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IRREDUCIBILITY OF SOME UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP WITH RESPECT TO THE POINCARÉ SUBSEMIGROUP, II

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Let $P(3)$ and $P_+(3)$ be the 3-dimensional space-time Poincaré group and the Poincaré subsemigroup, that is, $P(3) = R^3 \times_s SU(1, 1)$ and $P_+(3) = V_+(3) \times_s SU(1, 1)$ where $V_+(3) = \{x_0^2 - x_1^2 - x_2^2 \geq 0, x_0 \geq 0\}$. The multiplication is defined by the formula $(x, g)(x', g') = (x + g^{*-1}x'g^{-1}, gg')$ for $x, x' \in R^3$ and $g, g' \in SU(1, 1)$. Here $x = (x_0, x_1, x_2)$ is identified with the matrix $\begin{pmatrix} x_0 & x_2 - ix_1 \\ x_2 + ix_1 & x_0 \end{pmatrix}$.

The purpose of this paper is to give an affirmative answer to the problem if there is any irreducible unitary representation of $P(3)$ such that its restriction to the semigroup $P_+(3)$ is reducible. To be more precise, we shall determine all $P_+(3)$-invariant, closed proper subspaces for the irreducible unitary representations $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ $(\eta \in R, \varepsilon = 0, 1/2)$, which are associated with the one-sheeted hyperboloid $V_{iM}(3) = \{y_0^2 - y_1^2 - y_2^2 = -M^2\}$ $(M > 0)$. As for the other irreducible unitary representations of $P(3)$ it is easy to show that they are irreducible even when they are restricted to $P_+(3)$ (see [5], Theorem 5). Recall that all the irreducible unitary representations of the 2-dimensional space-time Poincaré group are irreducible even when they are restricted to the Poincaré subsemigroup ([5], Theorem 1). Using, among other things, the results in § 1, we shall show in the forthcoming Part III that the irreducible unitary representations of the 4-dimensional space-time Poincaré group whose irreducibility relative to the Poincaré subsemigroup remains unsettled in [5] are reducible as the representations of the semigroup.

The basic tools of our approach are i) the eigenfunction expansions for second order ordinary differential operators $\mathscr{L}_{k,\eta}$ (see (1.1)), which are connected with the Laplacian of $SU(1, 1)$, and ii) rephrased versions of the Hilbert transform and the Frobenius method for ordinary differential

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equations with a regular singularity.

This paper consists of two sections and an appendix. In § 1 we enumerate closed proper subspaces of $L^2(R)$ left invariant under the self-adjoint operator $\mathscr{L}_{k,\eta}$ and a semigroup $T_t = \exp(it\,\mathrm{sh}\,\tau)$ $(t \geq 0)$ of multiplication operators (Theorems 1.1–1.3). Toward the end of § 1 we shall determine nontrivial sequences $\{D_k\}_{k\in Z_+ + \varepsilon}$ $(\varepsilon = 0, 1/2)$ of subspaces such that i) D_k is a closed, proper subspace of $L^2(R)$ left invariant under $\mathscr{L}_{k,\eta}$ and $T_t(t \geq 0)$, ii) $F_{\pm,k,\eta}D_k \subset D_{k\pm1}$, where $F_{\pm,k,\eta} = -d/d\tau + (\pm k + 1/2)\,\mathrm{th}\,\tau \pm \eta/\mathrm{ch}\,\tau$ with domain $H_2(R)$, the Sobolev space of order 2 (Theorem 1.4). In § 2 we firstly define the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ $(\eta \in R, \varepsilon = 0, 1/2)$ of the group $P(3)$, and then describe all the $P_+(3)$-invariant, closed proper subspaces $\mathscr{D}^{\eta,\varepsilon}_{\pm}$ in $\mathfrak{H}^{\eta,\varepsilon}$ and $\mathscr{D}^{0,0}_{\pm1}$ in $\mathfrak{H}^{0,0}$. Namely, there are four such subspaces in $\mathfrak{H}^{0,0}$ in the special case $(\eta, \varepsilon) = (0, 0)$. It should be noted that Corollary 2.3 plays an important role in verifying that $SU(1, 1)$ leaves $\mathscr{D}^{\eta,\varepsilon}_{\pm}$ in $\mathfrak{H}^{\eta,\varepsilon}$ as well as $\mathscr{D}^{0,0}_{\pm1}$ in $\mathfrak{H}^{0,0}$. The appendix is devoted to a quick review of Frobenius method in our context.

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Notation and terminology

Z is the set of integers and $Z_+ = \{n \in Z; n \geq 0\}$.

R is the set of real numbers, $R_+ = \{\lambda \in R; \lambda > 0$ and $R^* = R\backslash\{0\}$.

C is the set of complex numbers, $C^* = C\backslash\{0\}$ and $T = \{z \in C, |z| = 1\}$.

More subsets of C is to be defined. $D_\tau = \{z \in C; |\mathrm{Im}\, z| < \pi/2\}$, $\bar{D}_\tau = \{z \in C; |\mathrm{Im}\, z| \leq \pi/2\}$ and $\dot{D}_\tau = \bar{D}_\tau\backslash\{\pm i\pi/2\}$. An element of these three sets will be denoted by τ. Throughout the paper $\sigma = \tau - i\pi/2$. A polynomial in $\log \sigma$ with holomorphic coefficients will be denoted by $h(\sigma, \log \sigma)$, that is, $h(\sigma, \log \sigma) = \sum_{n=0}^{m} (\log \sigma)^n\, h_n(\sigma)$, where $h_n(\sigma)$ are holomorphic around $\sigma = 0$. For a function $f(\sigma)$ we denote by $Rf(\sigma)$ the function $f(-\sigma)$. An integral $\int_R f(\tau)\, d\tau$ will be abbreviated to $\int f\, d\tau$ or $\langle f \rangle$. The relation $a \propto b$ for two elements a and b in a linear space means $a = cb$ for some c in C^*.

$M_{m,n}$, $m, n \in Z_+ + 1$, is the set of complex $m \times n$-matrices and $M_n = M_{n,n}$. M_n^+ (resp. M_n^{++}) stands for the set of non-negative (resp. positive) definite $n \times n$-matrices. I_n means the unit matrix in M_n. For a matrix $A = (a_{jk})$ in $M_{m,n}$, we set $\bar{A} = (\bar{a}_{jk})$, ${}^tA$ = the transpose of A, $A^* = {}^t\bar{A}$ and $A = \max_k \sum_{j=1}^{m} |a_{jk}|$.

$C^r(S)^n$ $(r = 0, 1, \cdots, \infty; n \in Z_+ + 1)$ for a C^∞-manifold S is the totality of C^n-valued C^r-functions on S. $C_0^r(S)^n = \{f \in C^r(S)^n; f$ is compactly supported$\}$. $C_0(S)^n = C_0^0(S)^n$. $H_r(R)$, $r \in Z_+$, is the Sobolev space of order r on R. $H_r(R)^n$ means the direct sum $\sum_{j=1}^n \oplus H_r(R)$. Of course $H_0(R) = L^2(R)$, the Hilbert space consisting of C-valued square integrable functions on R. Let (B, Σ) be a measurable space, where B is a Borel set of R^n and Σ is the set of all Borel sets in B. $L^2(B, \mu)$ is the usual L^2-space defined in terms of a measure μ on (B, Σ). Let $\rho(x)$ be a M_m^{++}-valued measurable functions on a Borel set B of R^n. Then $L^2(B, \rho)$ denotes the Hilbert space consisting of C^m-valued measurable functions f on B such that $\int_B f^*(x)\rho(x)f(x)\,dx$ is finite. Here dx is the Lebesgue measure.

Let L be a linear operator from H_1 to H_2. When both H_j, $1 \leq j \leq 2$, are Hilbert spaces, L^* means the (formal) adjoint of L. In this paper a Hilbert space is assumed to be separable. LH_1 is the range of L, namely, $LH_1 = \{Lh;\ h$ in H_1 belongs to the domain of $L\}$. For a subspace H_0 of H_1, $L|H_0$ denotes the restriction of L to the subspace H_0. Let D be a subset of a Hilbert space. Then $D^\perp$ is the set of all elements which are orthogonal to D. $\|\ \|$ and $\langle\ ,\ \rangle$ denote the norm and the inner product in a Hilbert space (C^n, $L^2(B, \mu)$, etc.) respectively. However, $\langle x, y\rangle = x_0y_0 - x_1y_1 - x_2y_2$ for $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$ in R^3. Recall that $\langle f\rangle$ is an abbreviation to the integral $\int_R f(\tau)\,d\tau$. A closed subspace D of a Hilbert space is said to be invariant under a selfadjoint operator L if $P_D L = LP_D$, where P_D denotes the orthogonal projection $H \to D$. As is well-known, D is invariant under L iff the one-parameter unitary group $\exp(itL)$ leaves D invariant.

$T_t = \exp(it\,\mathrm{sh}\,\tau)$ $(t \geq 0)$ is a continuous semigroup in $L^2(R)$ such that $T_t f(\tau) = \exp(it\,\mathrm{sh}\,\tau)f(\tau)$. $G_\alpha = (\alpha - i\,\mathrm{sh}\,\tau)^{-1}$ $(\mathrm{Re}\,\alpha > 0)$ are resolvent operators for the semigroup. By abuse of notation G_α also means the function $(\alpha - i\,\mathrm{sh}\,\tau)^{-1}$ of τ. Finally, f' means the derivative for either an absolutely continuous function f on R or a holomorphic function f.

§1. Invariant subspaces common to $\mathscr{L}_{k,\eta}$ and $T_t(t \geq 0)$

The purpose of this section is to determine all closed proper subspaces in $L^2(R)$ which stay invariant under the selfadjoint operator $\mathscr{L}_{k,\eta}$ with domain $H_2(R)$ and the semigroup $T_t(t \geq 0)$ on $L^2(R)$;

$$\mathscr{L}_{k,\eta} = -\,d^2/d\tau^2 + (1/4 - k^2 + \eta^2 + 2k\eta\,\mathrm{sh}\,\tau)/\mathrm{ch}^2\tau \qquad (k \in Z/2,\ \eta \in R)\,, \tag{1.1}$$

(1.2) $$T_t = e^{it\,\mathrm{sh}\,\tau}.$$

To this end, first the case $k=0$ or $1/2$ will be discussed. Then the general case can be dealt with by the aid of the following differential operator

(1.3) $$F_{\pm,k,\eta} = -d/d\tau + (\pm k + 1/2)\,\mathrm{th}\,\tau \pm \eta/\mathrm{ch}\,\tau .$$

Throughout the rest of this section the suffix η will frequently be omitted.

In case $(k, \eta) = (0, \eta)$ or $(1/2, 0)$ clearly $\mathscr{L}_k$ reduces to an operator of the following form.

(1.4) $$\mathscr{N}_\kappa = -d^2/d\tau^2 + \kappa/\mathrm{ch}^2\tau, \qquad \kappa \geq 0 .$$

We shall search for closed proper invariant subspaces common to $\mathscr{N}_\kappa$ and T_t $(t \geq 0)$. To begin with, denote by $\Phi = (\phi_1, \phi_2)$ the solution of an ordinary differential equation $(\mathscr{N}_\kappa - \lambda)\Phi = 0$ with initial value ${}^t({}^t\Phi, {}^t\Phi')_{\tau=0} = I_2$, the unit matrix. Since $\kappa/\mathrm{ch}^2\tau$ is integrable and $\mathscr{N}_\kappa$ is positive definite, there exists a so-called spectral density ρ on R_+ satisfying the following conditions i)~iii) [4].

i) ρ is an M_2^{++}-valued continuous function on R_+.

ii) The operator $\mathscr{F} : L^2(R) \to L^2(R_+, \rho)$ (refer to the Notation) defined by

(1.5) $$\mathscr{F}f(\lambda) = \lim_{N\to\infty} \int_{|\tau|<N} {}^t\Phi(\tau, \lambda) f(\tau) d\tau$$

is an onto isometry, whose inverse $\mathscr{F}^{-1}$ is given by

(1.6) $$\mathscr{F}^{-1}g(\tau) = \lim_{N\to\infty} \int_{0<\lambda<N} \Phi(\tau, \lambda)\, \rho(\lambda)\, g(\lambda)\, d\lambda .$$

iii) $\mathscr{F}\mathscr{N}_\kappa\mathscr{F}^{-1}g(\lambda) = \lambda g(\lambda)$ if $\lambda g(\lambda)$ lies in $L^2(R_+, \rho)$.

On the other hand the equation $(\mathscr{N}_\kappa - \lambda)\zeta = 0$ has a regular singularity at $\tau = i\pi/2$, that is, $\sigma = 0$. The Frobenius method yields linearly independent solutions $\zeta_\pm(\tau, \lambda)$ which, being holomorphic in $\dot{D}_\tau \times C$, admit the following expansions around $\tau = i\pi/2$;

(1.7)
$$\zeta_\pm = \sigma^{\alpha_\pm}\Big(\sum_{n=0}^{\infty} z_{\pm,n}\sigma^n\Big) \qquad \text{if } \kappa \neq 1/4,$$
$$\zeta_+ = \sigma^{1/2}\Big(\sum_{n=0}^{\infty} z_{+,n}\sigma^n\Big)$$
$$\zeta_- = \zeta_+ \log\sigma + \sigma^{1/2}\Big(\sum_{n=1}^{\infty} z_{-,n}\sigma^n\Big) \qquad \text{if } \kappa = 1/4,$$

where $\alpha_\pm = (1 \pm \sqrt{1-4\kappa})/2$ and $z_{\pm,0} = 1$. Set $\zeta = (\zeta_-, \zeta_+)$, and define $X(\lambda)$

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$\in M_2$ and $s_\pm(\lambda)$, $r_\pm(\lambda) \in M_{2,1}$ as follows.

$$(1.8) \qquad \zeta = \Phi X, \quad s_\pm = X^t v_\pm, \quad r_\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} s_\pm,$$

where $v_\pm = {}^t(1 \pm 1, 1 \mp 1)$ or ${}^t(0, 2)$ according as $\kappa \neq 1/4$ or not. Now we are in a position to introduce invariant subspaces

$$(1.9) \qquad D_\pm^\kappa = \mathscr{F}^{-1}\{g \in L^2(R_+, \rho);\ {}^t s_\pm(\lambda) g(\lambda) = 0 \text{ a.e.}\}.$$

Notice that $\mathscr{F} D_\pm^\kappa = \{r_\pm h \in L^2(R_+, \rho);\ h \in L^2(R_+, r^* \rho r)\}$. This is because ${}^t s_\pm r_\pm = 0$.

THEOREM 1.1. *Let D be a closed proper subspace of $L^2(R)$. Then D is invariant under the selfadjoint operator $\mathscr{N}_\kappa$ and the semigroup $T_t = e^{it\,\mathrm{sh}\,\tau}$ $(t \geq 0)$ iff it coincides with one of $D_\pm^\kappa$.*

For the proof we prepare two lemmas and two propositions.

LEMMA 1.1. (i) *The domain $D_\tau = \{|\mathrm{Im}\,\tau| < \pi/2\}$ is holomorphically isomorphic to a domain $\{\mathrm{Im}\,z \neq 0$ or $z \in (0, 1)\}$ via the map $z = (1 + i\,\mathrm{sh}\,\tau)/2$.*

(ii) *Let $f(\tau)$ be holomorphic in $\dot{D}_\tau$. Then $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in $\{\mathrm{Re}\,z < 1\}$ iff $f(\tau)$ can be expanded as $\sum_{n=0}^\infty c_n \sigma^{2n+1}$ near $\tau = i\pi/2$, where $\sigma = \tau - i\pi/2$.*

Proof. It is easy to see that z is a univalent function sending D_τ onto $\{\mathrm{Im}\,z \neq 0$ or $z \in (0, 1)\}$. Since the derivative z' does not vanish on D_τ, (i) follows. To verify (ii), assume that $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in a neighborhood of $z = 0$. Then $f(\tau)/\sqrt{z}$ is holomorphic too. Since $\sqrt{z}$ is a holomorphic odd function of σ in a vicinity of $\sigma = 0$, $f(\tau)$ has the desired expansion. Conversely, assume that f satisfies the condition. Then $F(z)$ $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in $\{\mathrm{Re}\,z < 1\}\backslash\{z \leq 0\}$. Notice that F admits an analytic continuation across the line $\{z < 0\}$, for $z = (1 + i\,\mathrm{sh}\,\tau)/2$ is a local isomorphism of $C\backslash\{i\pi n/2;\ n \in Z\}$. By the condition on f we see that $F(x + i0) = F(x - i0)$ for any negative $x > -\varepsilon$ $(\varepsilon > 0)$. Therefore $F(z)$ is holomorphic in $\{\mathrm{Re}\,z < 1\}\backslash\{0\}$. Since $F(z)$ is bounded in a punctured disc $\{0 < |z| < \varepsilon\}$, $z = 0$ is a removable singularity. This completes the proof of (ii). Q.E.D.

The next proposition is concerned with the Hilbert transform.

PROPOSITION 1.2. (i) *Assume that $F(z)$ is holomorphic in $\{\mathrm{Re}\,z < 1\}$. If the integral $\int |F(x + iy)|^p dy$ $(p > 1)$ is bounded on $x < 1 - \varepsilon$, $\varepsilon > 0$, then*

$$\int_{a-i\infty}^{a+i\infty} \frac{F(z)}{z-\alpha} dz = 0 \qquad \text{for } a < \min\{\operatorname{Re}\alpha, 1-\varepsilon\}.$$

(ii) *Assume that $F(z)$ is holomorphic in a strip $1/2 - 2\varepsilon < \operatorname{Re} z < 1/2 + 2\varepsilon$, $\varepsilon > 0$. If the integral $\int |F(x+iy)|^2 dy$ is bounded on $[1/2-\varepsilon, 1/2+\varepsilon]$, then $F(z)$ has the following integral representation in $1/2 - \varepsilon < \operatorname{Re} z < 1/2 + \varepsilon$.*

$$F(z) = \frac{1}{2\pi i}\left(- \int_{1/2-\varepsilon-i\infty}^{1/2-\varepsilon+i\infty} + \int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty}\right) \frac{F(\zeta)}{\zeta - z} d\zeta .$$

Proof. To prove (i), we apply a lemma [9, p. 125] to F to show that the integral in question is independent of a. On the other hand Hölder's inequality implies that the integral tends to zero as $a \to -\infty$. Now (i) follows. The statement (ii) is well-known [9, p. 130]. Q.E.D.

As to an estimate of the solution $\Phi(\tau, \lambda)$ we have the following

LEMMA 1.3. *Let $\Psi(\tau, \lambda) \in M_{1,2}$ be a solution of the following equation with initial value ${}^t({}^t\Psi, {}^t\Psi')_{\tau=0} = I_2$;*

$$\{-d^2/d\tau^2 + (a + b\operatorname{sh}\tau)/\operatorname{ch}^2\tau - \lambda\}\Psi(\tau, \lambda) = 0, \qquad a, b \in C.$$

Fix $\lambda_0 \in R_+$. Then for any $\varepsilon > 0$ there exist positive K and δ such that

i) $|\Psi(\tau, \lambda_0)| + |\Psi'(\tau, \lambda_0)| < K$ *on* $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| \geq 1\}$,

ii) $|\Psi(\tau, \lambda)| + |\Psi'(\tau, \lambda)| < Ke^{\varepsilon|\tau|}$ *on* $R \times \{|\lambda - \lambda_0| < \delta\}$.

Proof. We shall prove the existence of K satisfying only i), for we can argue similarly to show the existence of K and δ satisfying the condition ii). Put $S = \begin{pmatrix} 1 & 1 \\ \sqrt{-\lambda} & -\sqrt{\lambda} \end{pmatrix}$, and define χ by the relation ${}^t({}^t\Psi, {}^t\Psi') = S\left\{\exp\begin{pmatrix} \sqrt{-\lambda} & 0 \\ 0 & -\sqrt{-\lambda} \end{pmatrix}\tau\right\}\chi$. Then we note that $\chi(\tau, \lambda_0)$ is bounded on $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| = 1\}$ and that $\chi' = V(\tau)\chi$, where $|V(\tau)|$ is bounded by a function $v(\operatorname{Re}\tau)$ on $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| \geq 1\}$. Here v is integrable on $I = (-\infty, -1] \cup [1, \infty)$. Consequently the integral $\int_I |V(\tau + i\varepsilon)| d\tau$ is bounded on $|\varepsilon| \leq \pi/2$. Hence $\chi(\tau, \lambda_0)$ is bounded on $\bar{D}_\tau \backslash \{|\operatorname{Re}\tau| < 1\}$ (see Problem 1 [1, p. 97]), from which follows that $|\Psi(\tau, \lambda_0)| + |\Psi'(\tau, \lambda_0)|$ is bounded there. Q.E.D.

Let δ be an atomic measure on a finite subset Λ of R such that $\delta(\{\lambda\}) = 1$ for each $\lambda \in \Lambda$, ρ_2 be an M_2^{++}-valued Borel measurable function on a Borel set B of R. Set $H_p = L^2(\Lambda, \delta)$, $H_{ac} = L^2(B, \rho_2)$ and $H = H_p \oplus H_{ac}$. We denote by $e^{it\lambda}$, $t \in R$, the one-parameter unitary group acting on H as multiplication.

Then we have

PROPOSITION 1.4. *A closed subspace D of H is invariant under the one-parameter group $e^{it\lambda}$ iff there exist a subset Λ_0 of Λ, disjoint Borel subsets B_1, B_2 of B (Λ_0 and B_j may be a null set) and a Borel measurable function s on B_1 with values in $M_{2,1}\backslash\{0\}$ almost everywhere such that D coincides with*

$$L^2(\Lambda_0, \delta)\oplus\{{}^t(g_1, g_2) \in H_{ac}; (g_1, g_2)s = 0 \text{ a.e. on } B_1, (g_1, g_2) = 0 \text{ a.e. outside } B_1\}\oplus\{{}^t(g_1, g_2) \in H_{ac}; (g_1, g_2) = 0 \text{ a.e. outside } B_2\}. \tag{1.10}$$

Proof. It suffices to show that the conditions are necessary. We regard $e^{it\lambda}$ as a representation of R in H, and apply Theorem 8.6.6 [2] to this representation. Then there exist a subset Λ_0 of Λ and disjoint Borel sets of B such that the representation in D is unitarily equivalent to the following representation

$$\int_{\Lambda_0}^{\oplus} e^{it\lambda}d\delta(\lambda)\oplus\int_{B_1}^{\oplus} e^{it\lambda}d\lambda\oplus[2]\int_{B_2}^{\oplus} e^{it\lambda}d\lambda$$

in $\tilde{H} = L^2(\Lambda_0, \delta) \oplus L^2(B_1) \oplus [2]L^2(B_2)$. Let $U: \tilde{H} \to D$ be an onto isometry ensuring the equivalence. By Proposition 8.4.6 [2] U sends $L^2(\Lambda_0, \delta)$ in $\tilde{H}$ onto $L^2(\Lambda_0, \delta)$ in H_p while $L^2(B_1) \oplus [2]L^2(B_2)$ in $\tilde{H}$ into H_{ac}. Choose $f_i \in L^2(B_i)$, $i = 1, 2$, such that $f_j \neq 0$ a.e. on B_i, and denote by D_1, D_{21} and D_{22} the closed subspaces of H_{ac} cyclically generated by the vectors ${}^t(h_1, h_2) = U(0, f_1, 0, 0)$, ${}^t(h_{11}, h_{12}) = U(0, 0, f_2, 0)$ and ${}^t(h_{21}, h_{22}) = U(0, 0, 0, f_2)$ respectively. For the sake of simplicity assume that both B_1 and B_2 are non-null sets. In case either B_1 or B_2 is a null set, we can argue similarly. Note that (h_1, h_2) and (h_{i1}, h_{i2}) do not vanish a.e. on B_1 and B_2 respectively. Moreover, $\det(h_{ij}) \neq 0$ a.e. on B_2, for if it happened to vanish on a set of positive measure, the representation in $D_{21} \oplus D_{22}$ contains a subrepresentation of the multiplicity one, which contradicts Theorem 8.6.6 [2]. Since the Fourier transform for $L^1(R)$ is injective, it is not hard to see that $D_{21} \oplus D_{22}$ constitutes the third component of (1.10). Finally $D_1 = \{\text{rh} \in H_{ac}; h \in L^2(B_1, r^*\rho_2 r)\}$ coincides with the second component of (1.10) with $s=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}{}^t(h_1, h_2)$.

Q.E.D.

We are ready for the

Proof of Theorem 1.1. 1) We shall prove the sufficiency of the condition. To begin with, we note that $D_\pm^\kappa$ are closed proper subspaces variant under $\mathscr{N}_\kappa$. Indeed $\mathscr{F} \exp(it\mathscr{N}_\kappa)\mathscr{F}^{-1}$, $t \in R$, is the multiplication

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operator $e^{it\lambda}$ in $L^2(R_+, \rho)$. In order to see that T_t $(t \geq 0)$ leaves $D^\epsilon_\pm$ invariant, it suffices to show that the resolvent G_α (Re $\alpha > 0$) of the semigroup sends a dense subspace $\mathscr{F}^{-1}\{r_\pm h;\, h \in C_0(R_+)^1\}$ in $D^\epsilon_\pm$ into $D^\epsilon_\pm$, that is,

$$(1.11) \qquad {}^t s_\pm(\lambda)[\mathscr{F} G_\alpha \mathscr{F}^{-1} r_\pm h](\lambda) = 0, \qquad h \in C_0(R_+)^1 .$$

To verify (1.11) we shall show that

$$(1.12) \qquad \int {}^t s_\pm(\lambda)\, {}^t\Phi(\tau, \lambda)\, G_\alpha \Phi(\tau, \xi)\rho(\xi) r_\pm(\xi)\, d\tau = 0 .$$

Note that (1.11) follows from (1.12) immediately by integrating the both sides of (1.12) with respect to a signed measure $h(\xi)d\xi$ (we can safely change the order of integration on account of Lemma 1.3). To show (1.12), put, for positives λ and ξ,

$$I_{\alpha,\lambda,\xi} = \int {}^t\zeta(\tau, \lambda)\, G_\alpha \zeta(\tau, \xi)\, d\tau, \qquad \tilde\rho = X^{-1}\rho\, {}^tX^{-1} = (\tilde\rho_{ij}),$$
$$J_{\alpha,\lambda,\xi} = I_{\alpha,\lambda,\xi}\tilde\rho(\xi) .$$

Then, using the relation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\, {}^tY^{-1} \det Y$, the left side of (1.12) can be written as

$$(1.13) \qquad v_\pm J_{\alpha,\lambda,\xi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v_\pm \det X(\xi) .$$

See (1.8) for the definition of $v_\pm$, ζ and X. We shall show that

$$(1.14) \qquad I_{\alpha,\lambda,\xi} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ if } \kappa \neq 1/4, \qquad \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \text{ if } \kappa = 1/4,$$

$$(1.15) \qquad \tilde\rho = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ if } \kappa \neq 1/4, \qquad \begin{pmatrix} 0 & \tilde\rho_{12} \\ \tilde\rho_{12} & \tilde\rho_{22} \end{pmatrix} \text{ if } \kappa = 1/4,$$

to the effect that $J_{\alpha,\lambda,\xi}$ is diagonal or of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ according as $\kappa \neq 1/4$ or not, which proves (1.12) since (1.13) turns out to vanish. To see (1.14), let R be an operator assigning a function $f(\sigma)$ to $f(-\sigma)$ and $\mathscr{N}_\kappa(\sigma)$ be the differential operator $\mathscr{N}_\kappa$ expressed in terms of $\sigma = \tau - i\pi/2$. Then $R\mathscr{N}_\kappa(\sigma)R = \mathscr{N}_\kappa(\sigma)$. This relation gives rise to a symmetry of coefficients $z_{\pm,n}$ in (1.7). That is,

$$(1.16) \qquad z_{\pm,n}(-1)^n = z_{\pm,n} \text{ if } \kappa \neq 1/4, \quad z_{+,n}(-1)^n = z_{+,n} \text{ if } \kappa = 1/4 .$$

In particular ${}^t\zeta_\pm\zeta_\mp$ (resp. ${}^t\zeta_+\zeta_+$) can be expanded as $\sum_{n=0}^\infty c_n\sigma^{2n+1}$ near $\sigma = 0$ in the case $\kappa \neq 1/4$ (resp. $\kappa = 1/4$). Since $I_{\alpha,\lambda,\xi}$ is equal to

$$(1.17)\qquad \int_{1/2-i\infty}^{1/2+i\infty} \frac{{}^t\zeta(\tau,\lambda)\zeta(\tau,\xi)\{z(1-z)\}^{-1/2}}{z-\alpha}\,dz\,,\qquad z=(1+i\,\mathrm{sh}\,\tau)/2\,,$$

(1.14) follows from Proposition 1.2 (i) in view of Lemmas 1.1 and 1.3. Finally, to see (1.15), let g belong to $C_0(R_+)^2$. Since αG_α converges to the identity operator as $\alpha\to\infty$, there is a sequence α_n tending to ∞ such that $\alpha_n\mathscr{F}G_{\alpha_n}\mathscr{F}^{-1}g$ converge to g a.e. In other words

$$(1.18)\qquad \alpha_n\int_{R_+} I_{\alpha_n,\lambda,\xi}\,\tilde\rho(\xi)\,{}^tX(\xi)g(\xi)\,d\xi \longrightarrow {}^tX(\lambda)g(\lambda)\qquad \text{a.e. as } n\to\infty\,.$$

Set ${}^tXg={}^t(a,b)$. Then, if $\kappa\neq 1/4$, the first (resp. second) component of the left side of (1.18) does not depend on b (resp. a), while the right side of (1.18) is equal to ${}^t(\tilde\rho_{22}a-\tilde\rho_{12}b,\ -\tilde\rho_{21}a+\tilde\rho_{11}b)$. Thus $\tilde\rho_{12}=\tilde\rho_{21}=0$ if $\kappa\neq 1/4$. Similar argument, together with the fact that ρ is diagonal, yields $\tilde\rho_{11}=0$ if $\kappa=1/4$. This completes the proof of (1.15). 2) We shall show that the condition is necessary. Applying Proposition 1.4 to the one-parameter group $e^{it\lambda}$ on $L^2(R_+,\rho)$, we define Borel sets B_1, B_2 of R_+ and a Borel measurable function s with values in $M_{2,1}\backslash\{0\}$ a.e. on B_1. Since the image $G_\alpha D$ is dense in D, $\det(\mathscr{F}G_\alpha f_1,\mathscr{F}G_\alpha f_2)\neq 0$ a.e. on B_2 for some f_1, $f_2\in D$. If B_2 is not a null set, the determinant does not vanish a.e. on R_+, for it is holomorphic in a neighborhood of R_+. Therefore, if B_2 is not a null set, $D=L^2(R)$, which is a contradiction. Thus we may assume that $B_2=\phi$ and $B_1=R_+$ on account of the analyticity of $\mathscr{F}G_\alpha f(\lambda)$, $f\in D$. Set $r=\begin{pmatrix}0&1\\-1&0\end{pmatrix}s$. Then $\mathscr{F}D=\{rh\in L^2(R_+,\rho);\ h\in L^2(R_+,r^*\rho r)\}$. Consequently we can replace r and s by real analytic functions $\mathscr{F}G_{\alpha_0}f$, $f\in D\backslash\{0\}$ and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}r$ respectively. Since rh, $h\in C_0(R_+)^1$, belongs to $\mathscr{F}D$, we have ${}^ts(\lambda)[\mathscr{F}G_\alpha\mathscr{F}^{-1}rh](\lambda)=0$ on R_+. Letting h converge to the Dirac measure supported at $\xi\in R_+$, we obtain $\langle {}^ts(\lambda)\Phi(\tau,\lambda)G_\alpha\Phi(\tau,\xi)\rho(\xi)r(\xi)\rangle=0$. Namely,

$$(1.19)\qquad {}^t(X^{-1}s)(\lambda)J_{\alpha,\lambda,\xi}\begin{pmatrix}0&1\\-1&0\end{pmatrix}(X^{-1}s)(\xi)=0\,,\qquad \mathrm{Re}\,\alpha>0\,.$$

Put $X^{-1}s={}^t(a,b)$. Then (1.19) implies, by Proposition 1.2 (ii), that the following function of $z=(1+i\,\mathrm{sh}\,\tau)/2$

$$(a\zeta_-\zeta_-\rho_{11}b-b\zeta_+\zeta_+\rho_{22}a)/\sqrt{z(1-z)}\,,\qquad \kappa\neq 1/4\,,$$
$$a(\zeta_-\zeta_-\rho_{12}+\zeta_-\zeta_+\rho_{22})a/\sqrt{z(1-z)}\,,\qquad \kappa=1/4\,,$$

is holomorphic at $z=0$, from which it is immediate that

$$a(\lambda)b(\xi)=b(\lambda)a(\xi)=0 \text{ for } \kappa\neq 1/4, \text{ while } a(\lambda)a(\xi)=0 \text{ for } \kappa=1/4\,.$$

Since a as well as b is real analytic, either a or b must vanish identically if $\kappa \neq 1/4$, and $a = 0$ if $\kappa = 1/4$. Thus there exists a Borel measurable function $c_\pm$ with values in C^* such that $s = c_\pm s_\pm$ a.e. Q.E.D.

We return to the study of invariant closed subspaces common to $\mathscr{L}_0$ and T_t $(t \geq 0)$. In case $\alpha_\pm = 1/2 \pm i\eta$, denote by $\zeta_{0,\pm}$, ζ_0, X_0, $s_{0,\pm}$ and $r_{0,\pm}$, respectively, $\zeta_\pm$, ζ, X, $s_\pm$ and $r_\pm$ in (1.8). Then we define subspaces $D^\eta_{0,\pm}$ of $L^2(R)$ by

$$(1.20) \qquad D^\eta_{0,\pm} = \mathscr{F}_0^{-1}\{g \in L^2(R_+; \rho_0); {}^t s_{0,\pm}(\lambda) g(\lambda) = 0 \text{ a.e.}\},$$

where ρ_0 is the spectral density for $\mathscr{L}_0$ with respect to Φ_0 and $\mathscr{F}_0$ stands for the isometry associated with the eigenfunction expansion. Here, Φ_k, $k \in Z/2$, is the solution of the following ordinary differential equation;

$$(1.21) \qquad (\mathscr{L}_k - \lambda)\Phi_k(\tau, \lambda) = 0, \qquad {}^t({}^t\Phi_k, {}^t\Phi_k')_{\tau=0} = I_2 .$$

Thanks to Theorem 1.1 $D^\eta_{0,\pm}$ are invariant, closed proper subspaces for $\mathscr{L}_0$ and T_t $(t \geq 0)$, and there are no other closed proper subspaces with the invariant property.

We proceed to the study of invariant closed subspaces common to $\mathscr{L}_{1/2}$ and T_t $(t \geq 0)$.

LEMMA 1.5. *The selfadjoint operator $\mathscr{L}_{1/2,\eta}$, $\eta \in R$, has no eigenvalues.*

Proof. Consider a selfadjoint operator $M_{1/2,\eta} = i\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}d\Big/d\tau + i\eta\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Big/\operatorname{ch}\tau$ with domain $H_1(R)^2$ [6, p. 287]. We note that $(UM_{1/2,\eta}U^*)^2 = \mathscr{L}_{1/2,\eta} \oplus \mathscr{L}_{1/2,-\eta}$ for a unitary matrix $U = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\Big/\sqrt{2}$. This relation implies that an eigenvalue of $\mathscr{L}_{1/2,\pm\eta}$, if any, is equal to zero, because $\mathscr{L}_{1/2,\pm\eta}$ has no positive eigenvalues in virtue of Theorem 4 [4]. Now assume that f is an eigenvector corresponding to the eigenvalue zero, say, of $\mathscr{L}_{1/2,\eta}$. Then $(UM_{1/2,\eta}U^*)^2\,{}^t(f, 0) = 0$. This contradicts the fact that $M_{1/2,\eta}$ has no eigenvalues by Theorem 2 [4] Q.E.D.

Since the function $(1/4 - k^2 + \eta^2 + 2k\eta \operatorname{sh}\tau)/\operatorname{ch}^2\tau$ is integrable, the spectral matrix for $\mathscr{L}_k$ relative to Φ_k has an M_2^{++}-valued continuous density ρ_k on R_+ due to Theorem 4 [4]. On account of Lemma 1.5 we can define an onto isometry $\mathscr{F}_{1/2}$: $L^2(R) \to L^2(R_+, \rho_{1/2})$ and its inverse $\mathscr{F}_{1/2}^{-1}$ in a similar way as (1.5) and (1.6) respectively. To define invariant subspaces $D^\eta_{1/2,\pm}$ we first note that the equation (1.21) has a regular singularity at $\tau = i\pi/2$, the indicial roots at which are $1/2 \pm (i\eta - k)$. Therefore, the equation (1.21)

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for $k=1/2$ has linearly independent solutions $\zeta_{1/2,\pm}(\tau,\lambda)$ which, being holomorphic in $\dot{D}_\tau \times C$, admit the following expansion near $\sigma=0$.

$$\zeta_{k,\pm}=\sigma^{1/2\pm(i\eta-k)}\left(\sum_{n=0}^{\infty} z_{k,\pm,n}\,\sigma^n\right), \qquad z_{k,\pm,0}=1\,, \tag{1.22}$$

where $k=1/2$. It should be noted that $(\zeta_{1/2,-},\zeta_{1/2,+})=\Phi_{1/2}$ if $\eta=0$. Let us define $X_k(\lambda)\in M_2$, $s_{k,\pm}(\lambda)$, $r_{k,\pm}(\lambda)\in M_{2,1}$ in terms of Φ_k and $\zeta_{k,\pm}$ as in (1.8), and set, for $k=1/2$,

$$D^{\eta}_{k,\pm}=\mathscr{F}_k^{-1}\{g\in L^2(R_+,\rho_k);\ {}^t s_{k,\pm}(\lambda)g(\lambda)=0 \text{ a.e.}\}\,. \tag{1.23}$$

Then, repeating the argument in the proof of Theorem 1.1, we get the next theorem.

THEOREM 1.2. *Let D be a closed proper subspace of $L^2(R)$. Then the selfadjoint operator $\mathscr{L}_{1/2,\eta}$ and the semigroup T_t $(t\geq 0)$ leave D invariant iff D coincides with one of $D^{\eta}_{1/2,\pm}$.*

From now on we shall be concerned with a general $\mathscr{L}_k$. The following lemma shows close relations among the operators $\mathscr{L}_k$ and $F_{\pm,k}$ (see (1.3)).

LEMMA 1.6. *Let $F_{\pm,k}$ and $\mathscr{L}_k$ be the differential operators on $C^\infty(R)$.*

(i) $F_{\mp,k\pm1}F_{\pm,k}=-\mathscr{L}_k-(k\pm 1/2)^2$.

(ii) $\mathscr{L}_{k\pm1}F_{\pm,k}=F_{\pm,k}\mathscr{L}_k$.

(iii) $F^*_{\pm,k}=-F_{\mp,k\pm1}$, $\qquad F^*_{\pm,k}F_{\pm,k}=\mathscr{L}_k+(k\pm 1/2)^2$.

(iv) *If f satisfies $(\mathscr{L}_k-\lambda)f=0$, then $(\mathscr{L}_{k\pm1}-\lambda)F_{\pm,k}f=0$. In particular $F_{\pm,k}\Phi_k=\Phi_{k\pm1}X_{\pm,k}$, where*

$$X_{\pm,k}=\begin{pmatrix}\pm\eta & -1\\ \lambda+(k\pm 1/2)^2-\eta^2 & \pm\eta\end{pmatrix}.$$

Proof. Simple calculation is enough to verify (i)~(iii). The statement (iv) follows from (ii). Q.E.D.

As to eigenfunctions for $\mathscr{L}_k$ we assert

LEMMA 1.7. *Let $f_{\pm k,\pm k}$, $k>1/2$, be an absolute continuous function on R such that $F_{\mp,\pm k}f_{\pm k,\pm k}=0$. Set $f_{\pm k\pm m,\pm k}=F_{\pm,\pm k\pm m\mp1}\cdots F_{\pm,\pm k}f_{\pm k,\pm k}$, $m\in Z_+$.*

(i) *$f_{\pm k\pm m,\pm k}$ lies in $H_2(R)$, satisfies the equation*

$$\{\mathscr{L}_{\pm k\pm m}+(k\mp 1/2)^2\}f_{\pm k\pm m,\pm k}=0\,, \tag{1.24}$$

and takes the following form near $\sigma=0$.

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$$(1.25) \qquad \sigma^{1/2\pm i\eta-k-m}\Big(\sum_{n=0}^{\infty} z_n\sigma^n\Big), \quad z_0 \neq 0, \quad (-1)^n z_n = z_n .$$

(ii) *$f_{\pm k\pm m,\pm k}(\tau)$, as a function of $z=(1+i\,\mathrm{sh}\,\tau)/2$, is bounded on* $\{|z|\geq 2\}$.

Proof. The function $f_{\pm k,\pm k}$ is clearly a constant multiple of the function $(\mathrm{ch}\,\tau)^{\mp k+1/2}\exp\Big(\pm\eta\int_0^\tau 1/\mathrm{ch}\,t\,dt\Big)$ which lies in $L^2(R)$ as well as its derivative. By Lemma 1.6 (i) we note that $f_{\pm k,\pm k}$ is an eigenfunction of $\mathscr{L}_{\pm k}$ corresponding to the eigenvalue $-(k\mp 1/2)^2$. Since $1/2\pm(i\eta-k')$ and $1/2\pm(i\eta-k')$ are indicial roots at $\sigma=0$ for the equations $F_{\mp,k'}\,f=0$ and $(\mathscr{L}_{k'}-\lambda)\,f=0$ respectively, $f_{\pm k,\pm k}$ can be expanded as (1.25) for $m=0$. From now on only $f_{k,k}$ will be discussed. By Frobenius method, together with what we have proved, it can be easily seen that the equation (1.24) for $m=0$ has linearly independent solutions $\zeta_\pm$ such that

$$(1.26) \qquad \zeta_\pm = \sigma^{1/2\pm(i\eta-k)}\Big(\sum_{n=0}^{\infty} z_{\pm,n}\sigma^n\Big), \qquad z_{\pm,0}\neq 0,$$

where $\zeta_+\propto f_{k,k}$. Let $\mathscr{L}_k(\sigma)$ stand for $\mathscr{L}_k$ represented in terms of the variable σ. Using the relation $R\mathscr{L}_k(\sigma)R=\mathscr{L}_k(\sigma)$, we can show that $(-1)^n z_{\pm,n} = z_{\pm,n}$. It is now immediate that $(-1)^n z_n = z_n$ when $m=0$. This proves (i) for $m=0$. To show (i) for any m, we can proceed by induction on m, keeping in mind that $F_{+,k+m-1}\cdots F_{+,k}\zeta_+$ takes the form $\sigma^{1/2+i\eta-k-m}(\sum_{n=0}^{\infty} z_n\sigma^n)$, $z_0\neq 0$. To prove the statement (ii) we note that the equation (1.21) can be written as

$$(1.27) \qquad \Big\{\frac{d^2}{dz^2}+\frac{2z-1}{2(z^2-1)}\frac{d}{dz}+\frac{1/4-k^2+\eta^2-i(2z-1)}{4(z^2-1)^2}+\frac{\lambda}{4(z^2-1)}\Big\}\Psi_k = 0,$$

where $z=(1+i\,\mathrm{sh}\,\tau)/2$ and $\Psi_k(z,\lambda)=\Phi_k(\tau,\lambda)$. The indicial equation at $z=\infty$ for the above equation is $\alpha^2+\lambda=0$. Since $f_{\pm k\pm m,\pm k}$ satisfies (1.24), it assumes the form $z^{-k+1/2}(\sum_{n=0}^{\infty} y_n z^{-n})$, $y_0\neq 0$, near $z=\infty$. This is because $f_{\pm k\pm m,\pm k}(\tau)$ in $H_2(R)$ tends to zero as $\tau\to\pm\infty$ (i.e. $z\to 1/2\pm i\infty$). Q.E.D.

DEFINITION. Let notation be as in Lemma 1.7. We denote by $e_{\pm k\pm m,\pm k}$, $m\in Z_+$, the normalized eigenvector $f_{\pm k\pm m,\pm k}/\|f_{\pm k\pm m,\pm k}\|$ of $\mathscr{L}_{\pm k\pm m}$ corresponding to the eigenvalue $-(k\mp 1/2)^2$. Let Λ_k be the set of eigenvalues of $\mathscr{L}_k$ and $\tilde{E}_k$ be the Hilbert space $L^2(\Lambda_k,\delta_k)$, where δ_k is an atomic measure on Λ_k such that $\delta_k(\{\lambda\})=1$ for each $\lambda\in\Lambda_k$.

We already know that $\Lambda_k=\phi$ if $|k|\leq 1/2$. It will be proved in the following proposition that

$$\Lambda_k = \{-(j+1/2)^2; j = k, k+1, \cdots < -1/2\} \text{ if } k < -1/2,$$
$$= \{-(j-1/2)^2; j = k, k-1, \cdots < 1/2\} \text{ if } k > 1/2.$$

According to the eigenfunction expansion theorem for $\mathscr{L}_k$ (see [1, p. 251]) we can define an onto isometry $\mathscr{F}_k : L^2(R) \to L^2(R_+, \rho_k) \oplus \tilde{E}_k$ and its inverse $\mathscr{F}_k^{-1}$ as follows.

$$\text{(1.28)} \qquad \mathscr{F}_k f(\lambda) = \lim_{N\to\infty} \int_{|\tau|<N} {}^t\Phi_k(\tau, \lambda) f(\tau)\, d\tau \qquad \text{in } L^2(R_+, \rho_k),$$
$$\mathscr{F}_k f(\lambda) = \langle e_{k,j}, f\rangle \ \text{ for } \lambda = -\{j - (\operatorname{sign} k)1/2\}^2 \in \Lambda_k .$$

$$\text{(1.29)} \qquad \mathscr{F}_k^{-1} g(\tau) = \lim_{N\to\infty} \int_{0<|\tau|<N} \Phi_k(\tau, \lambda)\, \rho_k(\lambda)\, g(\lambda)\, d\lambda$$
$$\oplus \Sigma_j g(-\{j - (\operatorname{sign} k)1/2\}^2) e_{k,j} .$$

Here ρ_k is the spectral density for $\mathscr{L}_k$ relative to Φ_k. The next Proposition is concerned with the spectral property of $\mathscr{L}_k$.

PROPOSITION 1.8.

(i) *The set of eigenvalues Λ_k, $|k| > 1/2$, is given as above.*

(ii) $\rho_{k+1}(\lambda) = -X_{+,k}(\lambda)\rho_k {}^tX^{-1}_{-,k+1}(\lambda)$, $\lambda \in R_+$,

where $X_{\pm,k}$ stands for the same as in Lemma 1.6.

Proof. We shall prove the assertion (i) only for $k > 1/2$. Assume that an f in $H_2(R)\backslash\{0\}$ satisfies $(\mathscr{L}_k - \lambda)f = 0$ for $k = 1$ or $3/2$. Then $(\mathscr{L}_{k-1} - \lambda)F_{-,k}f = 0$ by Lemma 1.6 (ii). Particularly $F_{-,k}f$ belongs to $H_2(R)$. Since $\mathscr{L}_{k-1}$ has no eigenvalues, we conclude that $F_{-,k}f = 0$. Consequently a possible eigenvalue for $\mathscr{L}_k$ is $-(k-1/2)^2$ by Lemma 1.7. Conversely, the same lemma implies that $-(k-1/2)^2$ is really an eigenvalue. Recalling the well-known fact that the multiplicity of an eigenvalue for $\mathscr{L}_k$ is one, (i) has been proved in this case. Working by induction on k, we can complete the proof of (i). If g belongs to $C_0(R_+)^2$, $f = \mathscr{F}_k^{-1}g$ lies in the domain of $\mathscr{L}_k$ and tends to zero as $|\tau| \to \infty$. Integration by parts, together with Lemma 1.6 (iv), yields $\mathscr{F}_{k+1}F_{+,k}f = X^{*-1}_{-,k}g$. Therefore we can represent $F_{+,k}f$ in two ways;

$$\int_{R_+} \Phi_{k+1} X_{+,k}\, \rho_k\, g\, d\lambda = \int_{R_+} \Phi_{k+1}\, \rho_{k+1}\, X^{*-1}_{-,k+1} g\, d\lambda ,$$

which results in (ii), for $X_{-,k}$ is a real matrix. Q.E.D.

We are in a position to define invariant closed subspaces $D^{\eta}_{k,\pm}$ in $L^2(R)$. Since $s_{k,\pm}$ and $r_{k,\pm}$ for $k = 0, 1/2$ are defined in connection with

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$D^{\eta}_{k,\pm}$, $k = 0$, $1/2$, the following definition makes sense.

(1.30) $$s_{k,\pm} = X_{+,k-1}s_{k-1,\pm}, \qquad r_{k,\pm} = {}^tX_{-,k}r_{k-1,\pm}.$$

(1.31) $$D^{\eta}_{k,\pm} = \mathscr{F}^{-1}_k\{g \in L^2(R_+, \rho_k);\ {}^ts_{k,\pm}(\lambda)g(\lambda) = 0 \text{ a.e.}\} \oplus \mathscr{F}^{-1}_k \tilde{E}_{k,\pm},$$

where $\tilde{E}_{k,\pm} = \tilde{E}_k$ if $\pm k > 0$, while $\{0\}$ if $\pm k < 0$. The following is one of the main theorems in this section.

THEOREM 1.3. *Let D be a closed proper subspace of $L^2(R)$. Then the selfadjoint operator $\mathscr{L}_{k,\eta}$ and the semigroup T_t $(t \geq 0)$ leave D invariant iff D coincides with one of $D^{\eta}_{k,\pm}$.*

To prove the theorem we need a lemma.

LEMMA 1.9. *Let λ be positive.*

(i) ${}^ts_{k,\pm}(\lambda)r_{k,\pm}(\lambda) = 0.$

(ii) *If either $\eta \in R^*$ or $k \in Z + 1/2$, then*

$$\Phi_k(\tau, \lambda)s_{k,\pm}(\lambda) = O(\sigma^{1/2\pm(-i\eta+k)}),$$
$$\Phi_k(\tau, \lambda)\rho_k(\lambda)r_{k,\pm}(\lambda) = O(\sigma^{1/2\pm(i\eta-k)}).$$

If $\eta = 0$ and $k \in Z$, then

$$\Phi_k(\tau, \lambda)s_{k,\pm}(\lambda),\ \Phi_k(\tau, \lambda)\rho_k(\lambda)r_{k,\pm}(\lambda) = O(\sigma^{1/2+|k|}).$$

In the above $O(\sigma^\alpha)$ denotes a holomorphic function on $\dot{D}_\tau$ which assumes the form $\sigma^\alpha(\sum_{n=0}^{\infty} c_n\sigma^{2n})$, $c_0 \neq 0$, near $\sigma = 0$.

Proof. The relation (i) holds for $k = 0$, $1/2$. Since $X_{-,k}(\lambda)X_{+,k-1}(\lambda) = -\lambda - (k-1/2)^2$, (i) follows from the definition of $s_{k,\pm}$ and $r_{k,\pm}$. As to the statement (ii) only the functions $\Phi_k s_{k,\pm}$ will be examined. We recall that

$$\Phi_k s_{k,\pm} = 2\zeta_{k,+} \text{ if } (k, \eta) = (0, 0) \text{ while } 2\zeta_{k,\pm} \text{ if } k = 1/2 \text{ or } k = 0,\ \eta \in R^*.$$

Therefore (ii) is valid for $k = 0$, $1/2$. Assume that (ii) holds down to $k \leq 0$. To proceed by induction on k, we note that

$$F_{\pm,k}\left(\sum_{n=0}^{\infty} c_n\sigma^{\alpha+2n}\right) = \{1/2 \pm (-i\eta + k) - \alpha\}c_0\sigma^{\alpha-1} + \sum_{n=1}^{\infty} d_n\sigma^{\alpha+2n-1},$$
$$F_{-,k}\Phi_k(\tau, \lambda)s_{k,\pm}(\lambda) = -\{\lambda + (k-1/2)^2\}\Phi_{k-1}(\tau, \lambda)s_{k-1,\pm}(\lambda).$$

Let $\Phi_k s_{k,\pm}$ take the form $\sum_{n=0}^{\infty} c_n\sigma^{\alpha+2n}$, $c_0 \neq 0$. Then it can be easily seen that if $1/2 - (-i\eta + k) - \alpha$ vanishes, d_n is equal to zero unless $\mathrm{Re}(\alpha + 2n - 1) \geq \mathrm{Re}\{1/2 - (k-1) + i\eta\}$. This is due to the fact that $F_{-,k}\Phi_k s_{k,\pm}$

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is a nonzero solution of the equation $(\mathscr{L}_{k-1} - \lambda) f = 0$ whose indicial roots at $\sigma = 0$ are $1/2 \pm (k - 1 - i\eta)$. This proves (ii) for $k < 0$. In case $k > 0$, we can argue similarly, using the equality $F_{+,k}\Phi_k s_{k,\pm} = \Phi_{k+1} s_{k+1,\pm}$. Q.E.D.

Proof of Theorem 1.3. The proof is much like that of Theorem 1.1. We may assume that $k \neq 0$, $1/2$, and shall prove the theorem in the case $k > 0$. On account of Lemmas 1.1, 1.3, 1.7 and 1.9, Proposition 1.2 (i) yields the following equalities.

$$\int {}^t s_{k,+}(\lambda)\, {}^t\Phi_k(\tau, \lambda)\, G_\alpha \zeta_k(\tau, \xi)\, \sigma_k(\xi) r_{k,+}(\xi)\, d\tau = 0\,,$$

$$\int {}^t s_{k,+}(\lambda)\, {}^t\Phi_k(\tau, \lambda)\, G_\alpha e_{k,j}(\tau) d\tau = 0\,,$$

$$\int {}^t s_{k,-}(\lambda)\, {}^t\Phi_k(\tau, \lambda)\, G_\alpha \Phi_k(\tau, \xi)\, \rho_k(\xi)\, r_{k,-}(\xi)\, d\tau = 0\,,$$

$$\int e_{k,j}(\tau) G_\alpha \Phi_k(\tau, \xi)\, \rho_k(\xi)\, r_{k,-}(\xi)\, d\tau = 0\,,$$

where λ and ξ are positive. We can show, as in the proof of Theorem 1.1, that the first two and last two equalities imply the invariance of $D^\eta_{k,+}$ and $D^\eta_{k,-}$ under the semigroup T_t $(t \geq 0)$ respectively. Here we used the fact that $\bar{e}_{k,j} = c e_{k,j}$ for some constant c, $|c| = 1$. On the other hand, $\mathscr{L}_k$ clearly leaves $D^\eta_{k,\pm}$ invariant. Conversely, let D be a proper closed subspace with the desired invariant property. Arguing as in the proof of Theorem 1.1, we see that

$$D = \sum_{j \in I} \oplus \{e_{k,j}\} \oplus \mathscr{F}_k^{-1}\{g \in L^2(R_+, \rho_k);\ {}^t s(\lambda) g(\lambda) = 0 \text{ a.e.}\}$$

for some subset I of $\{k, k-1, \cdots, 1 \text{ or } 3/2\}$ and a real analytic function s on R_+ with values in $M_{2,1}\backslash\{0\}$ a.e. Denote by $\zeta_{k,\pm}(\tau, \lambda)$ linearly independent solutions of the equation $(\mathscr{L}_k - \lambda)\zeta = 0$ such that they are holomorphic in $\dot{D}_\tau \times C$ and have the following expansion near $\sigma = 0$.

$$\zeta_{k,\pm} = \sigma^{1/2 \pm (i\eta - k)}\Big(1 + \sum_{n=1}^{\infty} z_{k,\pm,2n}\sigma^{2n}\Big)\,, \text{ if } \eta \in R^* \text{ or } k \in Z + 1/2\,,$$

$$\zeta_{k,+} = \sigma^{1/2 + |k|}\Big(1 + \sum_{n=1}^{\infty} z_{k,+,2n}\,\sigma^{2n}\Big)\,, \text{ if } \eta = 0 \text{ and } k \in Z\,.$$

$$\zeta_{k,-} = (F_{+,k-1} \cdots F_{+,0}\zeta_{0,+}) \log \sigma + \sigma^{1/2 - |k|}\Big(\sum_{n=0}^{\infty} z_{k,-,n}\,\sigma^n\Big)\,,\ z_{k,0,-} \neq 0\,.$$

Set $\zeta_k = (\zeta_{k,-}, \zeta_{k,+})$, and define X_k by $\zeta_k = \Phi_k X_k$. Then, it can be shown, as in the proof of Theorem 1.1, that the symmetric matrix $X_k^{-1} \rho_k\, {}^t X_k^{-1}$ is diagonal in the case either $\eta \in R^*$ or $k \in Z + 1/2$ while the matrix assumes the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ in the case $\eta = 0$ and $k \in Z$. It is not hard to see that in the former case one of the components of $X_k^{-1}s$ must vanish identically while in the latter case the first component of $X^{-1}s$ must vanish (see the proof of Theorem 1.1). This means that there are, at most, two possibilities for s. Therefore, since $D_{k,\pm}^{\eta}$ possess the invariant property, there exists a C^*-valued measurable function c_+ or c_- such that $s = c_+ s_{k,+}$ or $c_- s_{k,-}$ a.e. on R_+. Suppose $s = c_+ s_{k,+}$. We must show that $I = \{k, k-1, \cdots, 1 \text{ or } 3/2\}$, provided $\eta \in R^*$ or $k + 1/2 \in Z$ (recall that $s_{k,+} = s_{k,-}$ in the case when $\eta = 0$ and $k \in Z$). On account of Lemmas 1.1, 1.3, 1.7 and 1.9, using Proposition 1.2 (ii), we can show that for any eigenvector $e_{k,j}$, there exists an α', $\operatorname{Re} \alpha' > 0$, satisfying

$$\langle e_{k,j}(\tau) G_{\alpha'} \Phi_k(\tau, \xi) \rho_k(\xi) r_{k,+}(\xi) \rangle \neq 0$$

so that $\langle e_{k,j}(\tau) G_{\alpha'} \mathscr{F}_k^{-1} r_{k,+} h \rangle \neq 0$ for some $h \in C_0(R_+)^1$. This means $D = D_{k,+}^{\eta}$, that is, $I = \{k, k-1, \cdots, 1 \text{ or } 3/2\}$, for D is $\mathscr{L}_k$-invariant. Next, assume $s = c_- s_{k,-}$. We must show that $I = \phi$, provided $\eta \in R^*$ or $k \in Z + 1/2$. To this end, we note that for any eigenvector $e_{k,j}$ and positive λ, there is an α', $\operatorname{Re} \alpha' > 0$, such that

$${}^t s_{k,-}(\lambda) \langle {}^t \Phi_k(\tau, \lambda) G_{\alpha'} e_{k,j}(\tau) \rangle \neq 0$$

on the same basis as above. This implies that $I = \phi$, since ${}^t s(\lambda)[\mathscr{F}_k G_\alpha f](\lambda) = 0$ a.e. for any $f \in D$. Finally, we note that for any eigenvectors $e_{k,i}$ and $e_{k,j}$, there exists an α', $\operatorname{Re} \alpha' > 0$, such that $\langle e_{k,i}, G_{\alpha'} e_{k,j} \rangle \neq 0$. This means $I = \phi$ or $\{k, k-1, \cdots, 1 \text{ or } 3/2\}$. Since $s_{k,-} = s_{k,+}$ in the case $\eta = 0$ and $k \in Z$, Theorem 1.3 has been shown for $k > 0$. In case $k < 0$, we can argue similarly. Q.E.D.

We set $W_k = L^2(R)$ for $k \in Z/2$ and regard $\mathscr{L}_k$ as a selfadjoint operator in W_k and $F_{\pm,k}$ as an operator sending W_k into $W_{k\pm 1}$. It is the next theorem that will be used in § 2.

THEOREM 1.4. *Let $\{D_k\}_{k \in Z+\varepsilon}$, $\varepsilon = 0, 1/2$, be a nontrivial sequence of closed subspaces of W_k. Then the sequence $\{D_k\}$ fulfils the following two conditions iff it coincides with one of*

$\{D_{k,-}^{\eta}\}$, $\{D_{k,+}^{\eta}\}$ *if* $\eta \in R^*$ *or* $1/2$,

$\{D_{k,\operatorname{sign}(-k+1/2)}^{0}\}$, $\{D_{k,-}^{0}\}$, $\{D_{k,+}^{0}\}$ *and* $\{D_{k,\operatorname{sign}(k+1/2)}^{0}\}$ *if* $\eta = \varepsilon = 0$.

i) D_k *is invariant under the selfadjoint operator* $\mathscr{L}_{k,\eta}$ *and the semi-group* T_t $(t \geq 0)$.

ii) $F_{\pm,k,\eta} D_k \subset D_{k\pm1}$, *where the domains of* $F_{\pm,k,\eta}$ *are* $H_2(R)$.

Proof. We shall first show the sufficiency of the condition. Assume that an f in $H_2(R)$ satisfies $\mathscr{F}_k f = r_{k,\pm} h$, $h \in L^2(R_+, r^*_{k,\pm}\rho_k r_{k,\pm})$. Then integration by parts yields

$$(1.32) \qquad \mathscr{F}_{k+1} F_{+,k} f = -r_{k+1,\pm} h, \ \mathscr{F}_{k-1} F_{-,k} f = \{\lambda + (k-1/2)^2\} r_{k-1,\pm} h\,.$$

Making use of Lemma 1.6, we can verify easily that for k, $|k| > 1/2$,

$$(1.33) \qquad F_{\pm,k} e_{k,j} = \pm(\mathrm{sign}\, k)\sqrt{(k \pm 1/2)^2 - \{j - (\mathrm{sign}\, k)1/2\}^2}\, e_{k\pm1,j}\,.$$

By (1.32) and (1.33) the sequences mentioned in the theorem satisfy the conditions i) and ii). Conversely, let $\{D_k\}$ be a nontrivial sequence satisfying i) and ii). In view of Theorem 1.3 and the relations (1.32) and (1.33), $\{D_k\}$ must coincide with one of the aforementioned sequences, provided some D_k is a proper subspace. Therefore it remains to show that all D_k are proper subspaces. To this end, suppose $D_k = L^2(R)$ for some k. Let us show that $D_{k\pm1} = L^2(R)$. In fact, on account of the equality $G_\alpha F_{\pm,k} = F_{\pm,k} G_\alpha + G'_\alpha$ it is not hard to see that if an f in $(D_{k\pm1})^\perp$ is orthogonal to the image $G_\alpha F_{\pm,k} C_0^\infty(R)$, then $f = 0$. Assume now that $D_{k-1} = \{0\}$ and $D_k \neq \{0\}$ for some k. This contradicts Theorem 1.3 and (1.32). Thus each D_k must be proper for the sequence $\{D_k\}$ to be nontrivial. Q.E.D.

Before concluding this section we shall rewrite the relation (1.32) in a more convenient manner. For this purpose, introduce Hilbert spaces $\tilde{D}^\eta_{k,\pm}$, $\hat{D}^\eta_{k,\pm}$ and an onto isometry $I^{\eta,\varepsilon}_{\pm,k} : \tilde{D}^\eta_{k,\pm} \to \hat{D}^\eta_{k,\pm}$, $k \in Z + \varepsilon$, as follows.

$$(1.34) \qquad \begin{aligned} \tilde{D}^\eta_{k,\pm} &= \{r_{k,\pm} h \in L^2(R_+, \rho_k);\, h \in L^2(R_+, r^*_{k,\pm}\rho_k r_{k,\pm})\} \oplus \tilde{E}_{k,\pm}\,. \\ \hat{D}^\eta_{k,\pm} &= L^2(R_+) \oplus \tilde{E}_{k,\pm}\,. \\ (I^{\eta,\varepsilon}_{\pm,k} r_{k,\pm} h)(\lambda) &= \langle r_{k,\pm}(\lambda), \rho_k(\lambda) r_{k,\pm}(\lambda)\rangle^{1/2} h(\lambda)\,, \qquad \lambda > 0\,, \\ I^{\eta,\varepsilon}_{\pm,k} | \tilde{E}_{k,\pm} &= \text{the identity operator.} \end{aligned}$$

Furthermore, for $F_{\pm,k}$ with domain $H_1(R)$, set

$$\begin{aligned} \hat{F}_{+,k,\pm} &= I^{\eta,\varepsilon}_{\pm,k+1} \mathscr{F}_{k+1} F_{+,k} (I^{\eta,\varepsilon}_{\pm,k} \mathscr{F}_k)^{-1}\,, \\ \hat{F}_{-,k,\pm} &= I^{\eta,\varepsilon}_{\pm,k-1} \mathscr{F}_{k-1} F_{-,k} (I^{\eta,\varepsilon}_{\pm,k} \mathscr{F}_k)^{-1}\,. \end{aligned}$$

Then (1.32) yields

$$(1.35) \qquad \hat{F}_{\pm,k,s} h(\lambda) = \mp\sqrt{\lambda + (k \pm 1/2)^2}\, h(\lambda)\,, \quad h \in C_0(R_+)^1\,, \quad s = + \text{ or } -\,.$$

This is because $\langle r_{k,\pm}(\lambda), \rho_k(\lambda) r_{k,\pm}(\lambda)\rangle = \{\lambda + (k-1/2)^2\}\langle r_{k-1,\pm}(\lambda), \rho_{k-1}(\lambda) r_{k-1,\pm}(\lambda)\rangle$ by virtue of the definition of $r_{k,\pm}$ and Proposition 1.8 (ii).

§ 2. $P_+(3)$-invariant subspaces for the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$

We begin by defining the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ of the group $P(3)$ (see the introduction for the definition of $P(3)$) associated with the one-sheeted hyperboloid $V_{iM}(3) = \{y_0^2 - y_1^2 - y_2^2 = -M^2\}$, $M > 0$, after Mackey [7]. Let G be $SU(1, 1)$, and ω_j, $1 \leq j \leq 3$, be one-parameter subgroup of G;

$$\omega_1(t) = \begin{pmatrix} \operatorname{ch} t/2 & \operatorname{sh} t/2 \\ \operatorname{sh} t/2 & \operatorname{ch} t/2 \end{pmatrix}, \qquad \omega_2(t) = \begin{pmatrix} \operatorname{ch} t/2 & i \operatorname{sh} t/2 \\ -i \operatorname{sh} t/2 & \operatorname{ch} t/2 \end{pmatrix},$$
$$\omega_3(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}.$$

G acts on R^3 as $y \cdot g = g^* y g$, where $y = (y_0, y_1, y_2)$ is identified with a matrix $\begin{pmatrix} y_0 & y_2 - iy_1 \\ y_2 + iy_1 & y_0 \end{pmatrix}$. It can be easily seen that the orbit of $\hat{y} = M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $V_{iM}(3)$ and that the isotropy group at $\hat{y}$ is $G_0 = \{\pm \omega_2(t);\ t \in R\}$. Let $\pi_{\eta,\varepsilon}$, $\eta \in R$, $\varepsilon = 0, 1/2$, be an irreducible unitary representation of G_0 such that $\pi_{\eta,\varepsilon}(\pm \omega_2(t)) = (\pm 1)^{2\varepsilon} \exp i\eta t$. We can identify the factor space $G_0\backslash G \simeq (R^3 \times_s G_0)\backslash(R^3 \times_s G)$ with $V_{iM}(3)$ via a projection p of G onto $V_{iM}(3)$ defined by $p(g) = g^* \hat{y} g$. As is well known, the measure $d\bar{y} = dy_1 dy_2/M|y_0|$ on $V_{iM}(3)$ is G-invariant. Let $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ be the set of C-valued measurable functions on $P(3)$ such that

$$f((x', g_0)(x, g)) = e^{i\langle x', \hat{y}\rangle} \pi_{\eta,\varepsilon}(g_0) f(x, g), \qquad g_0 \in G_0,$$

and that $|f(x, g)|^2$, which is a function on $V_{iM}(3)$, is integrable relative to the measure $d\bar{y}$. Then $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ equipped with the inner product $\langle f, h\rangle = \int \bar{f} h\, d\bar{y}$ give rise to a Hilbert space, which we denote by $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ again. Let $U^{\eta,\varepsilon}(x, g)$, $(x, g) \in P(3)$, be a linear operator on $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ defined by

$$[U^{\eta,\varepsilon}(x, g) f](x', g') = f((x', g')(x, g)).$$

It is well-known that $(U^{\eta,\varepsilon}, \tilde{\mathfrak{H}}^{\eta,\varepsilon})$ is an irreducible unitary representation of $P(3)$ associated with $V_{iM}(3)$ and $\pi_{\eta,\varepsilon}$. We prefer to realize this representation in $L^2(V_{iM}(3), d\bar{y})$. For this purpose, note that a map $p(\omega_1(\tau)\omega_3(\theta))$ of $R \times (0, 2\pi)$ into $V_{iM}(3)$ is a diffeomorphism onto an open dense set of $V_{iM}(3)$, and fix a Borel measurable section s_e of $V_{iM}(3)$ into G such that $s_e \circ p(\omega_1(\tau)\omega_3(\theta)) = \omega_1(\tau)\omega_3(\theta)$ for $(\tau, \theta) \in R \times (0, 2\pi)$. Then we can define an equivalent representation $(U^{\eta,\varepsilon}, L^2(V_{iM}(3), d\bar{y}))$ as follows.

$$(2.1)\qquad \begin{aligned} &U^{\eta,\varepsilon}(x, g)f(y) = e^{i\langle x',\hat{y}\rangle}\pi_{\eta,\varepsilon}(g_0)f(y\cdot g)\,, \\ &(0, s_e(y))(x, g) = (x', g_0)(0, s_e(y\cdot g))\,, \qquad g_0 \in G_0\,. \end{aligned}$$

Clearly $(\tau, \theta) \in R \times (0, 2\pi)$ is a system of coordinates on an open dense set of $V_{iM}(3)$. Simple calculation yields

$$(y_0, y_1, y_2) = M(\mathrm{sh}\,\tau, \mathrm{ch}\,\tau \sin\theta, \mathrm{ch}\,\tau\cos\theta)\,, \qquad d\bar{y} = \mathrm{ch}\,\tau\, d\tau\, d\theta\,.$$

Therefore, by identifying $L^2(V_{iM}(3), d\bar{y})$ with $\mathfrak{H}^{\eta,\varepsilon} = L^2(R \times (0, 2\pi), \mathrm{ch}\,\tau\, d\tau d\theta)$ in a trivial manner, we obtain a representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ equivalent to the one $(U^{\eta,\varepsilon}, \tilde{\mathfrak{H}}^{\eta,\varepsilon})$ above. From now on the former realization will be discussed. By (2.1) it is easy to see that

$$U^{\eta,\varepsilon}(t, 0, 0, e) = e^{iMt\,\mathrm{sh}\,\tau}\,.$$

Let ω_j, $1 \le j \le 3$, be an infinitesimal operator of the one-parameter unitary group $U^{\eta,\varepsilon}(0, \omega_j(t))$, and put

$$\Delta = -\omega_1^2 - \omega_2^2 + \omega_3^2\,, \quad F_\pm = -\omega_1 \mp i\omega_2\,, \quad H_3 = i\omega_3\,.$$

To be more precise, Δ stands for the selfadjoint extension of a symmetric operator $-\omega_1^2 - \omega_2^2 + \omega_3^2$ whose domain is the Gårding space, while the domains of $F_\pm$ are the intersection of the domains of ω_1 and ω_2. Using (2.1), we can easily get expressions for the restrictions $\omega_j | C_0^\infty(R \times (0, 2\pi))$. That is,

$$\begin{aligned} \omega_1 &= \cos\theta\,\partial_\tau - \mathrm{th}\,\tau \sin\theta\,\partial_\theta + i\eta\sin\theta/\mathrm{ch}\,\tau\,, \\ \omega_2 &= -\sin\theta\,\partial_\tau - \mathrm{th}\,\tau\cos\theta\,\partial_\theta + i\eta\cos\theta/\mathrm{ch}\,\tau\,, \\ \omega_3 &= \partial_\theta\,. \end{aligned}$$

In particular,

$$F_\pm = -e^{\mp i\theta}(\partial_\tau \mp \mathrm{th}\,\tau\,\partial_\theta \mp \eta/\mathrm{ch}\,\tau)\,.$$

Put $\mathscr{W}_k^{\eta,\varepsilon} = \{f \in \mathfrak{H}^{\eta,\varepsilon};\, H_3 f = kf\}$, $k \in Z/2$. Then $\mathfrak{H}^{\eta,\varepsilon} = \sum_k \oplus \mathscr{W}_k^{\eta,\varepsilon}$, since eigenvalues of H_3 lie in $Z/2$ (see Lemma 2.1). Furthermore, it is not hard to show that $\mathscr{W}_k^{\eta,\varepsilon} = \{0\}$, $k \notin Z + \varepsilon$, and

$$\mathscr{W}_k^{\eta,\varepsilon} = \{f(\tau)e^{-ik\theta};\, f \in L^2(R, \mathrm{ch}\,\tau)\},\ k \in Z + \varepsilon\,.$$

Now put $W_k = L^2(R)$, $k \in Z/2$, and define an onto isometry $J_k^{\eta,\varepsilon} : \mathscr{W}_k^{\eta,\varepsilon} \to W_k$ by $J_k^{\eta,\varepsilon}(f(\tau)e^{-ik\theta}) = f(\tau)\sqrt{\mathrm{ch}\,\tau/2\pi}$. Then an onto isometry $J^{\eta,\varepsilon} : \mathfrak{H}^{\eta,\varepsilon} \to W^\varepsilon = \sum_{k\in Z+\varepsilon} \oplus W_k$ arises naturally, namely $J^{\eta,\varepsilon} = \sum_{k \in Z+\varepsilon} \oplus J_k^{\eta,\varepsilon}$. It is immediate that

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$$J^{\eta,\varepsilon}U^{\eta,\varepsilon}(t/M, 0, 0, e)J^{\eta,\varepsilon-1} = e^{it\,\mathrm{sh}\,\tau}. \tag{2.2}$$

Using the explicit forms of ω_j, $1 \leq j \leq 3$, we obtain, for $k \in Z + \varepsilon$,

$$\begin{aligned} J_k^{\eta,\varepsilon}\Delta J_k^{\eta,\varepsilon-1} &= \mathscr{L}_{k,\eta} + 1/4\,, \\ J_{k\pm1}^{\eta,\varepsilon}F_{\pm}J_k^{\eta,\varepsilon-1} &= F_{\pm,k,\eta}\,. \end{aligned} \tag{2.3}$$

See (1.1) and (1.3) for the definition of $\mathscr{L}_{k,\eta}$ and $F_{\pm,k,\eta}$ respectively. To be more precise, we can verify the equality (2.3) only on $C_0^\infty(R)$. Since $J_k^{\eta,\varepsilon}\Delta J_k^{\eta,\varepsilon-1}$ is selfadjoint, the first equality in (2.3) follows from Theorem 4.3 [6, p. 287]. On the other hand, the second equality is understood to hold on $H_1(R)$. We regard $D_{k,\pm}^{\eta}$ (see (1.31)) as a subspace of W_k and introduce closed subspaces $\mathscr{D}_{\pm}^{\eta,\varepsilon} \subset W^{\varepsilon}$, $\varepsilon = 0$, $1/2$, and $\mathscr{D}_{\pm1}^{0,0} \subset W^0$ as follows.

$$\begin{aligned} \mathscr{D}_{\pm}^{\eta,\varepsilon} &= \textstyle\sum_{k\in Z+\varepsilon} \oplus\, J_k^{\eta,\varepsilon-1}D_{k,\pm}^{\eta}\,, \\ \mathscr{D}_{\pm1}^{0,0} &= \textstyle\sum_{k\in Z} \oplus\, J_k^{0,0-1}D_{k,\mathrm{sign}(\pm k+1/2)}^{0}\,. \end{aligned} \tag{2.4}$$

Now we are ready to state main theorems of this paper.

THEOREM 2.1. *Let $\mathscr{D}$ be a closed proper subspace of $\mathfrak{H}^{\eta,\varepsilon}$. Then $\mathscr{D}$ is $P_+(3)$-invariant iff it coincides with one of $\mathscr{D}_{\pm}^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm1}^{0,0}$, provided $(\eta, \varepsilon) =$ (0, 0)).*

THEOREM 2.2. *The representations of $SU(1, 1)$ realized in $\mathscr{D}_{\pm}^{\eta,\varepsilon}$, $\mathscr{D}_{-1}^{0,0}$ and $\mathscr{D}_{1}^{0,0}$ decompose into irreducible ones, respectively, as*

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)}d\eta \oplus \Sigma_{-k\in Z_+ +1+\varepsilon} \oplus T_{(k,\varepsilon)}^{\mp}\,,$$

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,0)}d\eta\,,$$

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,0)}d\eta \oplus \Sigma_{-k\in Z_+ +1} \oplus (T_{(k,0)}^{-} \oplus T_{(k,0)}^{+})\,.$$

See the following passage for the definition of the representation $T_{(-1/2+i\eta,\varepsilon)}$ and $T_{(k,\varepsilon)}^{\pm}$.

Remark. It is known [8] that the representation of $SU(1, 1)$ in $\mathfrak{H}^{\eta,\varepsilon}$ decomposes into irreducible ones as

$$[2]\int_{R_+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)}d\eta \oplus \Sigma_{-k\in Z_+ +1+\varepsilon} \oplus (T_{(k,\varepsilon)}^{-} \oplus T_{(k,\varepsilon)}^{+})\,.$$

The rest of this section will be devoted to the proof of the above theorems. We begin by reviewing some properties of irreducible unitary representations of $G = SU(1, 1)$. We retain the notation due to Vilenkin

[10, Chapter VI]. Thus $T_{(\ell,\varepsilon)}$ with either $(\ell, \varepsilon) = (-1/2 + i\eta, 0)$, $\eta \geq 0$, or $(\ell, \varepsilon) = (-1/2 + i\eta, 1/2)$, $\eta > 0$, stands for a representation belonging to the continuous series, while $T_{(\ell,0)}$ with $-1 < \ell < -1/2$ is a representation belonging to the supplementary series. In this paper the representation $T^{\pm}_{(\ell,\varepsilon)}$ with either $(\ell, \varepsilon) = (\ell, 0)$, $-\ell \in Z_+ + 1$, or $(\ell, \varepsilon) = (\ell, 1/2)$, $-\ell \in Z_+ + 1/2$, is said to belong to the discrete series, even though $T^{\pm}_{(-1/2,1/2)}$ is not a member of the discrete series in the sense that it is not contained in the regular representation of G as a direct sum component. Recall that $C^\infty(T)$ (resp. a subspace of $C^\infty(T)$) is dense in the representation space $H_{\ell,\varepsilon}$ (resp. $H^{\pm}_{\ell,\varepsilon}$) of $T_{(\ell,\varepsilon)}$ (resp. $T^{\pm}_{(\ell,\varepsilon)}$).

LEMMA 2.1. *For the irreducible unitary representation $T_{(\ell,\varepsilon)}$ or $T^{\pm}_{(\ell,\varepsilon)}$ of $G = SU(1,1)$, define operators ω_j, $1 \leq j \leq 3$, $F_\pm$, H_3, Δ and spaces $\mathscr{W}_k$, $k \in Z/2$, as for the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$. Then $\mathscr{W}_k = \{\exp\{-i(k-\varepsilon)\theta\}\}$ if $k \in Z + \varepsilon$ and if $\exp\{-i(k-\varepsilon)\theta\}$ lies in the representation space, while $\mathscr{W}_k = \{0\}$ otherwise. In addition,*

$$F_\pm e^{-i(k-\varepsilon)\theta} = (\pm k - \ell) e^{-i(k-\varepsilon\pm 1)\theta}, \qquad \Delta = -\ell(\ell+1).$$

Proof. The function $\exp\{-i(k-\varepsilon)\theta\}$ is known to lie in $\mathscr{W}_k$, if it belongs to the representation space. Since such functions form a complete orthogonal basis of the representation space, $\dim \mathscr{W}_k \leq 1$. Thus $\mathscr{W}_k$ is obtained. The remaining part of the lemma is well-known [10, p. 299 and p. 334]. The sign of $\ell(\ell+1)$ on p. 334, however, is misprinted. Q.E.D.

A corollary of the next proposition plays an important role in our discussion.

PROPOSITION 2.2. *Let the notation be as in Lemma 2.1. Each $i\omega_j$, $1 \leq j \leq 3$, restricted to the algebraic sum $\Sigma_{k\in Z/2} \oplus \mathscr{W}_k$ is essentially selfadjoint in the representation space.*

Proof. Let $H_{\ell,\varepsilon,c}$ be the algebraic sum $\Sigma_k \oplus \mathscr{W}_k$, and denote by $\dot{\omega}_j$ the restriction $\omega_j | H_{\ell,\varepsilon,c}$. Set, further, $C^\infty = C^\infty(T) \cap H_{\ell,\varepsilon}$, where T stands for the unit circle and $H_{\ell,\varepsilon}$ is the representation space. Since a function $T_{(\ell,\varepsilon)}(g)f(e^{i\theta})$ or $T^{\pm}_{(\ell,\varepsilon)}(g)f(e^{i\theta})$ is smooth on $G \times T$ for any $f \in C^\infty$, C^∞ lies in the domain of ω_j and invariant under $T_{(\ell,\varepsilon)}$ or $T^{\pm}_{(\ell,\varepsilon)}$. Here we used the fact that the uniform convergence in C^∞ implies the convergence in $H_{\ell,\varepsilon}$. Let $\mathring{\omega}_j$ be the restriction $\omega_j | C^\infty$. We shall show that $i\mathring{\omega}_j$ is essentially selfadjoint. Evidently $i\dot{\omega}_j$ is symmetric, so it remains to show that the

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image $(\omega_j - \alpha)C^\infty$ is dense in $H_{\ell,\varepsilon}$ for any α, $\operatorname{Re}\alpha \neq 0$. For this purpose, assume that an f in $H_{\ell,\varepsilon}$ is orthogonal to the image. Then, since $T_{(\ell,\varepsilon)}(g)$ or $T^{\pm}_{(\ell,\varepsilon)}(g)$ leaves C^∞ invariant, we have

$$\langle T_{(\ell,\varepsilon)}(\omega_j(t))(\omega_j - \alpha)\phi, f\rangle = 0\,, \quad \phi \in C^\infty\,,$$

or a similar relation for $T^{\pm}_{(\ell,\varepsilon)}$. Multiply the both sides by $e^{-\alpha t}$, and integrate on R_+ or $-R_+$ according as $\operatorname{Re}\alpha$ is positive or negative. Then it follows that $\langle\phi, f\rangle = 0$, which implies $f = 0$, as desired. Thus $i\mathring{\omega}_j$ is essentially selfadjoint. To complete the proof, it suffices to show that the closure of $\dot{\omega}_j$ is an extension of $\mathring{\omega}_j$, for $\dot{\omega}_j \subset \mathring{\omega}_j$. To this end, we note first that $\dot{\omega}_j$ is a differential operator with smooth coefficients on T. Secondly, the partial sum of the Fourier series for any $f \in C^\infty$ lies in $H_{\ell,\varepsilon,c}$ and they and their derivatives uniformly converge to f and its derivative respectively. Now clearly the closure of $\dot{\omega}_j$ is an extension of $\mathring{\omega}_j$. Q.E.D.

COROLLARY 2.3. *For the irreducible unitary representations $T_{(\ell,\varepsilon)}$ belonging to the continuous series and $T^{\pm}_{(\ell,\varepsilon)}$ belonging to the discrete series in our sense, define ℓ^2-spaces $\ell^2_{\ell,\varepsilon}$ and $\ell^{2\pm}_{\ell,\varepsilon}$ as follows.*

$$\ell^2_{\ell,\varepsilon} = \{(a_k)_{k\in Z+\varepsilon};\ \Sigma_k|a_k|^2 < \infty\}\,,$$
$$\ell^{2\pm}_{\ell,\varepsilon} = \{(a_k)_{k\in Z+\varepsilon,\ \mp k+\ell\geq 0};\ \Sigma_k|a_k|^2 < \infty\}\,.$$

Put $\ell^2_{\ell,\varepsilon,c} = \{(a_k) \in \ell^2_{\ell,\varepsilon};\ a_k = 0,\ |k| > n,$ for some $n \in Z_+\}$, and define $\ell^{2\pm}_{\ell,\varepsilon,c}$ similarly. Then operators $i\dot{\omega}_j$, $1 \leq j \leq 2$, with domain $\ell^2_{\ell,\varepsilon,c}$ (resp. $\ell^{2\pm}_{\ell,\varepsilon,c}$) are essentially selfadjoint in $\ell^2_{\ell,\varepsilon}$ (resp. $\ell^{2\pm}_{\ell,\varepsilon}$), where $\dot{\omega}_j$ are defined as follows. Let $f_k = (a_{k'})$ be an element of either $\ell^2_{\ell,\varepsilon}$ or $\ell^{2\pm}_{\ell,\varepsilon}$ such that $a_k = 1$ and $a_{k'} = 0$, $k' \neq k$, and set $\dot{F}_\pm = -\dot{\omega}_1 \mp i\dot{\omega}_2$. We require

$$\begin{aligned}\dot{F}_\pm f_k &= \mp\sqrt{\eta^2 + (k \pm 1/2)^2}\,f_{k\pm1} && \text{in } \ell^2_{-1/2+i\eta,\varepsilon}\,,\\ &\quad \mp\sqrt{(k \mp \ell)(k \pm \ell \pm 1)}\,f_{k\pm1} && \text{in } \ell^{2+}_{\ell,\varepsilon}\,,\\ &\quad \pm\sqrt{(k \mp \ell)(k \pm \ell \pm 1)}\,f_{k\pm1} && \text{in } \ell^{2-}_{\ell,\varepsilon}\,.\end{aligned}$$

Proof. Let the notation be as in Lemma 2.1, and set

$$e_k = m_k \exp\{-i(k - \varepsilon)\theta\}/\|\exp\{-i(k - \varepsilon)\theta\}\| \in H_{\ell,\varepsilon}, \text{ where } |m_k| = 1\,.$$

In case (ℓ, ε) is a parameter of the continuous series, we can choose m_k so that $m_k/m_{k-1} = -|k + \ell|/(k + \ell)$. In other cases, set $m_k = 1$. Then it can be easily seen that the restriction of ω_j, $j = 1, 2$, in Proposition 2.2 is unitarily equivalent to $\dot{\omega}_j$ in the above lemma. Q.E.D.

The next lemma is concerned with a pair of one-parameter unitary groups.

LEMMA 2.4. *Let H_j, $j = 1, 2$, be Hilbert spaces, and $U_j(t)$ be one-parameter continuous unitary groups on H_j with the infinitesimal operators $\Omega_j = dU_j(t)/dt_{t=0}$. If H_1 is a closed subspace of H_2 and there exists an essentially selfadjoint operator $i\dot{\Omega}$ such that $\dot{\Omega} \subset \Omega_j$, $j = 1, 2$, then $U_1(t) = U_2(t)$ on H_1.*

Proof. Let Ω be the closure of $\dot{\Omega}$. Then $i\Omega$ is selfadjoint and clearly $\Omega \subset \Omega_j$. Consequently, for any $n \in Z_+ + 1$ and $h \in H_1$ we have

$$\Omega(1 - n^{-1}\Omega)^{-1}h = \Omega_j(1 - n^{-1}\Omega_j)^{-1}h\,, \qquad j = 1,\ 2.$$

That is, $\Omega(1 - n^{-1}\Omega)^{-1} = \Omega_j(1 - n^{-1}\Omega_j)^{-1}$ on H_1. By the representation theorem for the continuous semigroup [11, p. 248] we get

$$(\exp t\Omega)h = \lim_{n\to\infty}\{\exp t\Omega(1 - n^{-1}\Omega)^{-1}\}h = (\exp t\Omega_j)h\,, \qquad h \in H_1\,.$$

Q.E.D.

We return to the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$. Recall the definition of the subspaces $\tilde{D}^{\eta}_{k,\pm}$, $\hat{D}^{\eta}_{k,\pm}$ and the isometry $I^{\eta,\varepsilon}_{\pm,k}$ introduced in (1.34). Let us define auxilary Hilbert spaces $D^{\eta,\varepsilon}_{\pm}$, $D^{0,0}_{\pm1}$, $\tilde{D}^{\eta,\varepsilon}_{\pm}$, $\tilde{D}^{0,0}_{\pm1}$, $\hat{D}^{\eta,\varepsilon}_{\pm}$ and $\hat{D}^{0,0}_{\pm1}$ as follows.

$$\begin{aligned} D^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus D^{\eta}_{k,\pm}\,, &\quad D^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus D^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,,\\ \tilde{D}^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus \tilde{D}^{\eta}_{k,\pm}\,, &\quad \tilde{D}^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus \tilde{D}^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,,\\ \hat{D}^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus \hat{D}^{\eta}_{k,\pm}\,, &\quad \hat{D}^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus \hat{D}^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,. \end{aligned}$$

In terms of the isometries $\mathscr{F}_k : D^{\eta}_{k,\pm} \to \tilde{D}^{\eta}_{k,\pm}$ and $I^{\eta,\varepsilon}_{\pm,k} : \tilde{D}^{\eta}_{k,\pm} \to \hat{D}^{\eta}_{k,\pm}$ we can define onto isometries $\mathscr{F}^{\eta,\varepsilon}_{\pm} : D^{\eta,\varepsilon}_{\pm} \to \tilde{D}^{\eta,\varepsilon}_{\pm}$, $\mathscr{F}^{0,0}_{\pm1} : D^{0,0}_{\pm1} \to \tilde{D}^{0,0}_{\pm1}$, $I^{\eta,\varepsilon}_{\pm} : \tilde{D}^{\eta,\varepsilon}_{\pm} \to \hat{D}^{\eta,\varepsilon}_{\pm}$ and $I^{0,0}_{\pm1} : \tilde{D}^{0,0}_{\pm1} \to \hat{D}^{0,0}_{\pm1}$ in an obvious manner. Let $\hat{D}^{\eta,\varepsilon}_{\pm,c}$ be a dense subspace $\{(h_k) \in \hat{D}^{\eta,\varepsilon}_{\pm};\ h_k \in C_0(R_+) \oplus \tilde{E}_{k,\pm},\ h_k = 0 \text{ for large } |k|\}$, and put

$$\mathscr{D}^{\eta,\varepsilon}_{\pm,c} = (I^{\eta,\varepsilon}_{\pm}\mathscr{F}^{\eta,\varepsilon}_{\pm}J^{\eta,\varepsilon})^{-1}\hat{D}^{\eta,\varepsilon}_{\pm,c}\,.$$

Similarly we define $\hat{D}^{0,0}_{\pm1,c}$ and $\mathscr{D}^{0,0}_{\pm1,c}$.

LEMMA 2.5. *Let ω_j, $j = 1, 2$, be the infinitesimal operator of $U^{\eta,\varepsilon}(0, \omega_j(t))$. Then the restriction $i\omega_j|\mathscr{D}^{\eta,\varepsilon}_{\pm,c}$ is essentially selfadjoint in $\mathscr{D}^{\eta,\varepsilon}_{\pm}$. In case $(\ell, \varepsilon) = (0, 0)$, so is the restriction $i\omega_j|\mathscr{D}^{0,0}_{\pm1,c}$ in $\mathscr{D}^{0,0}_{\pm1}$.*

Proof. Only the operator $i\omega_j|\mathscr{D}^{0,0}_{1,c}$, $1 \le j \le 2$, is to be discussed. Denote it by $i\dot{\omega}_j$, and set $\hat{\omega}_j = I^{0,0}_1\mathscr{F}^{0,0}_1J^{0,0}\dot{\omega}_j(I^{0,0}_1\mathscr{F}^{0,0}_1J^{0,0})^{-1}$, $\hat{F}_{\pm} = -\hat{\omega}_1 \mp i\hat{\omega}_2$. First,

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suppose k is a negative integer, we recall the definition of $e_{k,n}$ given after Lemma 1.7. Evidently $\{\mathscr{F}_k e_{k,n}; n = k, k+1, \cdots, -1\}$ is a basis of $\tilde{E}_k$. On account of (1.33) a closed subspace $\hat{E}_n$, $-n \in Z_+ + 1$, of $\hat{D}_1^{0,0}$ spanned by $\{\mathscr{F}_k e_{k,n}; k = n, n-1, \cdots\}$ is invariant under $\hat{F}_\pm$. Moreover, Corollary 2.3, together with (1.33), implies that $i\hat{\omega}_j$ is essentially selfadjoint in $\hat{E}_n$. As one can see easily, this assertion is valid even for $n \in Z_+ + 1$. It remains, therefore, to show the essentially selfadjointness of $i\hat{\omega}_j$ in $\Sigma_{k\in Z} \oplus L^2(R_+) \subset \hat{D}_1^{0,0}$. To this end, let $C_{0,c}$ be the algebraic sum $\Sigma_{k\in Z} \oplus C_0(R_+)$, and we shall prove that the image $(i\hat{\omega}_j - z)C_{0,c}$, $\operatorname{Im} z \neq 0$, is dense in $\Sigma_{k\in Z} \oplus L^2(R_+)$. If $h = (h_{k'})$ is an element of $C_{0,c}$ such that $h_{k'} = 0$ for $k' \neq k$, then we have by (1.35) the following.

$$i\hat{\omega}_j h(\lambda) = (\cdots, 0, a_{jk}(\lambda)h_k(\lambda), 0, b_{jk}(\lambda)h_k(\lambda), 0, \cdots),$$

where a_{jk} and b_{jk} are smooth functions on R_+. We consider an operator $i\hat{\omega}_j(\lambda)$ in $\ell^2 = \Sigma_{k\in Z} \oplus C$ with domain $\ell_c^2 = \{(a_k) \in \ell^2; a_k = 0 \text{ for large } |k|\}$ such that

$$i\hat{\omega}_j(\lambda)e_k = (\cdots, 0, a_{jk}(\lambda), 0, b_{jk}(\lambda), 0, \cdots)$$

for $e_k = (\cdots, 0, 0, 1, 0, 0, \cdots)$. It follows from (1.35) and Corollary 2.3 that $i\hat{\omega}_j(\lambda)$ is essentially selfadjoint. Suppose an h in $\Sigma_{k\in Z} \oplus L^2(R_+)$ is orthogonal to $(i\hat{\omega}_j - z)C_{0,c}$, $\operatorname{Im} z \neq 0$. Then we obtain

$$a_{jk}(\lambda)h_{k-1}(\lambda) - z^* h_k(\lambda) + b_{jk}(\lambda)h_{k+1}(\lambda) = 0 \text{ a.e. on } R_+ .$$

Since $i\hat{\omega}_j(\lambda)$ is essentially selfadjoint in ℓ^2, $(h_k(\lambda))$ is a zero vector in ℓ^2 a.e. This means $h = 0$ in $\Sigma_{k\in Z} \oplus L^2(R_+)$. We have shown that $i\hat{\omega}_j$ is essentially selfadjoint in $\Sigma_{k\in Z} \oplus L^2(R_+)$, for it is symmetric. Q.E.D.

We are ready for the proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. We shall prove the sufficiency first. Set $\mathscr{D}_{k,\pm}^{\eta,\varepsilon} = \mathscr{D}_\pm^{\eta,\varepsilon} \cap \mathscr{W}_k^{\eta,\varepsilon}$, $\mathscr{D}_{k,\pm 1}^{0,0} = \mathscr{D}_{\pm 1}^{0,0} \cap \mathscr{W}_k^{0,0}$. It is evident that $U^{\eta,\varepsilon}(0, \omega_3(t))$ leaves $\mathscr{D}_{k,\pm}^{\eta,\varepsilon}$ (and $\mathscr{D}_{k,\pm 1}^{0,0}$ as well, provided $(\eta, \varepsilon) = (0, 0)$) invariant. By (2.2) and Theorem 1.3 $U^{\eta,\varepsilon}(t, 0, 0, e)$, $t \geq 0$, also leaves $\mathscr{D}_\pm^{\eta,\varepsilon}$ invariant. We note that $P_+(3)$ is topologically generated by the subsemigroup $\{(t, 0, 0, e); t \geq 0\}$ and the subgroup $\{(0, g); g \in G\}$, and that so is G by one-parameter groups $\omega_j(t)$, $j = 2, 3$. To complete the proof of sufficiency, it is enough to show that $U^{\eta,\varepsilon}(0, \omega_2(t))$ keeps $\mathscr{D}_\pm^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm 1}^{0,0}$ as well, if $(\eta, \varepsilon) = (0, 0)$) invariant. But this fact is an immediate consequence of Lemmas 2.4 and 2.5. Secondly, we shall show the necessity of the condition. Assume that $\mathscr{D}$ is a $P_+(3)$-invariant closed proper subspace of $\mathfrak{H}^{\eta,\varepsilon}$. Since $(t, 0, 0, e) \in P(3)$ commutes with $(0, \omega_3(s)) \in P(3)$, $\mathscr{D}_k^{\eta,\varepsilon} = \mathscr{D} \cap \mathscr{W}_k^{\eta,\varepsilon}$ is invariant under $U^{\eta,\varepsilon}(t, 0, 0, e)$, $t \geq 0$. Moreover, $\mathscr{D}$ being G-invariant, we have

$$\varDelta \mathscr{D}_k^{\eta,\varepsilon} \subset \mathscr{D}_k^{\eta,\varepsilon}, \qquad F_\pm \mathscr{D}_k^{\eta,\varepsilon} \subset \mathscr{D}_{k\pm 1}^{\eta,\varepsilon}, \qquad k \in Z/2.$$

Thus $\mathscr{D}$ must coincide with one of $\mathscr{D}_\pm^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm 1}^{0,0}$, provided $(\eta, \varepsilon) = (0, 0)$) in virtue of (2.2), (2.3) and Theorem 1.4. Q.E.D.

Proof of Theorem 2.2. Let $\mathscr{D}_{k,\pm}^{\eta,\varepsilon}$ and $\mathscr{D}_{k,\pm 1}^{0,0}$ be the same as in the above proof. First consider the case $\varepsilon = 1/2$. Then $\mathscr{D}_{k,\pm}^{\eta,\varepsilon} = \{0\}$, $k \in Z$ and

$$\text{(2.5)} \qquad \begin{aligned} &\dim (\mathscr{D}_{k,-}^{\eta,\varepsilon} \ominus F_+ \mathscr{D}_{k-1,-}^{\eta,\varepsilon}) = 0, \qquad k \in Z_+ + \varepsilon, \\ &\dim (\mathscr{D}_{k,-}^{\eta,\varepsilon} \ominus F_- \mathscr{D}_{k+1,-}^{\eta,\varepsilon}) = 0 \quad \text{or} \quad 1 \\ &\qquad \text{according as } -k = 1/2 \text{ or } -k \in Z_+ + 3/2. \end{aligned}$$

These relations imply that among the representations belonging to the discrete series only the representations $T_{(k,\varepsilon)}^+$, $-k \in Z_+ + 3/2$, are contained with multiplicity one in $\mathscr{D}_-^{\eta,\varepsilon}$. Since the following unitary equivalences hold

$$(\varDelta - 1/4) | \mathscr{D}_{1/2,-}^{\eta,\varepsilon} \simeq \mathscr{L}_{1/2,\eta} | D_{1/2,-}^{\eta} \simeq \int_{R_+}^{\oplus} \lambda \, d\lambda,$$

the representations $T_{(-1/2+i\eta,\varepsilon)}$, $\eta > 0$, are contained in $\mathscr{D}_-^{\eta,\varepsilon}$ as

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)} d\eta.$$

Consequently the representation $(U^{\eta,\varepsilon}, \mathscr{D}_-^{\eta,\varepsilon})$ of G admits a decomposition as stated in Theorem 2.1. We can argue similarly for the representation of G in $\mathscr{D}_+^{\eta,\varepsilon}$. Secondly, assume that $\varepsilon = 0$. We shall confine our discussion to the representation $(U^{0,0}, \mathscr{D}_1^{0,0})$. Since $\mathscr{W}_k^{\eta,\varepsilon} = \{0\}$ for $k \notin Z + \varepsilon$, $\mathscr{D}_{k,1}^{0,0} = \{0\}$, $k \in Z + 1/2$. Moreover, $\dim (\mathscr{D}_{k,1}^{0,0} \ominus F_\pm \mathscr{D}_{k\mp 1,1}^{0,0}) = 1$ for $k \in Z \backslash \{0\}$. This means that among the representations in the discrete series only $T_{(k,0)}^{\mp}$, $-k \in Z_+ + 1$, are contained with multiplicity one in $\mathscr{D}_1^{0,0}$. On account of the following unitary equivalences

$$(\varDelta - 1/4) | \mathscr{D}_{0,1}^{0,0} \simeq \mathscr{L}_{0,0} | D_{0,+}^{0} \simeq \int_{R_+}^{\oplus} \lambda \, d\lambda.$$

We conclude that the representations $T_{(-1/2+i\eta,0)}$, $\eta \geq 0$, are contained as

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,0)} \, d\eta.$$

We have verified Theorem 2.2 for the representation in $\mathscr{D}_1^{0,0}$. Q.E.D.

Appendix

The first lemma is concerned with an n-th order equation assuming the following form.

$$z^n w^{(n)} + z^{n-1} c_1(z, \lambda) w^{(n-1)} + \cdots + c_n(z, \lambda) w = 0\,, \tag{A.1}$$

where c_j, $1 \leq j \leq n$, are holomorphic in $\{|z| < \delta_1\} \times \{|\lambda| < \delta_2\}$, $c_j(0, \lambda)$ being constant.

LEMMA A.1. (i) *If the above equation has a solution of the form* $z^\alpha(1 + zh(z, \log z))$, *then* α *is an indicial root, that is,*

$$(\alpha-1)\cdots(\alpha-n+1)+c_1(0,\lambda)(\alpha-1)\cdots(\alpha-n+2)+\cdots+c_n(0,\lambda)=0\,. \tag{A.2}$$

(ii) *Suppose* α_j, $1 \leq j \leq k$, *are roots of* (A.2) *such that* $\alpha_j - \alpha_{j+1}$ *is a positive integer and that there are no other roots in* $Z_+ + \alpha_k$. *Assume further that* α_j, $1 \leq j < k$, *is a simple root while* α_k *is an* m_k*-ple root. Then there exists a system of solutions* $w_j(z, \lambda)$, $1 \leq j \leq k + m_k - 1$, *such that* w_j, *being holomorphic in* $\{0 < |z| < \varepsilon;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$ *for some positive* ε *depending on* δ_2, *takes the following form.*

$$\begin{aligned} &z^{\alpha_1}(1 + zh(z))\,, && j = 1\,,\\ &z^{\alpha_j}(1 + zh(z, \log z))\,, && 2 \leq j \leq k\,,\\ &z^{\alpha_k}((\log z)^{j-k} + zh(z, \log z))\,, && k < j < k + m_k\,, \end{aligned}$$

where $h(z)$ *and* $h(z, \log z)$ *stand for, respectively, a holomorphic function and a polynomial in* $\log z$ *with holomorphic coefficients.*

Proof. To verify (i), it suffices to compare the coefficients of z^α on the both sides of (A.1). The Frobenius method yields (ii) [1, p. 133]. Indeed, put $L = z^n d^n/dz^n + z^{n-1} c_1 d/dz^{n-1} + \cdots + c_n$, and denote by $f(\alpha)$ the polynomial on the left side of (A.2). As is well known, we can find a formal series

$$\phi_j(z, \lambda, \alpha) = z^\alpha \sum_{p=0}^{\infty} d_{jp}(\lambda, \alpha) z^p\,, \qquad d_{j0} = (\alpha - \alpha_j)^{j-1}\,,$$

such that $L\phi_j = f(\alpha) z^\alpha (\alpha - \alpha_j)^{j-1}$. Take δ so small that there is no roots of $f(\alpha)$ in $\{|\alpha - \alpha_j| < \delta\}$ except for α_j. Then it can be shown that $d_{jp}(\lambda, \alpha)$ is homomorphic and $|d_{jp}(\lambda, \alpha)| < K^{2p+1}$, $K > 0$, in $\{|\alpha - \alpha_j| < \delta\} \times \{|\lambda| < \delta_2\}$. Setting $\alpha_j = \alpha_k$ for $j > k$, it suffices to put

$$w_j(z, \lambda) = (\partial/\partial\alpha)^{j-1}_{\alpha=\alpha_j}\phi_j(z, \lambda, \alpha), \qquad 1 \leq j < k + m_k .$$

By Osgood's lemma [3] w_j is holomorphic in $\{0 < |z| < 1/K;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$. Q.E.D.

Next consider a differential equation

$$d/dz\, w = A(z, \lambda)w\,, \qquad A(z, \lambda) = \sum_{m=-1}^{\infty} A_m(\lambda) z^m\,, \tag{A.3}$$

where $A(z, \lambda)$ is an M_n-valued holomorphic function on $\{0 < |z| < \delta_1\} \times \{|\lambda| < \delta_2\}$, $A_{-1}(0, \lambda)$ being constant.

LEMMA A.2. (i) *If the above equation has a solution of the form* $z^\alpha(p + zh(z, \log z))$, *then* $(A_{-1} - \alpha)p = 0$.

(ii) *Assume that* α_j, $1 \leq j \leq k$, *are characteristic roots of* A_{-1} *such that* $\alpha_j - \alpha_{j+1}$ *is a positive integer and that there are no other characteristic roots in* $Z_+ + \alpha_k$. *Assume further that* α_j, $1 \leq j < k$, *is a simple root. Then there exists a system of solutions* $w_j(z, \lambda)$, $1 \leq j \leq k$, *such that* w_j, *being holomorphic in* $\{0 < |z| < \varepsilon;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$ *for some positive* ε *depending on* δ_2, *takes the following form.*

$$z^{\alpha_1}(p_1 + zh(z)) \text{ for } j = 1, \quad z^{\alpha_j}(p_j + zh(z, \log z)) \text{ for } 1 < j \leq k\,,$$

where $(A_{-1} - \alpha_j)p_j = 0$. *The functions* $h(z)$ *and* $h(z, \log z)$ *stand for the same as in Lemma* A.1.

Proof. Compare the coefficients of $z^{\alpha-1}$ on the both sides of (A.3). Then (i) follows. The Frobenius method yields (ii) [1, pp. 136–137]. To be more precise, let $\psi(z, \lambda, \alpha, s_0)$ be a formal series $\sum_{m=0}^{\infty} s_m z^{m+\alpha}$ such that $\psi' - A\psi = (\alpha - A_{-1})s_0 z^{\alpha-1}$, where ψ' denotes the formal series $\sum_{m=0}^{\infty}(\alpha + m)z^{\alpha+m-1}$. Then each component of s_m $(m \geq 1)$, is a rational function of α. Let δ be small enough so that only α_j is a characteristic root of A_{-1} in $\{|\alpha - \alpha_j| < \delta\}$. When $s_0 = p_1$, there exists a positive K such that $|s_m(\lambda, \alpha)| < K^{2m+1}$ in $\{|\lambda| < \delta_2\} \times \{|\alpha - \alpha_j| < \delta\}$. We can set $w_1(z, \lambda) = \psi(z, \lambda, \alpha_1, p_1)$. When $s_0 = (\alpha - \alpha_j)^{j-1}p_j$ $(j > 1)$, $s_m(\lambda, \alpha)$ is holomorphic and $|s_m(\lambda, \alpha)| < K^{2m+1}$ in $\{|\lambda| < \delta_2\} \times \{|\alpha - \alpha_j| < \delta\}$ for some positive K depending on δ_2. In this case, set

$$w_j(z, \lambda) = (\partial/\partial\alpha)^{j-1}_{\alpha=\alpha_j}\psi(z, \lambda, \alpha, s_0)\,, \qquad j > 1\,.$$

The desired analyticity follows from Osgood's lemma [3]. Q.E.D.

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