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THE ANALYSIS OF WARNER BOUNDEDNESS

JERZY KĄKOL[1], STEPHEN A. SAXON[2] AND AARON R. TODD[3]

[1] *Faculty of Mathematics and Informatics, A. Mickiewicz University, 60-769 Poznań, Matejki 48–49, Poland* (kakol@math.amu.edu.pl)
[2] *Department of Mathematics, University of Florida, PO Box 118105, Gainesville, FL 32611-8105, USA* (saxon@math.ufl.edu)
[3] *Department of Mathematics, Baruch College, CUNY, New York, NY 10010, USA* (artbb@cunyvm.cuny.edu)

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Abstract In answer to Jarchow's 1981 text, we recently characterized when $C_{\mathrm{c}}(X)$ is a *df*-space, finding along the way attractive analytic characterizations of when the Tychonov space X is pseudocompact. Analogues now reveal how exquisitely Warner boundedness lies between these two notions. To illustrate, X is pseudocompact, X is Warner bounded or $C_{\mathrm{c}}(X)$ is a *df*-space if and only if for each sequence $(\mu_n)_n \subset C_{\mathrm{c}}(X)'$ there exists a sequence $(\varepsilon_n)_n \subset (0,1]$ such that $(\varepsilon_n \mu_n)_n$ is weakly bounded, is strongly bounded or is equicontinuous, respectively. Our characterizations and proofs add to and simplify Warner's.

Keywords: Warner bounded; pseudocompact; docile locally convex space (LCS); compact–open topology

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1. Introduction

Let E be a Hausdorff locally convex space (LCS) over the reals $\mathbb{R}$. One may consult [**10**] and [**4**] for standard terminology and results. By E'_β and E'_σ we denote the strong and weak duals of E, and $C_{\mathrm{c}}(X)$ is the space $C(X)$ of continuous real-valued functions on the Tychonov (completely regular Hausdorff) space X endowed with the compact–open topology. To denote the subspace of bounded functions in $C(X)$, we write $C^{\mathrm{b}}(X)$, and $C^{\mathrm{b}}_{\mathrm{u}}(X)$ indicates $C^{\mathrm{b}}(X)$ endowed with the uniform Banach topology having unit ball $[X,1] = \{f \in C(X) : |f(x)| \leqslant 1 \text{ for all } x \in X\}$. The Tychonov space X is *pseudocompact* if $C(X) = C^{\mathrm{b}}(X)$, i.e. if $[X,1]$ is absorbing in $C(X)$, and hence a barrel in $C_{\mathrm{c}}(X)$.

An LCS E is *docile* [**6**–**9**] if every infinite-dimensional subspace of E contains an infinite-dimensional bounded set. This is weaker than the Fréchet–Urysohn property [**6**, Theorem 4.1], itself weaker than metrizability. Recall that E is *Fréchet–Urysohn* if for every subset A of E, each x in the closure of A is the limit of a sequence from A. Trivially, E is docile if each sequence $(u_n)_n \subset E$ is *boundable*, i.e. admits $(\varepsilon_n)_n \subset (0,1]$ such that $(\varepsilon_n u_n)_n$ is bounded in E. We prove the converse for $E = C_{\mathrm{c}}(X)'_\sigma$; for arbitrary E we do not know the answer. In [**8**] and [**9**] we showed that the bounding cardinal $\mathfrak{b}$ (see [**12**]) is the largest cardinal such that the product space $\mathbb{R}^\kappa$ is docile when $\kappa < \mathfrak{b}$, and gave proof for the following.

Theorem 1.1. *The following assertions about a Tychonov space X are equivalent.*

(a) *X is pseudocompact.*

(b) *$C_c(X)'_\sigma$ is docile.*

(c) *$C_c(X)$ does not contain (a copy of) $\mathbb{R}^\mathbb{N}$.*

(d) *$C_c(X)$ does not contain a dense barrelled subspace of $\mathbb{R}^\mathbb{N}$.*

Thus, (a) $\Rightarrow$ each $(\mu_n)_n \subset C_c(X)'_\sigma$ is boundable (take $\varepsilon_n = (\|\mu_n\|+1)^{-1}$) $\Rightarrow$ (b) $\Rightarrow$ (a). This proves equivalence of docility and boundability in $C_c(X)'_\sigma$ and justifies the abstract's characterization of pseudocompactness.

Parts (b)–(d) suggest stronger analogues:

(b′) $C_c(X)'_\beta$ is docile (docility is preserved by continuous linear images);

(c′) $C_c(X)'_\beta$ does not contain φ, the strong dual of $\mathbb{R}^\mathbb{N}$ (φ is a useful $\aleph_0$-dimensional space with its strongest locally convex topology (cf. [**5**–**7**, **10**, **11**]));

(d′) $C_c(X)$ does not contain a dense subspace of $\mathbb{R}^\mathbb{N}$.

We shall see that each of (b′)–(d′) is equivalent to Warner boundedness of X, and if $C_c(X)$ is replaced by an arbitrary LCS E, we have (b′) $\Rightarrow$ (c′) $\Rightarrow$ (d′). Examples in the last section show that, in the larger LCS universe, neither arrow reverses and (d′) is generally much weaker than (b′).

In [**8**] we proved that $C_c(X)$ is a *df*-space if and only if $C_c(X)'_\beta$ is

(e) a Banach space, or

(f) a Fréchet space, or

(g) docile and locally complete, or

(h) has the equicontinuity property of the abstract.

Weaker analogues come to mind. Roughly half of Theorem 11 in [**14**] may be interpreted as saying that X is Warner bounded if and only if $C_c(X)'_\beta$ is

(e′) normable, or

(f′) metrizable;

(g) is another motivation for (b′); and (h) compels us to consider boundability as in the abstract.

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2. Analytic characterizations

Buchwalter [**1**] (see also [**13**]) calls a Tychonov space X *Warner bounded* if for every disjoint sequence $(U_n)_n$ of non-empty open sets in X there exists a compact $K \subset X$ such that $U_n \cap K \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Please note: if 'disjoint' is omitted, the concept remains the same due to regularity of X (cf. the theorem on p. 5 of [**2**]). Two observations light an amazingly direct path to analytic characterizations of Warner boundedness.

Lemma 2.1.

(a) *An LCS E contains a dense subspace of $\mathbb{R}^{\mathbb{N}}$ if and only if there is a sequence of non-zero $w_n \in E$ such that every continuous seminorm on E vanishes at w_n for almost all n.*

(b) *If E contains a dense subspace of $\mathbb{R}^{\mathbb{N}}$, then E'_β contains φ.*

Proof. (a) Given $(w_n)_n$ as above, one routinely finds a biorthogonal sequence $(v_n, u_n)_n \subset F \times F'$, where $(v_n)_n$ is a subsequence of $(w_n)_n$ spanning a subspace F of E. By hypothesis on $(w_n)_n$ and the Hahn–Banach Theorem, F' is spanned by the u_n. Clearly, F is isomorphic to the span G of the canonical unit vectors in $\mathbb{R}^{\mathbb{N}}$.

Conversely, for a dense subspace G of the metrizable $\mathbb{R}^{\mathbb{N}}$, there exists a sequence $(p_n)_n$ of continuous seminorms on G such that each continuous seminorm on G is majorized by some p_n. Since G is infinite dimensional and has its weak topology, each $G_n = \bigcap_{i \leqslant n} p_i^{-1}(0)$ is a finite-codimensional, hence infinite-dimensional, subspace of G, and we may choose $w_n \in G_n \backslash \{0\}$ to produce the required sequence.

(b) Since the strong dual of $\mathbb{R}^{\mathbb{N}}$ is φ, this is a straightforward application of the Hahn–Banach Theorem, the bipolar theorem and Grothendieck's result which says that every bounded set in the completion $\hat{F}$ of a separable metrizable LCS F is contained in the closure in $\hat{F}$ of a bounded set in F. □

Theorem 2.2. *The Tychonov space X is Warner bounded if and only if $C_{\mathrm{c}}(X)$ does not contain a dense subspace of $\mathbb{R}^{\mathbb{N}}$.*

Proof. If $(f_n)_n$ is a sequence of non-zero members of $C(X)$ almost all of which vanish on any compact subset of X, then there exist non-empty open sets U_n in X such that f_n is non-zero at each point of U_n, and thus each compact set in X misses U_n for almost all n, i.e. X is not Warner bounded.

Conversely, if $(U_n)_n$ is a sequence of non-empty open sets almost all of which miss each compact set in X, then there exists a sequence $(f_n)_n$ of non-zero members of $C(X)$ such that each f_n vanishes off U_n, and thus each continuous seminorm on $C_{\mathrm{c}}(X)$ is zero at almost all the f_n. □

The result seems strikingly natural, translating word for word between analysis and topology via Lemma 2.1 (a). Yet Theorem 2.2 was overlooked for 45 years in the absence of Theorem 1.1. Now, with the added aid of Lemma 2.1 (b), half of Warner's fundamental Theorem 11 [**14**] quickly follows.

Theorem 2.3 (Warner). *The following assertions are equivalent.*

(1) *X is Warner bounded.*

(2) *$[X,1]$ is bornivorous (absorbs bounded sets) in $C_{\mathrm{c}}(X)$.*

(3) *$C_{\mathrm{c}}(X)$ has a fundamental sequence of bounded sets (FSBS).*

Proof. (1) $\Rightarrow$ (2). Let B be a bounded set in $C_{\mathrm{c}}(X)$; equivalently, B is uniformly bounded on any compact subset of X. If B is not absorbed by $[X,1]$, then for each n there is some $f_n \in B$ and $x_n \in X$ such that $|f_n(x_n)| > n$. Continuity provides an open neighbourhood U_n of x_n such that $|f_n(x)| > n$ for all $x \in U_n$. By Warner boundedness, there is a compact $K \subset X$ with $K \cap U_n \neq \emptyset$ infinitely often, contradicting the fact that $\{f_n : n \in \mathbb{N}\} \subset B$ must be uniformly bounded on K.

(2) $\Rightarrow$ (3). Since $[X,1]$ is a bounded bornivore, $(n[X,1])_n$ is an FSBS for $C_{\mathrm{c}}(X)$.

(3) $\Rightarrow$ (1). By (3), $C_{\mathrm{c}}(X)'_\beta$ is metrizable, and no metrizable space contains the non-docile φ. Thus (1) holds by Lemma 2.1 (b) and Theorem 2.2. □

Warner's remaining two characterizations and the conveniently added part (6) also follow quickly.

Theorem 2.4 (Warner). *The previous three conditions are equivalent to the following three.*

(4) *Every Cauchy sequence in $C_{\mathrm{c}}(X)$ is a Cauchy sequence in $C^{\mathrm{b}}_{\mathrm{u}}(X)$.*

(5) *X is pseudocompact and $C_{\mathrm{c}}(X)$ is sequentially complete.*

(6) *X is pseudocompact and $C_{\mathrm{c}}(X)$ is locally complete.*

Proof. (1) $\Rightarrow$ (4). Since (1) $\Rightarrow$ (2), X is pseudocompact, so that $C(X) = C^{\mathrm{b}}(X)$. It suffices to show that any null sequence $(f_n)_n$ in $C_{\mathrm{c}}(X)$ is also null in the dominating Banach space $C_{\mathrm{u}}(X)$. But if $(f_n)_n$ is not null in $C_{\mathrm{u}}(X)$, then there exist a subsequence $(g_n)_n$, an $\varepsilon > 0$ and, by continuity, a sequence $(U_n)_n$ of non-empty open sets such that $|g_n(x)| > \varepsilon$ for all $x \in U_n$. Since $(g_n)_n$ converges to 0 uniformly on compact sets, compact sets miss U_n for almost all n, contradicting (1).

(4) $\Rightarrow$ (5). The dominating $C_{\mathrm{u}}(X)$ is complete.

(5) $\Rightarrow$ (6). In general, sequentially complete implies locally complete.

(6) $\Rightarrow$ (2). Barrels are bornivorous in locally complete spaces [**10**, 5.1.10]. □

Since (a) $\Leftrightarrow$ (b), we may replace the first part of (5) and (6) with '$C_{\mathrm{c}}(X)'_\sigma$ is docile'. If we then replace $C_{\mathrm{c}}(X)$ with an arbitrary LCS E, statements (5) and (6) still make sense, but become non-equivalent.

Example 2.5. There is a locally complete LCS E with E'_σ docile such that E is not sequentially complete. Indeed, $E = (c_0, \sigma(c_0, \ell^1))$ is locally but not sequentially complete [**10**, 5.1.12], and E'_σ is docile, being dominated by a Banach space.

Corollary 2.6. *The following assertions are equivalent.*

(1) *X is Warner bounded.*

(2) *$C_c(X)'_\beta$ is normable.*

(3) *$C_c(X)'_\beta$ is metrizable.*

(4) *$C_c(X)'_\beta$ is Fréchet–Urysohn.*

(5) *$C_c(X)'_\beta$ is docile.*

(6) *$C_c(X)'_\beta$ does not contain φ.*

(7) *$C_c(X)$ does not contain a dense subspace of $\mathbb{R}^{\mathbb{N}}$.*

As noted already, (2) and (3) interpret Theorem 2.3 and are due to Warner. Note also that $(i) \Rightarrow (i+1)$ for $i = 2,3,4,5,6$, even when $C_c(X)$ is replaced with an arbitrary LCS E. For E arbitrary, however, the ensuing Examples 3.1–3.4 show that $(j) \not\Rightarrow (j-1)$ for $j = 7,6,5,4,3$, making our conditions (7)–(4) decidedly weaker than Warner's conditions (3) and (2).

Since strong dual boundability fits between (2) and (5), the corollary confirms the abstract's characterization of Warner boundedness. We now enjoy several new analytical perspectives on the well-known fact that

$$X \text{ is pseudocompact} \Leftarrow X \text{ is Warner bounded} \Leftarrow C_c(X) \text{ is a } df\text{-space.}$$

Examples 3.5 and 3.6 deny the converses.

3. Examples

Example 3.1. Let E be the topological direct sum of $\mathfrak{d}$ or more copies of an infinite-dimensional normed space, where $\mathfrak{d}$ is the dominating cardinal. Then E has a continuous norm and therefore does not contain a dense subspace of $\mathbb{R}^{\mathbb{N}}$, yet E'_β is the product of $\mathfrak{d}$ or more copies of an infinite-dimensional Banach space and therefore contains φ by [**12**, Theorem 2] (see also [**11**, Theorem 1.4]).

Example 3.2. Let E be a space with its strongest locally convex topology and with dimension κ at least as large as the bounding cardinal $\mathfrak{b}$. Then E'_β is the product space $\mathbb{R}^\kappa$, which cannot contain φ since $\mathbb{R}^\kappa$ has its weak topology, yet is not docile since $\kappa \geqslant \mathfrak{b}$ (see the example at the end of [**9**]).

Example 3.3. Let κ be an uncountable cardinal and let $\mathbb{R}^{[\kappa]}$ be, without topology, the vector space consisting of those members of $\mathbb{R}^\kappa$ which have only finitely many non-zero coordinates, and identify $\mathbb{R}^\kappa$ with the algebraic dual of $\mathbb{R}^{[\kappa]}$ in the usual way. The subspace $\mathbb{R}^\kappa_0$ consisting of those members of $\mathbb{R}^\kappa$ which have at most countably many non-zero coordinates is sufficiently large to separate points of $\mathbb{R}^{[\kappa]}$. Let E_1 and E_2 be $\mathbb{R}^{[\kappa]}$ endowed with locally convex topologies that yield $E'_1 = \mathbb{R}^\kappa_0$ and $E'_2 =$ the span of $\mathbb{R}^\kappa_0$

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and x, where x is a fixed member of $\mathbb{R}^\kappa \backslash \mathbb{R}_0^\kappa$. It is clear that only finite-dimensional subsets of E_1 and E_2 can be bounded, and thus the topology for their strong duals is that induced by the product topology for $\mathbb{R}^\kappa$; it is well known that this topology, in the first case, is Fréchet–Urysohn but not metrizable (see, for example, [**5**, § 2]), and in the second case is not Fréchet–Urysohn but is docile, since every infinite-dimensional subspace of $(E_2)'_\beta$ meets $(E_1)'_\beta$ in an infinite-dimensional subspace that is Fréchet–Urysohn and hence docile [**6**, Theorem 4.1].

Example 3.4. The strong dual of φ is the metrizable, non-normable $\mathbb{R}^{\mathbb{N}}$.

Example 3.5. Haydon's example [**3**] of an infinite pseudocompact space Y in which every compact set is finite distinguishes between pseudocompact and Warner-bounded spaces. For such Y the space $C_c(Y)$ contains dense subspaces of $\mathbb{R}^{\mathbb{N}}$ but no dense barrelled subspaces of $\mathbb{R}^{\mathbb{N}}$ (Theorems 1.1 and 2.2).

Example 3.6. Let $X = [0, \omega_1] \times [0, \omega] \backslash \{(\omega_1, \omega)\}$ be the Tychonov plank, and let $(U_n)_n$ be a sequence of non-empty open sets in X. Each U_n contains some $(\alpha_n, \beta_n) \in [0, \omega_1) \times [0, \omega]$ by density of the latter set in X. Thus the compact set $K = [0, \alpha] \times [0, \omega] \subset X$ meets each U_n, where $\alpha = \sup_n \alpha_n$, and it follows that X is Warner bounded.

With $x_n = (\omega_1, n)$, it is clear that the sequence $(x_n)_n$ has no accumulation points in X. In particular, the sequence is not relatively compact, and therefore $C_c(X)$ is not a df-space by part (8) of the main theorem of [**8**]. One could also argue that the sequence $(\mu_n)_n$ of evaluation functionals corresponding to $(x_n)_n$ fails the equicontinuity condition stated in the abstract.

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