$, $\xi \in C_c(\mathbf{R}, A) \subseteq L^2(\mathbf{R},\mathcal{H})$, and $f \in A$ we define

$$\big(\widetilde{\pi}(f)\,\xi\big)(s) = \alpha_s^{-1}(f)\cdot\xi(s) \quad \text{and} \quad (\lambda_t\xi)(s) = \xi(s-t).$$

Thus, $\widetilde{\pi}\times\lambda$ is a faithful representation of $A \rtimes_\alpha \mathbf{R}$ on $L^2(\mathbf{R},\mathcal{H})$. It is well known that for μ irrational, $\mathcal{M} = \big(\widetilde{\pi}\times\lambda(A\rtimes_\alpha \mathbf{R})\big)''$ is a II_∞ factor [CMX]. In general $\mathcal{M}$ is a semifinite von Neumann algebra and $\widetilde{\pi}: A \to \mathcal{M}$. Now if δ is the densely defined derivation on A generating the representation $\alpha: \mathbf{R} \to \operatorname{Aut}(A)$ and we let $u \in U(A)$ be the function $u(z,w) = w$, then u is a smooth element for δ and $\delta(u) = -(2\pi i\mu)u$. Thus by Theorem 9.2, the Toeplitz operator $T_u := P\widetilde{\pi}(u)P$ is Fredholm relative to the trace $\widehat{\tau}$ in the semifinite von Neumann algebra $\mathcal{N} = P\mathcal{M}P$ and its index is given by

$$\widehat{\tau}\text{-}\operatorname{Index}(T_u) = \frac{-1}{2\pi i}\tau(\delta(u)u^*) = \mu.$$

10.2 General Kronecker Examples

Recall that $Z = C(X)$ is any commutative unital C^*-algebra with a faithful state ω and $\theta \in Z_{sa}$ is any self-adjoint element in Z. Recall that $A = C(\mathbf{T}^2, Z) = C(X)\otimes C(\mathbf{T}^2)$

and $\tau: A \to Z$ is given by the "slice-map" $\tau = \mathrm{id}_Z \otimes \varphi$ where φ is the trace on $C(\mathbf{T}^2)$ given by Haar measure. That is, for $f \in A = C(\mathbf{T}^2, Z)$ we have

$$\tau(f) = \int_{\mathbf{T}^2} f(z,w)\, d(z,w) \in Z,$$

and τ is a faithful, tracial, conditional expectation of A onto Z. Recall that $\overline{\omega} := \omega \circ \tau = \omega \otimes \varphi$ is a faithful (tracial) state $\overline{\omega}$ on A. We use the element $\theta \in Z_{sa}$ to define a τ-invariant action $\{\alpha_t\}$ of $\mathbf{R}$ on A, $\alpha_t(f)(x,z,w) = f(x, e^{-2\pi i t}z, e^{-2\pi i\theta(x)t}w)$, for $f \in A$, $t \in \mathbf{R}$, $x \in X$, and $z, w \in \mathbf{T}$.

Let $(\pi, \mathcal{H})$ be the GNS representation of A induced by $\overline{\omega}$. Then there is a continuous unitary representation $\{U_t\}$ of $\mathbf{R}$ on $\mathcal{H}$ so that (π, U) is covariant for α on A. Also, $\{U_t\}$ implements a uw-continuous "extension" of α to $\overline{\alpha}$ acting on $\mathfrak{A} := \pi(A)''$. Moreover, letting $\mathfrak{Z} := \pi(Z)''$, there exists a unique, faithful, unital, uw-continuous $\mathfrak{Z}$-trace $\overline{\tau} : \mathfrak{A} \to \mathfrak{Z}$ "extending" τ, and $\overline{\alpha}$ leaves $\overline{\tau}$ invariant. That is, in this representation on $\mathcal{H}$, we have that Standing Assumptions are also satisfied. We simplify our notation and write $L^2(X)$, $L^2(\mathbf{T}^2)$, $L^\infty(X)$, and $L^\infty(\mathbf{T}^2)$ for $L^2(X,\omega)$, $L^2(\mathbf{T}^2,\varphi)$, $L^\infty(X,\omega)$, and $L^\infty(\mathbf{T}^2,\varphi)$, respectively.

Then in this representation one easily verifies that $\mathcal{H} = L^2(X) \otimes L^2(\mathbf{T}^2)$ as Hilbert spaces and $\mathfrak{Z} = L^\infty(X) \otimes 1$ and $A_{\mathfrak{Z}} = L^\infty(X) \otimes C(\mathbf{T}^2)$ as C^*-algebras and $\mathfrak{A} = L^\infty(X) \overline{\otimes} L^\infty(\mathbf{T}^2)$ as von Neumann algebras.

Identifying $\mathfrak{Z} = L^\infty(X)$, our $L^\infty(X)$-Hilbert algebra is

$$\mathcal{A} = C_c(\mathbf{R}, L^\infty(X) \otimes C(\mathbf{T}^2))$$

with the $\overline{\alpha}$-twisted convolution multiplication and $L^\infty(X)$-valued inner product for $f, g \in \mathcal{A}$ given by

$$\begin{aligned}\widehat{\tau}(\mathrm{Ind}(f)^* \,\mathrm{Ind}(g)) = \langle f, g\rangle &= \int_{\mathbf{R}} \overline{\tau}\big((f(t))^* g(t)\big)\, dt \\ &= \int_{\mathbf{R}} \Big(\int_{\mathbf{T}^2} (f(t)[(z,w)])^* g(t)[(z,w)]\, d(z,w)\Big)\, dt.\end{aligned}$$

Now consider the following unitary v in A: $v(x,z,w) = w$. Then

$$\alpha_t(v)(x,z,w) = e^{-2\pi i\theta(x)t} w$$

and so $\delta(v)(x,z,w) = -2\pi i\theta(x)w$. Hence, $(\delta(v)v^*)(x,z,w) = -2\pi i \cdot \theta(x)$. Since the trace τ on A is just the slice map $\mathrm{id}_Z \otimes \varphi$, we see that $\tau(\delta(v)v^*) = -2\pi i \cdot \theta$. Hence, by Theorem 9.2, the Toeplitz operator T_v is Fredholm relative to the trace $\widehat{\tau}$ on $\mathcal{N} = P(\mathrm{Ind}(A \rtimes \mathbf{R})'')P$, and

$$\widehat{\tau}\text{- Index}(T_v) = \frac{-1}{2\pi i}\tau(\delta(v)v^*) = \theta \in C(X) = Z \hookrightarrow Z \otimes C(\mathbf{T}^2) = A.$$

10.3 Fiberings of Toeplitz Operators

Recall that for any fixed $x \in X$ (where $Z = C(X)$) the evaluation map at x yields a homomorphism from $A = Z \otimes C(\mathbf{T}^2)$ to $C(\mathbf{T}^2)$ that defines a morphism from Example 10.2 to Example 10.1 which carries θ to $\mu := \theta(x)$. Moreover this morphism carries v to $u = v(x)$. So that $\mathrm{Index}(T_u) = \mu = \theta(x) = (\mathrm{Index}(T_v))(x)$. That is, the Toeplitz operator T_v fibers over X as the Toeplitz operators $T_{\theta(x)}$ and moreover for each $x \in X$, $\mathrm{Index}(T_{v(x)}) = (\mathrm{Index}(T_v))(x)$. so the Index fibers accordingly.

Similarly for any fixed $x \in X$ (where $Z = C(X)$), the evaluation map at x yields a homomorphism from $A = Z \otimes A_\theta$ to A_θ which defines a morphism from

$$Z \otimes A_\theta, Z, \mathrm{id} \otimes \tau_\theta, \alpha^\eta)$$

to $(A_\theta, \mathbf{C}, \tau_\theta, \alpha^{\eta(x)})$. This morphism carries $1 \otimes V$ to V. Since $\mathrm{Index}(T_{1\otimes V}) = \eta$ and $\mathrm{Index}(T_V) = \eta(x)$, we see that $\mathrm{Index}(T_{1\otimes V})(x) = \mathrm{Index}(T_V) = \mathrm{Index}(T_{1\otimes V}(x))$.

10.4 C^*-algebra of the Integer Heisenberg Group

Recall that $A = C^*(H)$ is the C^*-algebra of the Integer Heisenberg group viewed as the universal C^*-algebra generated by three unitaries U, V, W satisfying $WU = UW$, $WV = VW$, and $UV = WVU$. In this case $Z = C^*(W)$ is the centre of A and also equals $C^*(C)$, the C^*-algebra generated by $C = \langle W \rangle$, the centre of H. The trace $\tau: C^*(H) \to C^*(C)$ on functions in $l^1(H) \subset C^*(H)$ is just given by restriction to C. Our Hilbert space $\mathcal{H} = l^2(H)$ is acted on by the left regular representation of $C^*(H)$. The restriction of this action to $Z = C^*(C)$ on $l^2(H) = \bigoplus_{(n,m)\in\mathbf{Z}^2} l^2(C \cdot (V^n U^m))$ is unitarily equivalent to $1_{\mathbf{Z}^2} \otimes \pi_C(C)$ on $\bigoplus_{(n,m)\in\mathbf{Z}^2} l^2(C)$. In this labelling of the cosets, multiplication by W acts the same on each coset: it increases the power of W by one. Multiplication by V acts as the identification of $l^2(C \cdot (V^n U^m))$ with

$$l^2(C \cdot (V^{n+1} U^m)),$$

that is, it acts as a permutation of the copies of $l^2(C)$ while acting on the basis elements as the identity on $l^2(C)$. However, multiplication by U not only maps $l^2(C \cdot (V^n U^m))$ to $l^2(C \cdot (V^n U^{m+1}))$, but it also acts on the basis elements of $l^2(C)$ by sending W^k to W^{k+1}. In this representation we recall that the map $\tau: C^*(H) \to C^*(C)$ is given by $\tau(x) = 1_{\mathbf{Z}^2} \otimes ExE$, where E is the projection of $l^2(H)$ onto $l^2(C)$. Thus we have an action $\alpha: \mathbf{R} \to \mathrm{Aut}(A)$ that fixes $Z = C^*(W)$ and leaves the Z-valued trace τ invariant. A short calculation using Theorem 9.2 then gives us the nontrivial index

$$\widehat{\tau}\text{-}\mathrm{Index}(T_{V^n U^m W^p}) = (n\theta + m) \in Z = C^*(W).$$

A Fredholm Theory Relative to a $\mathfrak{Z}$-valued Trace on a von Neumann Algebra

We let $\mathcal{N}$ denote a semifinite von Neumann algebra and let $\mathfrak{Z}$ denote a unital von Neumann subalgebra of the centre of $\mathcal{N}$. We suppose that we have a faithful, normal, semifinite $\mathfrak{Z}$-trace ϕ defined on $\mathcal{N}_+$ as in Definition 6.1. We will show that using ϕ as a dimension function we can adapt M. Breuer's arguments in [Br1, Br2] to obtain a Fredholm theory involving a $\mathfrak{Z}$-valued index with the usual algebraic and topological stability properties, and in which the role of the compact operators is replaced by the norm-closed ideal $\mathcal{K}_{\mathcal{N}}^{\phi}$ generated by the projections of ϕ-finite trace.

A projection E in $\mathcal{N}$ will be called ϕ-finite if $\phi(E) \in \mathfrak{Z}_+$. Since ϕ is faithful, it is clear that any ϕ-finite projection is also finite in the Murray-von Neumann sense. An operator $T \in \mathcal{N}$ is called ϕ-Fredholm if the projection N_T on $\ker(T)$ is ϕ-finite and there is a ϕ-finite projection E with $\mathrm{range}(1 - E) \subseteq \mathrm{range}(T)$. Since ϕ-finite projections are finite, every ϕ-Fredholm operator is Fredholm in Breuer's sense. If T

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is ϕ-Fredholm, the ϕ-index of T is by definition

$$\phi\text{- Index}(T) := \phi(N_T) - \phi(N_{T^*}).$$

We shall see below that T^* is also ϕ-Fredholm so that ϕ-Index(T) is a well-defined self-adjoint element of $\mathfrak{Z}$.

We observe, as we did in [PhR], that the ideal $\mathcal{K}_{\mathcal{N}}^{\phi}$ can also be described as the closure of any of

- the span of the ϕ-finite projections in $\mathcal{N}$,
- the span of the ϕ-finite elements in $\mathcal{N}$,
- the algebra of elements $T \in \mathcal{N}$ whose range projection R_T is ϕ-finite.

This ideal is clearly contained in Breuer's ideal $\mathcal{K}$ generated by all the finite projections in $\mathcal{N}$.

Now the further remarks and proofs concerning how Breuer's arguments carry over to this situation follow verbatim from [PhR, Appendix B]. So we obtain the analogues of Breuer's theorems exactly as we did in [PhR].

Theorem A.1 *Let ϕ be a faithful, normal, semifinite $\mathfrak{Z}$-trace on the von Neumann algebra $\mathcal{N}$ and let $\mathcal{K}_{\mathcal{N}}^{\phi}$ be the norm-closed ideal in $\mathcal{N}$ generated by the ϕ-finite projections.*

(i) *(The Fredholm alternative) If $T \in \mathcal{K}_{\mathcal{N}}^{\phi}$, then $(1-T)$ is ϕ-Fredholm and*

$$\phi\text{-Index}(1-T) = 0.$$

(ii) *(Atkinson's Theorem) An operator $T \in \mathcal{N}$ is ϕ-Fredholm if and only if $T + \mathcal{K}_{\mathcal{N}}^{\phi}$ is invertible in $\mathcal{N}/\mathcal{K}_{\mathcal{N}}^{\phi}$.*

(iii) *If S and T are ϕ-Fredholm, then so are S^* and ST, and we have*

$$\phi\text{-Index}(S^*) = -(\phi\text{-Index}(S)) \quad \text{and} \quad \phi\text{-Index}(ST) = \phi\text{-Index}(S) + \phi\text{-Index}(T).$$

The following corollary is proved exactly as [PhR, Corollary B2].

Corollary A.2 *The set $\mathcal{F}_\phi(\mathcal{N})$ of ϕ-Fredholm operators is open in the norm topology of $\mathcal{N}$, and the index map $T \mapsto \phi\text{-}Index(T)$ is locally constant on $\mathcal{F}_\phi(\mathcal{N})$.*

The following trace formula for the index goes back to Calderón for pseudodifferential operators. The general Type I case is due to Hörmander [H] but Connes also has an elegant proof [Co]. One of the authors generalised Hörmander's proof to the case of a factor of Type II_∞ [Ph, Theorem A7]. It is this latter proof that goes through essentially verbatim to our present setting, so we refer the reader to [Ph, Appendix A] for the proof.

Theorem A.3 *Let ϕ be a faithful, normal, semifinite $\mathfrak{Z}$-trace on the von Neumann algebra $\mathcal{N}$, and let $S, T \in \mathcal{N}$ so that $R_1 = 1 - ST$ and $R_2 = 1 - TS$ are both n-summable for some integer $n > 0$. Then T is a ϕ-Fredholm operator and*

$$\phi\text{-Index}(T) = \phi(R_1^n) - \phi(R_2^n).$$

Corollary A.4 *Let A be a unital C^*-algebra and let $Z \subseteq Z(A)$ be a unital C^*-subalgebra of the centre of A. Let $\tau\colon A \to Z$ be a faithful, unital Z-trace which is invariant* *under a continuous action α of* $\mathbf{R}$*. Then for any $a \in A^{-1} \cap \operatorname{dom}(\delta)$, the Toeplitz operator T_a is Fredholm relative to the trace $\widehat{\tau}$ on $\mathcal{N} = P(\operatorname{Ind}(A \rtimes \mathbf{R})'')P$, and*

$$\widehat{\tau}\text{-}\operatorname{Index}(T_a) = \widehat{\tau}([T_a, T_{a^{-1}}]).$$

Proof We let $T = T_a$ and $S = T_{a^{-1}}$ and $\phi = \widehat{\tau}$ in the statement of the previous theorem. Then $R_1 = 1 - T_{a^{-1}}T_a = T_{a^{-1}a} - T_{a^{-1}}T_a \in \mathcal{N}^{\widehat{\tau}}$ by Corollary 9.4, and similarly $R_2 \in \mathcal{N}^{\widehat{\tau}}$. Then by the previous theorem, T_a is $\widehat{\tau}$-Fredholm and

$$\widehat{\tau}\text{-}\operatorname{Index}(T_a) = \widehat{\tau}(R_1) - \widehat{\tau}(R_2) = \widehat{\tau}([T_a, T_{a^{-1}}]). \qquad \blacksquare$$

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