Ergod. Th. & Dynam. Sys., (2021), **41**, 3807–3820
doi:10.1017/etds.2020.123

Isomorphisms between determinantal point processes with translation-invariant kernels and Poisson point processes

SHOTA OSADA

Institute of Mathematics for Industry,
Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
(e-mail: s-osada@math.kyushu-u.ac.jp)

(*Received* 3 *October* 2019 *and accepted in revised form* 18 *October* 2020)

Abstract. We prove the Bernoulli property for determinantal point processes on $\mathbb{R}^d$ with translation-invariant kernels. For the determinantal point processes on $\mathbb{Z}^d$ with translation-invariant kernels, the Bernoulli property was proved by Lyons and Steif [Stationary determinantal processes: phase multiplicity, bernoullicity, and domination. *Duke Math. J.* **120** (2003), 515–575] and Shirai and Takahashi [Random point fields associated with certain Fredholm determinants II: fermion shifts and their ergodic properties. *Ann. Probab.* **31** (2003), 1533–1564]. We prove its continuum version. For this purpose, we also prove the Bernoulli property for the tree representations of the determinantal point processes.

Key words: Bernoulli property, determinantal point processes, tree representations
2020 Mathematics Subject Classification: 37A50 (Primary); 60G55 (Secondary)

1. *Introduction and the main result*

We consider an isomorphism problem of measure-preserving dynamical systems among translation-invariant point processes on $\mathbb{R}^d$ such as the homogeneous Poisson point processes and the determinantal point processes with translation-invariant kernel functions.

The homogeneous Poisson point process is a point process in which numbers of particles on disjoint subsets obey independently Poisson distributions. It is parameterized using intensity $r > 0$. From the general theory of Ornstein and Weiss [9], homogeneous Poisson point processes are isomorphic to each other regardless of the value of r.

The determinantal point process is a point process for which the determinants of its kernel function give its correlation functions. It describes a repulsive particle system and appears in various mathematical systems such as uniform spanning trees, the zeros of a hyperbolic Gaussian analytic function with a Bergman kernel, and the eigenvalue distribution of random matrices.

These two classes of point processes have different properties in correlations among particles. For example, determinantal point processes have negative associations [4]. The sine point process is a typical example of a translation-invariant determinantal point process that has number rigidity [1]. In contrast, Poisson point processes do not have this property because the particles are regionally independent. Nevertheless, we prove that they are isomorphic to each other.

We start by recalling the isomorphism theory.

An automorphism S of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a bi-measurable bijection such that $\mathbb{P} \circ \mathsf{S}^{-1} = \mathbb{P}$. Let $\mathsf{S}_G = \{\mathsf{S}_g : g \in G\}$ be a group of automorphisms of $(\Omega, \mathcal{F}, \mathbb{P})$ parametrized by a group G. A measure-preserving dynamical system of G-action is the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$. We call $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ the G-action system for short.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ and $(\Omega', \mathcal{F}', \mathbb{P}', \mathsf{S}'_G)$ be G-action systems. A factor map is a measurable map $\phi : \Omega \to \Omega'$ such that

$$\mathbb{P} \circ \phi^{-1} = \mathbb{P}', \quad \phi \circ \mathsf{S}_g(x) = \mathsf{S}'_g \circ \phi(x) \quad \text{for each } g \in G \text{ and a.e. } x \in \Omega.$$

In this case, we call $(\Omega', \mathcal{F}', \mathbb{P}', \mathsf{S}'_G)$ the ϕ-factor of $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ or simply a factor of $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$. An isomorphism is a bi-measurable bijection $\phi : \Omega \to \Omega'$ such that both ϕ and ϕ^{-1} are factor maps. If there exists an isomorphism $\phi : \Omega \to \Omega'$, then $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ and $(\Omega', \mathcal{F}', \mathbb{P}', \mathsf{S}'_G)$ are said to be isomorphic.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ be a G-action system with a measurable map ϕ from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$. Then $(\Omega', \mathcal{F}_\phi, \mathbb{P}_\phi, \mathsf{S}^\phi_G)$ is a G-action system. Here, $(\Omega', \mathcal{F}_\phi, \mathbb{P}_\phi)$ is the completion of $(\Omega', \sigma[\phi], \mathbb{P} \circ \phi^{-1})$, and $\mathsf{S}^\phi_G = \{\phi \circ \mathsf{S}_g \circ \phi^{-1} : g \in G\}$. We also call the G-action system $(\Omega', \mathcal{F}_\phi, \mathbb{P}_\phi, \mathsf{S}^\phi_G)$ the ϕ-factor of $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$.

A typical system with a discrete group action is a Bernoulli shift. A G-action Bernoulli shift is a system formed from the direct product of a probability space over G and the canonical shift. Ornstein [6, 7] proved that the $\mathbb{Z}$-action Bernoulli shifts with equal entropy are isomorphic to each other. We call a system $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ Bernoulli if $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_G)$ is isomorphic to a Bernoulli shift. Ornstein and Weiss [9] extended the isomorphism theory to amenable group actions. As a consequence of the general theory, all the homogeneous Poisson point processes on $\mathbb{R}^d$ are isomorphic to each other regardless of their intensity. We call $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{R}^d})$ Bernoulli if $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{R}^d})$ is isomorphic to a homogeneous Poisson point process. We also refer to Kalikow and Weiss [2] for $d = 1$, who constructed an explicit isomorphism between the time-one map of a homogeneous Poisson point process and a Bernoulli shift with infinite entropy. In this paper, we do not consider to construct explicit isomorphisms.

Let X be a locally compact Hausdorff space with countable basis. We denote by $\mathsf{Conf}(X)$ the set of all non-negative integer-valued Radon measures on X. We equip $\mathsf{Conf}(X)$ with the vague topology, under which $\mathsf{Conf}(X)$ is a Polish space. We call a Borel probability measure μ on $\mathsf{Conf}(X)$ a point process on X. We say that μ is simple when $\xi(\{x\}) \in \{0, 1\}$ for each $x \in X$ for a.e. $\xi \in \mathsf{Conf}(X)$.

Let μ be a point process on X. Throughout this paper, we write the completion of μ by the same symbol. We also write $(\mathsf{Conf}(X), \mu, \mathsf{T}_G)$ as the G-action system made of the completion of $(\mathsf{Conf}(X), \mathcal{B}(\mathsf{Conf}(X)), \mu)$ and a G-action group of automorphisms T_G.

A homogeneous Poisson point process with intensity $r > 0$ is the point process on $\mathbb{R}^d$ satisfying:

(1) $\xi(A)$ has a Poisson distribution with mean $r|A|$ for each $A \in \mathcal{B}(\mathbb{R}^d)$;

(2) $\xi(A_1), \ldots, \xi(A_k)$ are independent for any disjoint subsets $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d)$. Here, $\xi(A)$ is the number of particles on A for $\xi \in \mathsf{Conf}(X)$ and $|A|$ is the Lebesgue measure of A.

A determinantal point process μ on X is a point process associated with a kernel function $K : X \times X \to \mathbb{C}$ and a Radon measure λ on X, for which the n-point correlation function with respect to λ is given by

$$\rho_n(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n \tag{1.1}$$

for each $n \in \mathbb{N}$. See Definition 4.1 for the definition of the n-point correlation function. We call μ a (K, λ)-determinantal point process. If the context is clear, we omit λ, calling μ a K-determinantal point process. Throughout this paper, we assume that λ is the Lebesgue measure if $X = \mathbb{R}^d$.

Now, we state the main theorem.

THEOREM 1.1. *Let $\hat{K} \in L^1(\mathbb{R}^d)$ be such that $\hat{K}(t) \in [0, 1]$ for almost every (a.e.) $t \in \mathbb{R}^d$. Let μ^K be a determinantal point process on $\mathbb{R}^d$ with translation-invariant kernel K such that*

$$K(x, y) = \int_{\mathbb{R}^d} \hat{K}(t) e^{2\pi i (x-y)\cdot t}\, dt. \tag{1.2}$$

Then $(\mathsf{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{R}^d})$ is Bernoulli. Here, $\mathsf{T}_a : \sum_i \delta_{x_i} \mapsto \sum_i \delta_{x_i+a}$ for $a \in \mathbb{R}^d$ and $\mathsf{T}_{\mathbb{R}^d} = \{\mathsf{T}_a : a \in \mathbb{R}^d\}$.

We remark that the assumption for K in Theorem 1.1 implies the following conditions (1)–(4) with $X = \mathbb{R}^d$ and the Lebesgue measure λ.

(1) $K : X \times X \to \mathbb{C}$ is Hermitian symmetric.

(2) For each compact set $A \subset X$, the integral operator K on $L^2(A, \lambda)$ is of trace class.

(3) $\mathsf{Spec} K \subset [0, 1]$.

(4) $K(x, y) = K(x - y, 0)$.

Under assumptions (1)–(3), there exists a unique (K, λ)-determinantal point process μ with the kernel function K [**11**, **13**].

The K-determinantal point process μ satisfying (1)–(4) above is translation invariant because its n-correlation functions are translation invariant.

For determinantal point processes on $\mathbb{Z}^d$ with translation-invariant kernel and the counting measure, Lyons and Steif [**5**] and Shirai and Takahashi [**12**] independently proved the Bernoulli property, the latter giving a sufficient condition for the weak Bernoulli property under the assumption $K : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ satisfying (1), (2), $\mathsf{Spec}(K) \subset (0, 1)$, and (4). We recall that the weak Bernoulli property is stronger than the Bernoulli property.

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

Lyons and Steif [5] proved the Bernoulli property for the case K satisfying (1)–(4). Theorem 1.1 is its continuum version.

One of the ideas in [5] is using the dbar distance, which is a metric on the set of $\mathbb{Z}^d$-action systems; the Bernoulli property is closed under this metric [8, 9, 14]. However, the dbar distance does not work for systems with infinite entropy because entropy is continuous with respect to the dbar distance. In general, a translation-invariant point process on $\mathbb{R}^d$ has infinite entropy. Therefore, we cannot apply the dbar distance to our case. Therefore, we construct point processes on a discrete set that approximate the determinantal point process on $\mathbb{R}^d$. We prove the Bernoulli property of the discrete point processes. In turn, we can prove the Bernoulli property of the determinantal point process on $\mathbb{R}^d$ via its tree representations [10].

To prove Theorem 1.1, we apply the general theory given by Ornstein and Weiss [9]. We quote them in the form applicable to the $\mathbb{R}^d$- and $\mathbb{Z}^d$-actions. We also refer to [8] for the $\mathbb{Z}$- and $\mathbb{R}$-actions, and [14] for the $\mathbb{Z}^d$-action.

The outline of this paper is as follows. In §2, we recall notions related to the Bernoulli property. In §3, we introduce the kernel functions that approximate the determinantal kernel K in Theorem 1.1 uniformly on any compact set on $\mathbb{R}^d$. In §4, we introduce the tree representations of the determinantal point processes on $\mathbb{R}^d$. We combine these representations with the kernels introduced in §3. The tree representations are determinantal point processes on $\mathbb{Z}^d \times \mathbb{N}$ and are translation invariant with respect to the first coordinate. In §5, we prove the Bernoulli property of the tree representations using the properties of the dbar distance introduced in §2. In §6, we prove Theorem 1.1 using the Bernoulli property of the tree representations.

2. *Notions related to the Bernoulli property*

In this section, we collect properties of point processes without determinantal structure and notions related to the Bernoulli property.

We first recall the notion of monotone coupling. For $\zeta^i = \{\zeta^i_z\}_{z\in\mathbb{Z}^d} \in \{0, 1\}^{\mathbb{Z}^d}$ $(i = 1, 2)$, we write $\zeta^1 \le \zeta^2$ if $\zeta^1_z \le \zeta^2_z$ for each $z \in \mathbb{Z}^d$. We equip $\{0, 1\}^{\mathbb{Z}^d}$ with the product topology. We call a continuous function $f : \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ a monotone function on $\{0, 1\}^{\mathbb{Z}^d}$ if $\zeta^1 \le \zeta^2$ implies that $f(\zeta^1) \le f(\zeta^2)$. Let $\mathcal{B}$ be the Borel σ-field of $\{0, 1\}^{\mathbb{Z}^d}$. For probability measures μ and ν on $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B})$, we write $\mu \le \nu$ if for each monotone function f,

$$\int_{\{0,1\}^{\mathbb{Z}^d}} f\, d\mu \le \int_{\{0,1\}^{\mathbb{Z}^d}} f\, d\nu.$$

Let ν_1 and ν_2 be probability measures on $\{0, 1\}^{\mathbb{Z}^d}$. We say that a probability measure γ on $\{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d}$ is a monotone coupling of ν_1 and ν_2 if the following hold.

(1) $\gamma(A \times \{0, 1\}^{\mathbb{Z}^d}) = \nu_1(A)$ for $A \in \mathcal{B}$.

(2) $\gamma(\{0, 1\}^{\mathbb{Z}^d} \times B) = \nu_2(B)$ for $B \in \mathcal{B}$.

(3) $\gamma(\{(\zeta^1, \zeta^2) \in \{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d}; \zeta^1 \le \zeta^2\}) = 1$.

LEMMA 2.1. (E.g. [3]) *For probability measures μ and ν on $\{0, 1\}^{\mathbb{Z}^d}$, the following statements are equivalent.*

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

(1) $\mu \le \nu$.
(2) *There exists a monotone coupling of* μ *and* ν.

We naturally regard a simple point process μ on $\mathbb{Z}^d \times \mathbb{N}$ as a probability measure on $\{0, 1\}^{\mathbb{Z}^d \times \mathbb{N}}$, denoted by the same symbol μ. We write $\mu \le \nu$ for simple point processes μ and ν if the corresponding probability measures on $\{0, 1\}^{\mathbb{Z}^d \times \mathbb{N}}$ satisfy $\mu \le \nu$. We introduce the notion of monotone coupling for simple point processes on $\mathbb{Z}^d \times \mathbb{N}$ from that of the corresponding probability measures on $\{0, 1\}^{\mathbb{Z}^d \times \mathbb{N}}$ in an obvious fashion.

Fix $N \in \mathbb{N}$. We set $[N] = \{1, \ldots, N\}$. Let $Q^N = \{Q^N_{z,l} : (z, l) \in \mathbb{Z}^d \times [N]\}$ be a partition of $\mathbb{Z}^d \times \mathbb{N}$ such that

$$Q^N_{z,l} = \begin{cases} \{(z, l)\} & \text{for } l \in [N-1], \\ \{(z, m) \in \mathbb{Z}^d \times \mathbb{N}; m \ge l\} & \text{for } l = N \end{cases} \tag{2.1}$$

for each $(z, l) \in \mathbb{Z}^d \times [N]$. For $\xi \in \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N})$, we set

$$\omega^N_{z,l}(\xi) = 1_{\{\xi(Q^N_{z,l}) \ge 1\}}.$$

Let $\varpi_N : \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}) \to \{0, 1\}^{\mathbb{Z}^d \times [N]}$ denote the map

$$\xi \mapsto \{\omega^N_{z,l}(\xi)\}_{(z,l) \in \mathbb{Z}^d \times [N]}. \tag{2.2}$$

We denote the image measure $\nu \circ \varpi_N^{-1}$ by ν_N for a point process ν on $\mathbb{Z}^d \times \mathbb{N}$.

PROPOSITION 2.2. *Let* μ *and* ν *be simple point processes on* $\mathbb{Z}^d \times \mathbb{N}$. *Assume that* $\mu \le \nu$. *Then* $\mu_N \le \nu_N$.

Proof. By assumption and Lemma 2.1, there exists a monotone coupling γ of μ and ν. Let $\gamma_N(\xi, \eta) = \gamma \circ (\varpi_N(\xi), \varpi_N(\eta))^{-1}$. Then γ_N is a monotone coupling of μ_N and ν_N. From this and Lemma 2.1, we obtain the claim. □

We recall the notion of being finitely dependent, which is a sufficient condition for the Bernoulli property. See, e.g., [5].

Definition 2.3. Let Ω be a countable set.

(1) A probability measure ν on $\Omega^{\mathbb{Z}^d}$ is called r-dependent if, for each $R, S \subset \mathbb{Z}^d$,

$$\inf\{\mathsf{d}(z, w); z \in R, w \in S\} \ge r \Rightarrow \sigma[\pi_R] \text{ and } \sigma[\pi_S] \text{ are independent.}$$

Here, $\mathsf{d}(z, w)$ is the graph distance on $\mathbb{Z}^d$ and $\pi_R : \Omega^{\mathbb{Z}^d} \to \Omega^R$ is the projection given by $\{\omega_z\}_{z \in \mathbb{Z}^d} \mapsto \{\omega_z\}_{z \in R}$.

(2) ν is called finitely dependent if ν is r-dependent for some $r \in \mathbb{N}$.

Let $\mathcal{P}_{\mathrm{inv}}(M)$ be the set of translation-invariant probability measures on $[M]^{\mathbb{Z}^d}$. For $x, y \in \mathbb{Z}^d$, define $x < y$ if $x_i < y_i$ for $i = \min\{j = 1, \ldots, d; x_j \ne y_j\}$. For $P, Q \subset \mathbb{Z}^d$, we set $P < Q$ if $x < y$ for all $x \in P$ and $y \in Q$.

Definition 2.4. (Very weak Bernoulli) We call $\nu \in \mathcal{P}_{\mathrm{inv}}(M)$ very weak Bernoulli if for each $\epsilon > 0$, there is a rectangle $R \subset \mathbb{Z}^d$ such that if, for any finite set $Q = \{x_1, \ldots, x_m\} < R$,

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

there exists an $\mathcal{A} \subset \sigma[\pi_Q]$ satisfying (2.3) and (2.4),

$$\nu\Big(\bigcup_{A\in\mathcal{A}} A\Big) > 1-\epsilon, \tag{2.3}$$

$$\inf_{\gamma\in\Gamma(\nu|_R,\nu_A|_R)} \mathsf{E}^{\gamma}\Big[\frac{1}{\#R}\sum_{z\in R} 1_{\{X_z\neq Y_z\}}\Big] < \epsilon \quad \text{for } A\in\mathcal{A}. \tag{2.4}$$

Here, ν_A denotes the conditional probability measure under A, $\nu|_R = \nu\circ\pi_R^{-1}$, and $\nu_A|_R = \nu_A\circ\pi_R^{-1}$. Furthermore, $\Gamma(\nu|_R, \nu_A|_R)$ is the collection of the couplings of $\nu|_R$ and $\nu_A|_R$, and $((X_z)_{z\in R}, (Y_z)_{z\in R}) \in [M]^R \times [M]^R$.

LEMMA 2.5. (E.g. [5]) *If $\nu \in \mathcal{P}_{\mathrm{inv}}(M)$ is finitely dependent, then ν is very weak Bernoulli.*

The very weak Bernoulli property is equivalent to the Bernoulli property for elements of $\mathcal{P}_{\mathrm{inv}}(M)$.

LEMMA 2.6. [8, 9, 14] *For $\nu \in \mathcal{P}_{\mathrm{inv}}(M)$, the following statements are equivalent.*
(1) *ν is very weak Bernoulli.*
(2) *ν is Bernoulli.*

From Lemma 2.5 and Lemma 2.6, we obtain the following result.

PROPOSITION 2.7. (E.g. [5]) *If $\nu \in \mathcal{P}_{\mathrm{inv}}(M)$ is finitely dependent, then ν is Bernoulli.*

Let μ and $\nu \in \mathcal{P}_{\mathrm{inv}}(M)$. Define $\bar{d} : \mathcal{P}_{\mathrm{inv}}(M) \times \mathcal{P}_{\mathrm{inv}}(M) \to [0, 1]$ by

$$\bar{d}(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \gamma(\{(\zeta,\omega)\in[M]^{\mathbb{Z}^d}\times[M]^{\mathbb{Z}^d}; \zeta_0\neq\omega_0\}). \tag{2.5}$$

Then $\bar{d}$ gives a metric on $\mathcal{P}_{\mathrm{inv}}(M)$. The Bernoulli property is closed under $\bar{d}$.

LEMMA 2.8. [8, 9, 14] *Let ν and $\{\nu_n : n \in \mathbb{N}\}$ be elements of $\mathcal{P}_{\mathrm{inv}}(M)$. Suppose that $\lim_{n\to\infty} \bar{d}(\nu_n, \nu) = 0$ and that each ν_n is Bernoulli. Then ν is Bernoulli.*

We quote Theorem 5 in III.6 in [9].

LEMMA 2.9. [8, 9] *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{Z}^d})$ be an ergodic system. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be an increasing sequence of $\mathsf{S}_{\mathbb{Z}^d}$-invariant sub-$\sigma$-fields. Let $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n$ be the completion of $\sigma[\bigcup_{n\in\mathbb{N}} \mathcal{F}_n]$. Assume that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ satisfies (2.6) and (2.7):*

$$\bigvee_{n\in\mathbb{N}} \mathcal{F}_n = \mathcal{F}, \tag{2.6}$$

$$\textit{the } \mathcal{F}_n\textit{-factor is Bernoulli for each } n. \tag{2.7}$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{Z}^d})$ is Bernoulli.

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

PROPOSITION 2.10. *Let* $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu, \mathsf{T}_{\mathbb{Z}^d})$ *be ergodic. Let* ν *be simple. Suppose that there exists a sequence* $\{\nu_r : r \in \mathbb{N}\}$ *of point processes on* $\mathbb{Z}^d \times \mathbb{N}$ *such that*

$$\nu_{r,N} \text{ is Bernoulli for each } r \text{ and } N \in \mathbb{N}, \tag{2.8}$$

$$\lim_{r\to\infty} \bar{d}(\nu_{r,N}, \nu_N) = 0 \quad \text{for each } N \in \mathbb{N}. \tag{2.9}$$

Here, $\nu_{r,N} = \nu_r \circ \varpi_N^{-1}$ *and* $\nu_N = \nu \circ \varpi_N^{-1}$. *Then* ν *is Bernoulli.*

Proof. Recall that $Q^N = \{Q^N_{z,l} : (z, l) \in \mathbb{Z}^d \times [N]\}$ is a partition of $\mathbb{Z}^d \times \mathbb{N}$. Here, $Q^N_{z,l}$ is defined in (2.1). Then Q^N becomes finer as $N \to \infty$ and $\bigvee_{N\in\mathbb{N}} Q^N$ separates points of $\mathbb{Z}^d \times \mathbb{N}$ by construction. Here, $\bigvee_{N\in\mathbb{N}} Q^N$ is the refinement of partitions $\{Q^N\}_{N\in\mathbb{N}}$. From this, we obtain that $\{\sigma[\varpi_N]\}_{N\in\mathbb{N}}$ is increasing and $\bigvee_{N\in\mathbb{N}} \sigma[\varpi_N]$ separates points of $\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N})$. Hence, $\{\sigma[\varpi_N]\}_{N\in\mathbb{N}}$ satisfies (2.6).

From the assumptions (2.8) and (2.9) and Lemma 2.8, ν_N is Bernoulli. Hence, $\{\sigma[\varpi_N]\}_{N\in\mathbb{N}}$ satisfies (2.7).

From the above and Lemma 2.9, the claim holds. □

3. *Approximations of the determinantal kernel*

In this section, we introduce three approximations of the kernel K introduced in (1.2).

For $r > 0$, let $w_r : \mathbb{R}^d \to \mathbb{R}$ be the product of the tent function such that

$$w_r(x) = \prod_{j=1}^{d} (1 - |x_j|/r) 1_{\{|x_j|<r\}}(x).$$

We denote by $\hat{w}_r$ its Fourier transform

$$\hat{w}_r(t) = \int_{\mathbb{R}^d} w_r(x) e^{2\pi i x\cdot t} dx = r^{-d} \prod_{j=1}^{d} \left(\frac{\sin \pi r t_j}{\pi t_j} \right)^2.$$

Let $\hat{K} \in L^1(\mathbb{R}^d)$ be such that $\hat{K}(t) \in [0, 1]$ for a.e. $t \in \mathbb{R}^d$. Set $\hat{K}_r(t) = \hat{K} * \hat{w}_r(t)$. Then $\hat{K}_r(t) \in [0, 1]$ for a.e. $t \in \mathbb{R}^d$. Let

$$\underline{K}_r(x, y) = \int_{\mathbb{R}^d} (\hat{K}_r(t) \wedge \hat{K}(t)) e^{2\pi i (x-y)\cdot t}\, dt, \tag{3.1}$$

$$K_r(x, y) = \int_{\mathbb{R}^d} \hat{K}_r(t) e^{2\pi i (x-y)\cdot t}\, dt, \tag{3.2}$$

$$\overline{K}_r(x, y) = \int_{\mathbb{R}^d} (\hat{K}_r(t) \vee \hat{K}(t)) e^{2\pi i (x-y)\cdot t}\, dt. \tag{3.3}$$

Here, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for $a, b \in \mathbb{R}$, respectively. Then $\underline{K}_r$, K_r, and $\overline{K}_r$ satisfy (1)–(4) before Theorem 1.1.

For $K : X \times X \mapsto \mathbb{C}$, we denote $O \le K$ if K is non-negative definite as an integral operator on $L^2(\mathbb{R}^d)$ and $K_1 \le K_2$ if $K_2 - K_1$ is non-negative definite.

LEMMA 3.1. *Let* $\underline{K}_r$, K_r, *and* $\overline{K}_r$ *be as* (3.1), (3.2), *and* (3.3), *respectively. Then*

$$\underline{K}_r \le K \le \overline{K}_r, \tag{3.4}$$

$$\underline{K}_r \le K_r \le \overline{K}_r. \tag{3.5}$$

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

Proof. By construction, we see that

$$\hat{K}_r(t) \wedge \hat{K}(t) \le \hat{K}(t) \le \hat{K}_r(t) \vee \hat{K}(t),$$
$$\hat{K}_r(t) \wedge \hat{K}(t) \le \hat{K}_r(t) \le \hat{K}_r(t) \vee \hat{K}(t).$$

From (3.1)–(3.3) combined with the above inequalities, we obtain (3.4) and (3.5). □

4. *Tree representations of determinantal point processes*

In this section, we introduce the tree representations of determinantal point processes on $\mathbb{R}^d$. Then we apply them to the determinantal point processes associated with the kernels introduced in §3. Before doing so, we recall the definition and well-known facts about determinantal point processes.

Let μ be a point process on X. A locally integrable symmetric function $\rho^n : X^n \to [0, \infty)$ is called the n-point correlation function of μ (with respect to a Radon measure λ on X) if

$$\mathsf{E}^{\mu}\left[\prod_{i=1}^{k} \frac{\xi(A_i)!}{(\xi(A_i) - n_i)!}\right] = \int_{A_1^{n_1}\times\cdots\times A_k^{n_k}} \rho^n(x_1, \ldots, x_n)\lambda(dx_1)\cdots\lambda(dx_n) \tag{4.1}$$

for any disjoint Borel subsets $A_1, \ldots, A_k$ and for any $n_i \in \mathbb{N}$, $i = 1, \ldots, k$, such that $\sum_{i=1}^{k} n_i = n$. Let $K : X \times X \to \mathbb{C}$. We call μ a determinantal point process with kernel K and Radon measure λ if the n-point correlation function ρ^n of μ with respect to λ satisfies (1.1) for each n.

Assume that $K : X \times X \to \mathbb{C}$ satisfies

$$\overline{K(x, y)} = K(y, x), \tag{4.2}$$

$$\mathsf{Spec}(K) \subset [0, 1], \tag{4.3}$$

$$K_A \text{ is of trace class for any compact } A \subset X. \tag{4.4}$$

Here, K in (4.3) is an integral operator on $L^2(X, \lambda)$ such that $Kf(x) = \int_X K(x, y)\lambda(dy)$ and K_A in (4.4) is its restriction on $L^2(A, \lambda)$. Then there exists a unique determinantal point process on X with kernel function K.

Next, we introduce the tree representations of the determinantal point processes. Let μ^K be the determinantal point process on $\mathbb{R}^d$ with kernel function K satisfying (4.2)–(4.4). First, we introduce a partition of $\mathbb{R}^d$ and the associated orthonormal basis on $L^2(\mathbb{R}^d)$. Let $P = \{P_z : z \in \mathbb{Z}^d\}$ be a partition of $\mathbb{R}^d$ such that each P_z is relatively compact and

$$P_{z+w} = P_z + w \text{ for } z, w \in \mathbb{Z}^d.$$

Here, $A + x = \{a + x; a \in A\}$ for $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Let $\Phi = \Phi_P = \{\phi_{z,l}\}_{(z,l)\in\mathbb{Z}^d\times\mathbb{N}}$ be an orthonormal basis on $L^2(\mathbb{R}^d)$ such that $\mathsf{supp}\phi_{z,l} \subset P_z$ and

$$\phi_{z+w,l}(x) = \phi_{z,l}(x - w). \tag{4.5}$$

For the kernel function K above, let $K^{\Phi} : (\mathbb{Z}^d \times \mathbb{N}) \times (\mathbb{Z}^d \times \mathbb{N}) \to \mathbb{C}$ be such that

$$K^{\Phi}(z, l; w, m) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi_{z,l}(x)K(x, y)\phi_{w,m}(y)\, dx\, dy. \tag{4.6}$$

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

LEMMA 4.1. *Assume that* K *satisfies* (4.2)–(4.4) *with respect to* $L^2(\mathbb{R}^d)$. *Then* K^Φ *satisfies* (4.2)–(4.4) *with respect to the counting measure on* $\mathbb{Z}^d \times \mathbb{N}$.

Proof. By assumption and (4.6), K^Φ satisfies (4.2) and (4.4). Equation (4.3) follows from [**10**]. □

From Lemma 4.1 and the general theory in [**11**, **13**], there exists a determinantal point process $\nu^{K,\Phi}$ on $\mathbb{Z}^d \times \mathbb{N}$ associated with K^Φ. We call $\nu^{K,\Phi}$ the tree representation of μ^K with respect to Φ.

LEMMA 4.2. [**10**] *Let* $\pi : \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}) \to \mathsf{Conf}(\mathbb{Z}^d)$ *be such that*

$$\eta \mapsto \pi(\eta) = \sum_{z \in \mathbb{Z}^d} \eta(\{z\} \times \mathbb{N})\delta_z.$$

Then, for $A \in \sigma[\{\xi \in \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}); \xi(P_z) = n\}; z \in \mathbb{Z}^d, n \in \mathbb{N}]$,

$$\nu^{K,\Phi} \circ \pi^{-1}(A) = \mu^K(A).$$

Proof. From [**10**, Theorem 2, p. 427], we easily obtain the claim. □

We apply the tree representations for the translation-invariant kernels on $\mathbb{R}^d$ introduced in §3.

Assume that K is given by (1.2). Then K is translation invariant. Hence, by construction, K^Φ is translation invariant with respect to the first coordinate $\mathbb{Z}^d$. From this, we see that $\nu^{K,\Phi}$ is translation invariant with respect to the first coordinate.

Define $\underline{K}_r^\Phi$, K_r^Φ, and $\overline{K}_r^\Phi$ similarly as (4.6) with replacement of K with $\underline{K}_r$, K_r, and $\overline{K}_r$ in (3.1)–(3.3), respectively. By construction, $\underline{K}_r$, K_r, and $\overline{K}_r$ satisfy (4.2)–(4.4). Hence, $\underline{K}_r^\Phi$, K_r^Φ, and $\overline{K}_r^\Phi$ satisfy (4.2)–(4.4) with respect to the counting measure on $\mathbb{Z}^d \times \mathbb{N}$ by Lemma 4.2. Furthermore, $\underline{K}_r^\Phi$, K_r^Φ, and $\overline{K}_r^\Phi$ are translation invariant with respect to the first coordinate $\mathbb{Z}^d$.

Let $\underline{\nu}_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, and $\overline{\nu}_r^{K,\Phi}$ be $\underline{K}_r^\Phi$-, K_r^Φ- and $\overline{K}_r^\Phi$-determinantal point processes, respectively. We remark that a determinantal point process ν on $\mathbb{Z}^d$ has no multiple points with probability 1. Hence, we can regard ν as a probability measure on $\{0, 1\}^{\mathbb{Z}^d}$. We quote the following result.

LEMMA 4.3. [**4**] *Let* $K_i : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ *satisfy* (4.2)–(4.4) $(i = 1, 2)$. *Assume that* $K_1 \le K_2$. *Let* ν^{K_1} *and* ν^{K_2} *be the determinantal point processes with* K_1 *and* K_2, *respectively. Then there exists a monotone coupling of* ν^{K_1} *and* ν^{K_2}.

Applying Lemma 4.3, we obtain the following result.

LEMMA 4.4. *Let* $\underline{\nu}_r^{K,\Phi}$, $\nu^{K,\Phi}$, $\nu_r^{K,\Phi}$, *and* $\overline{\nu}_r^{K,\Phi}$ *be determinantal point processes on* $\mathbb{Z}^d \times \mathbb{N}$ *as above. Then*

$$\underline{\nu}_r^{K,\Phi} \le \nu^{K,\Phi} \le \overline{\nu}_r^{K,\Phi}, \tag{4.7}$$

$$\underline{\nu}_r^{K,\Phi} \le \nu_r^{K,\Phi} \le \overline{\nu}_r^{K,\Phi}. \tag{4.8}$$

Proof. Recall that Φ is the orthonormal basis of $L^2(\mathbb{R}^d)$ given in (4.5). Let $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{Z}^d \times \mathbb{N})$ be the unitary operator such that $U(\phi_{z,n}) = e_{z,n}$, where $\{e_{z,n}\}_{(z,n)\in\mathbb{Z}^d\times\mathbb{N}}$ is the canonical orthonormal basis of $L^2(\mathbb{Z}^d \times \mathbb{N})$. Then, by Lemma 1 in §3 of [10], we see that $K^\Phi = UKU^{-1}$. From this and Lemma 3.1, we obtain

$$\underline{K}_r^\Phi \le K^\Phi \le \overline{K}_r^\Phi, \tag{4.9}$$

$$\underline{K}_r^\Phi \le K_r^\Phi \le \overline{K}_r^\Phi. \tag{4.10}$$

From (4.9) and (4.10) combined with Lemma 4.3, we conclude (4.7) and (4.8). □

Recall that $\underline{K}_r^\Phi$, K_r^Φ, and $\overline{K}_r^\Phi$ are translation invariant with respect to the first coordinate. Hence, $\underline{\nu}_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, and $\overline{\nu}_r^{K,\Phi}$ are also translation invariant with respect to the first coordinate. We regard $\mathsf{T}_{\mathbb{Z}^d} = \{\mathsf{T}_a : a \in \mathbb{Z}^d\}$ as a translation on $\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N})$ such that

$$\mathsf{T}_a : \sum_i \delta_{(z_i,l_i)} \mapsto \sum_i \delta_{(z_i+a,l_i)} \quad \text{for } a \in \mathbb{Z}^d.$$

Then $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \underline{\nu}_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, and $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \overline{\nu}_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ are $\mathbb{Z}^d$-action systems.

5. *Bernoulli property of* $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$

We continue the setting of §4. Let K^Φ be the kernel defined by (4.6). Let $\nu^{K,\Phi}$ be the K^Φ-determinantal point process as before. The purpose of this section is to prove the Bernoulli property for $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$.

Let ϖ_N be the map defined by (2.2). Let $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ denote the ϖ_N-factor of $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$. Here, $\mathsf{T}_{\mathbb{Z}^d}$ in $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ is the shift of $\{0,1\}^{\mathbb{Z}^d\times[N]}$ such that for each $a \in \mathbb{Z}^d$,

$$\mathsf{T}_a : \omega = \{\omega_{z,l}\}_{(z,l)\in\mathbb{Z}^d\times[N]} \mapsto \{\omega_{z+a,l}\}_{(z,l)\in\mathbb{Z}^d\times[N]}.$$

We also denote ϖ_N-factors of $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \underline{\nu}_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, and $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \overline{\nu}_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ by $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \underline{\nu}_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, and $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \overline{\nu}_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$, respectively. We shall prove that $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_N^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ is Bernoulli.

LEMMA 5.1. *Let* $\underline{\nu}_r^{K,\Phi}$, $\nu^{K,\Phi}$, $\nu_r^{K,\Phi}$, *and* $\overline{\nu}_r^{K,\Phi}$ *be determinantal point processes on* $\mathbb{Z}^d \times \mathbb{N}$ *as in Lemma 4.4. Let* $\underline{\nu}_{r,N}^{K,\Phi}$, $\nu_N^{K,\Phi}$, $\nu_{r,N}^{K,\Phi}$, *and* $\overline{\nu}_{r,N}^{K,\Phi}$ *be their* ϖ_N*-factors, respectively. Then*

$$\underline{\nu}_{r,N}^{K,\Phi} \le \nu_N^{K,\Phi} \le \overline{\nu}_{r,N}^{K,\Phi},$$

$$\underline{\nu}_{r,N}^{K,\Phi} \le \nu_{r,N}^{K,\Phi} \le \overline{\nu}_{r,N}^{K,\Phi}.$$

Proof. From Proposition 2.2 and Lemma 4.4, we obtain the claim. □

LEMMA 5.2. $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ *is Bernoulli.*

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

Proof. We identify $\{0,1\}^{\mathbb{Z}^d\times[N]}$ with $[2^N]^{\mathbb{Z}^d}$ and $\nu_{r,N}^{K,\Phi}$ with an element of $\mathcal{P}_{\mathrm{inv}}(2^N)$, respectively. We shall prove that $\nu_{r,N}^{K,\Phi}$ is finitely dependent. For this, it only remains to prove that $\nu_r^{K,\Phi}$ is finitely dependent because $(\{0,1\}^{\mathbb{Z}^d\times[N]}, \nu_{r,N}^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ is the ϖ_N-factor of $(\mathsf{Conf}(\mathbb{Z}^d\times\mathbb{N}), \nu_r^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$.

Let d be the graph distance as before. Let $r_0>0$ be such that for each $z, w\in\mathbb{Z}^d$,

$$\mathsf{d}(z,w)\geq r_0 \Rightarrow \inf\{|z_i-w_i|; i=1,\ldots,d\}\geq r.$$

For $P, Q\subset\mathbb{Z}^d\times\mathbb{N}$, we define a pseudo-distance by

$$\mathsf{d}(P,Q)=\inf\{\mathsf{d}(z,w); (z,l)\in P, (w,m)\in Q\}.$$

Let $P, Q\subset\mathbb{Z}^d\times\mathbb{N}$ be finite sets such that $\mathsf{d}(P,Q)\geq r_0$. Then

$$K_r^{\Phi}(z,l;w,m)=0 \quad \text{for } (z,l)\in P, (w,m)\in Q\,. \tag{5.1}$$

For $P\subset\mathbb{Z}^d\times\mathbb{N}$, we define a cylinder set by

$$1^P=\{\omega\in\mathsf{Conf}(\mathbb{Z}^d\times\mathbb{N}); \omega(\{(z,l)\})=1 \text{ for all } (z,l)\in P\}.$$

By construction, $1^P\cap 1^Q=1^{P\cup Q}$. Therefore,

$$\begin{aligned}\nu_r^{K,\Phi}(1^P\cap 1^Q)&=\nu_r^{K,\Phi}(1^{P\cup Q})\\ &= \det[K_r^{\Phi}(z,l;w,m)]_{(z,l),(w,m)\in P\cup Q}\\ &= \det[K_r^{\Phi}(z,l;w,m)]_{(z,l),(w,m)\in P}\det[K_r^{\Phi}(z,l;w,m)]_{(z,l),(w,m)\in Q}\\ &=\nu_r^{K,\Phi}(1^P)\nu_r^{K,\Phi}(1^Q).\end{aligned}\tag{5.2}$$

The third equality follows from (5.1).

Let $R, S\subset\mathbb{Z}^d$ be such that $\mathsf{d}(R\times\mathbb{N}, S\times\mathbb{N})\geq r_0$. From (5.2) and the π-λ theorem,

$$\nu_r^{K,\Phi}(A\cap B)=\nu_r^{K,\Phi}(A)\nu_r^{K,\Phi}(B)$$

for each $A\in\sigma[\pi_{R\times\mathbb{N}}]$ and $B\in\sigma[\pi_{S\times R}]$. Hence, $\nu_{r,N}^{K,\Phi}$ is r_0-dependent.

From this and Proposition 2.7, the claim holds. □

LEMMA 5.3. *For each N,*

$$\lim_{r\to\infty}\bar{d}(\nu_N^{K,\Phi},\nu_{r,N}^{K,\Phi})=0. \tag{5.3}$$

Proof. Because $\bar{d}$ is a metric on $\mathcal{P}_{\mathrm{inv}}(M)$,

$$\bar{d}(\nu_N^{K,\Phi},\nu_{r,N}^{K,\Phi})\leq\bar{d}(\underline{\nu}_{r,N}^{K,\Phi},\nu_N^{K,\Phi})+\bar{d}(\underline{\nu}_{r,N}^{K,\Phi},\nu_{r,N}^{K,\Phi}), \tag{5.4}$$

$$\bar{d}(\nu_N^{K,\Phi},\nu_{r,N}^{K,\Phi})\leq\bar{d}(\nu_N^{K,\Phi},\overline{\nu}_{r,N}^{K,\Phi})+\bar{d}(\nu_{r,N}^{K,\Phi},\overline{\nu}_{r,N}^{K,\Phi}). \tag{5.5}$$

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

From Lemma 2.1 and Lemma 5.1, there exists a monotone coupling γ_N of $\underline{\nu}_{r,N}^{K,\Phi}$ and $\nu_N^{K,\Phi}$. By definition (2.5) of $\bar{d}$, we deduce that

$$\begin{aligned}\bar{d}(\underline{\nu}_{r,N}^{K,\Phi}, \nu_N^{K,\Phi}) &\le \gamma_N(\{(\omega_1,\omega_2)\ ;\ \omega_1(\{0\}\times\{l\}) \ne \omega_2(\{0\}\times\{l\}) \text{ for } \exists l \in [N]\}) \\ &\le \sum_{l\in[N]} \gamma_N(\{(\omega_1,\omega_2)\ ;\ \omega_1(\{0\}\times\{l\}) \ne \omega_2(\{0\}\times\{l\})\}) \\ &= \sum_{l\in[N]} \{\nu_N^{K,\Phi}(\omega_1(\{0\}\times\{l\})=1) - \underline{\nu}_{r,N}^{K,\Phi}(\omega_2(\{0\}\times\{l\})=1)\}. \qquad (5.6)\end{aligned}$$

The last equation follows from the fact that γ_N is a monotone coupling of $\underline{\nu}_{r,N}^{K,\Phi}$ and $\nu_N^{K,\Phi}$. Because of Lemma 5.1, (5.6) is true for $(\underline{\nu}_{r,N}^{K,\Phi}, \nu_{r,N}^{K,\Phi})$, $(\nu_N^{K,\Phi}, \overline{\nu}_{r,N}^{K,\Phi})$, and $(\nu_{r,N}^{K,\Phi}, \overline{\nu}_{r,N}^{K,\Phi})$. From this combined with (5.4) and (5.5), we obtain

$$\begin{aligned}\bar{d}(\nu_N^{K,\Phi}, \nu_{r,N}^{K,\Phi}) &\le \sum_{l\in[N]} \{\overline{\nu}_{r,N}^{K,\Phi}(\omega_1(\{0\}\times\{l\})=1) - \underline{\nu}_{r,N}^{K,\Phi}(\omega_2(\{0\}\times\{l\})=1)\} \\ &= \sum_{l\in[N-1]} \{\overline{\nu}_r^{K,\Phi}(\omega_1(\{0\}\times\{l\})=1) - \underline{\nu}_r^{K,\Phi}(\omega_2(\{0\}\times\{l\})=1)\} \\ &\quad + \overline{\nu}_r^{K,\Phi}(\omega_1(\{0\}\times\mathbb{N}\backslash[N]) \ge 1) - \underline{\nu}_r^{K,\Phi}(\omega_2(\{0\}\times\mathbb{N}\backslash[N]) \ge 1). \quad (5.7)\end{aligned}$$

The last equation follows from the definitions of $\overline{\nu}_r^{K,\Phi}$ and $\underline{\nu}_r^{K,\Phi}$.

For (z,l) and (w,m),

$$\begin{aligned}&|\overline{K}_r^{\Phi}(z,l;w,m) - \underline{K}_r^{\Phi}(z,l;w,m)| \qquad (5.8)\\ &= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} \{\overline{K}_r(x,y) - \underline{K}_r(x,y)\}\phi_{z,l}(x)\phi_{w,m}(y)\,dx\,dy \right| \\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |\overline{K}_r(x,y) - \underline{K}_r(x,y)||\phi_{z,l}(x)\phi_{w,m}(y)|\,dx\,dy \\ &= \int_{\mathrm{supp}\phi_{z,l}\times\mathrm{supp}\phi_{w,m}} |\overline{K}_r(x,y) - \underline{K}_r(x,y)||\phi_{z,l}(x)\phi_{w,m}(y)|\,dx\,dy. \qquad (5.9)\end{aligned}$$

Because $\overline{K}_r, \underline{K}_r \in L^2_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\phi_{z,l}$ and $\phi_{w,m}$ are orthonormal bases on $L^2(\mathbb{R}^d)$ with relatively compact support, the Schwarz inequality implies that

$$(5.9) \le \left(\int_{\mathrm{supp}\phi_{z,l}\times\mathrm{supp}\phi_{w,m}} |\overline{K}_r(x,y) - \underline{K}_r(x,y)|^2\,dx\,dy \right)^{1/2}. \qquad (5.10)$$

Because $\hat{K}_r \to \hat{K}$ in $L^1(\mathbb{R}^d)$ as $r\to\infty$, $\underline{K}_r$ and $\overline{K}_r$ converge to K uniformly on any compact set. Hence, the right-hand side of (5.10) goes to 0 as $r\to\infty$. This implies that (5.8) goes to 0 as $r\to\infty$. Hence, for each compact set $R \subset \mathbb{Z}^d\times\mathbb{N}$,

$$\max\{|\overline{K}_r^{\Phi}(z,l;w,m) - \underline{K}_r^{\Phi}(z,l;w,m)|\ ;\ (z,l),(w,m)\in R\} \to 0 \quad \text{as } r\to\infty.$$

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

From this and Proposition 3.10 in [11],

$$\overline{\nu}_r^{K,\Phi},\ \underline{\nu}_r^{K,\Phi} \to \nu^{K,\Phi} \text{ weakly as } r \to \infty. \tag{5.11}$$

Finally, (5.7) and (5.11) imply (5.3). □

THEOREM 5.4. $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ *is Bernoulli.*

Proof. From Proposition 2.10, Lemma 5.2 and Lemma 5.3, the claim holds. □

6. *Proof of Theorem 1.1*

The purpose of this section is to complete the proof of Theorem 1.1.

We quote a general fact of isomorphism theory.

LEMMA 6.1. [8, 9] *Let $(\Omega', \mathcal{F}', \mathbb{P}', \mathsf{S}'_{\mathbb{Z}^d})$ be a factor of $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{Z}^d})$. If $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{Z}^d})$ is Bernoulli, then $(\Omega', \mathcal{F}', \mathbb{P}', \mathsf{S}'_{\mathbb{Z}^d})$ is Bernoulli.*

For $n \in \mathbb{N}$, let $P_n = \{P_{n,z} : z \in \mathbb{Z}^d\}$ be a partition of $\mathbb{R}^d$ such that

$$P_{n,z} = \prod_{i=1}^{d} \left[\frac{z_i}{2^{n-1}}, \frac{z_i+1}{2^{n-1}}\right), \quad z = (z_1, \ldots, z_d) \in \mathbb{Z}^d.$$

Let $\Pi_{P_n} : \mathsf{Conf}(\mathbb{R}^d) \to \mathsf{Conf}(\mathbb{Z}^d)$ be such that

$$\xi \mapsto \sum_{z \in \mathbb{Z}^d} \xi(P_{n,z})\delta_z.$$

Then $\Pi_{P_n} \circ \mathsf{T}_z(\xi) = \mathsf{T}_z \circ \Pi_{P_n}(\xi)$ for each $z \in \mathbb{Z}^d$ and $\xi \in \mathsf{Conf}(\mathbb{R}^d)$. Let $\mu^K_{P_n} = \mu^K \circ \Pi^{-1}_{P_n}$. Then $(\mathsf{Conf}(\mathbb{Z}^d), \mu^K_{P_n}, \mathsf{T}_{\mathbb{Z}^d})$ is the Π_{P_n}-factor of $(\mathsf{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{Z}^d})$.

LEMMA 6.2. $(\mathsf{Conf}(\mathbb{Z}^d), \mu^K_{P_n}, \mathsf{T}_{\mathbb{Z}^d})$ *is Bernoulli.*

Proof. Let $\Phi_n = \{\phi^n_{z,l}\}_{(z,l) \in \mathbb{Z}^d \times \mathbb{N}}$ be an orthonormal basis on $L^2(\mathbb{R}^d)$ such that $\phi^n_{z+w,l}(x) = \phi^n_{z,l}(x-w)$ and $\operatorname{supp} \phi^n_{z,l} = P_{n,z}$. Let $\nu^{K,\Phi}$ be the tree representation of μ^K with respect to Φ_n. Let $\pi : \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}) \to \mathsf{Conf}(\mathbb{Z}^d)$ be such that

$$\eta \mapsto \pi(\eta) = \sum_{z \in \mathbb{Z}^d} \eta(\{z\} \times \mathbb{N})\delta_z.$$

By construction, $\pi \circ \mathsf{T}_z(\eta) = \mathsf{T}_z \circ \pi(\eta)$ for each $z \in \mathbb{Z}^d$ and $\eta \in \mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N})$. From Lemma 4.2,

$$\nu^{K,\Phi} \circ \pi^{-1} = \mu^K_{P_n}.$$

Hence, $(\mathsf{Conf}(\mathbb{Z}^d), \mu^K_{P_n}, \mathsf{T}_{\mathbb{Z}^d})$ is the π-factor of $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$. From Theorem 5.4, $(\mathsf{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathsf{T}_{\mathbb{Z}^d})$ is Bernoulli. From this and Lemma 6.1, the claim holds. □

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press

LEMMA 6.3. *$(\mathsf{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{Z}^d})$ is Bernoulli.*

Proof. By construction, the sequence of partitions $\{P_n : n \in \mathbb{N}\}$ is increasingly finer and separates the points of $\mathbb{R}^d$. From this, we obtain that $\{\sigma[\Pi_{P_n}]\}_{n\in\mathbb{N}}$ is increasing and $\bigvee_{n\in\mathbb{N}} \sigma[\Pi_{P_n}]$ separates the points of $\mathsf{Conf}(\mathbb{R}^d)$. Putting this result together with Lemma 6.2 and Lemma 2.9 implies the claim. □

We quote Theorem 10 of III.6 in [9].

LEMMA 6.4. [9] *For an $\mathbb{R}^d$-action system $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{R}^d})$, let $\mathsf{S}_{\mathbb{Z}^d} = \{\mathsf{S}_g : g \in \mathbb{Z}^d\}$ be the limitation on $\mathbb{Z}^d$-action of $\mathsf{S}_{\mathbb{R}^d}$. If $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{Z}^d})$ is Bernoulli with infinite entropy, then $(\Omega, \mathcal{F}, \mathbb{P}, \mathsf{S}_{\mathbb{R}^d})$ is Bernoulli.*

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 6.3, $(\mathsf{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{Z}^d})$ is Bernoulli. Because the restriction of μ^K on $[0, 1)^d$ is a non-atomic probability measure, the entropy of $(\mathsf{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{Z}^d})$ is infinite. Putting this and Lemma 6.4 together implies the claim. □

Acknowledgment. This work is supported by JSPS KAKENHI Grant Number 18J20465.

REFERENCES

[1] A. I. Bufetov and T. Shirai. Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces. *Proc. Japan Acad. Ser. A Math. Sci.* **93**(1) (2017), 1–5.
[2] S. Kalikow and B. Weiss. Explicit codes for some infinite entropy Bernoulli shifts. *Ann. Probab.* **20**(1) (1992), 397–402.
[3] T. M. Liggett. *Interacting Particle Systems (Grundlehren der Mathematischen Wissenschaften, 276)*. Springer, New York, 1985.
[4] R. Lyons. Determinantal probability measures. *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 167–212.
[5] R. Lyons and J. E. Steif. Stationary determinantal processes: phase multiplicity, bernoullicity, and domination. *Duke Math. J.* **120** (2003), 515–575.
[6] D. S. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Adv. Math.* **4** (1970), 337–352.
[7] D. S. Ornstein. Two Bernoulli shifts with infinite entropy are isomorphic. *Adv. Math.* **5** (1970), 339–348.
[8] D. S. Ornstein. *Ergodic Theory, Randomness, and Dynamical Systems*. Yale University Press, New Haven, CT, 1974.
[9] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Anal. Math.* **48** (1987), 1–144.
[10] H. Osada and S. Osada. Discrete approximations of determinantal point processes on continuous spaces: tree representations and the tail triviality. *J. Stat. Phys.* **170** (2018), 421–435.
[11] T. Shirai and Y. Takahashi. Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson processes. *J. Funct. Anal.* **205** (2003), 414–463.
[12] T. Shirai and Y. Takahashi. Random point fields associated with certain Fredholm determinants II: fermion shifts and their ergodic properties. *Ann. Probab.* **31** (2003), 1533–1564.
[13] A. Soshnikov. Determinantal random point fields. *Russian Math. Surveys* **55** (2000), 923–975.
[14] J. E. Steif. Space–time bernoullicity of the lower and upper stationary processes for attractive spin systems. *Ann. Probab.* **19**(2) (1991), 609–635.

https://doi.org/10.1017/etds.2020.123 Published online by Cambridge University Press