Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1052–1062

MULTIPLICATIONS ON A SPACE WITH FINITELY MANY NON-VANISHING HOMOTOPY GROUPS

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Copeland [**1**] proved that if X is a CW complex then the set of homotopy classes of multiplications is in 1-1 correspondence with the loop $[X \wedge X, X]$. But in general $[X \wedge X, X]$ is very hard to compute.

Here we study the problem of finding how many different (up to homotopy) multiplications can be put on a space with finitely many non-vanishing homotopy groups. We reduce this problem to computing quotients of certain cohomology groups and to determining the primitive elements (Theorem 3.6). Our approach to this problem uses Postnikov decompositions and multiplier arguments. The results agree with those obtained by various authors for special cases: Copeland [**1**] proved that if Y is an H-space and has exactly two non-vanishing homotopy groups, one in dimension p and one in dimension q where $1 < p < q$, then there is a one-to-one correspondence between $H^q(Y \wedge Y, \pi_q(Y))$ and the multiplications (up to homotopy) on Y. McCarty [**3**] proved that there is a one-to-one correspondence between $\mathrm{Hom}[\pi_q(Y) \times \pi_q(Y); \pi_{2q}(Y)]$ and the multiplications (up to homotopy) on Y, provided $\pi_i(Y) = 0$ except for $1 \leqq q \leqq i \leqq 2q$ for some q.

In § 1 we shall establish a relation between the multiplications (up to homotopy) and H-equivalence classes of multipliers (see Definition 1.10). In § 2 we set up a correspondence between H-equivalence classes and certain cohomology groups. In § 3 we reach our main goal, namely by using quotients of certain cohomology groups and knowledge of primitive elements we determine the multiplications (up to homotopy). Finally in § 4 we shall cite two special cases, showing that our results include the known ones in this area.

We restrict ourselves to the CW category.

1. Relation between multiplications and multipliers.

Definition 1.1. An H-space is a triple $(X, *, m)$ where $(X, *)$ is a space with base point $*$ and $m{:}X \times X \to X$ is a mapping which satisfies $m(x, *) = m(*, x) = x$ for any $x \in X$.

Let PX, LX and ΩX be the Moore free path space, path space with $*$ as initial point and loop space of X, respectively.

In general we write $f{:}\,(X_1, *, m_1) \to (X_2, *, m_2)$ if $f \circ m_1 \simeq m_2 \circ (f \times f)$, i.e. if f preserves the multiplication. However, since $X_1 \vee X_1$ is a retractile of

Received June 9, 1971 and in revised form, September 28, 1971.

$X_1 \times X_1$ in the CW category (see [**2**; **4**]), $f \circ m_1 \simeq m_2 \circ (f \times f)$ implies that $f \circ m_1 \simeq m_2 \circ (f \times f)$ rel $X_1 \vee X_1$. Therefore without loss of generality we may adopt the following definition of H-map which has been used by Zabrodsky [**6**] and Stasheff [**5**].

Definition 1.2. An H-map from $(X_1, *, m_1)$ to $(X_2, *, m_2)$ is a pair (f, F) where $f: (X_1, *) \to (X_2, *)$ and $F: X_1 \times X_1 \to PX_2$ are such that $e_0 F = m_2 \circ (f \times f)$, $e_\infty F = f \circ m_1$ and $e_t F(x, *) = e_t F(*, x) = f(x)$, where e_t is the evaluation at t and e_∞ is the evaluation at the end point. F is called the *multiplier* of the H-map (f, F). If $e_t F = e_0 F$ for all t, we call f a multiplicative map.

The following Theorems (1.3), (1.5), (1.6) and (1.7) are due to Stasheff [**5**].

THEOREM 1.3. *Let (B, m) and (K, n) be H-spaces. Let the fibering $\Omega K \to E \to B$ be induced from $\Omega K \to LK \to K$ by a map $f: B \to K$. If (f, F) is an H-map, then there exists a multiplication $s: E \times E \to E$ such that the projection $\pi: (E, s) \to (B, m)$ is multiplicative and*

$$\begin{tikzcd} \Omega K \times E \arrow[r] \arrow[d] & E \times E \arrow[r, "s"] & E \arrow[d] \\ PK \times PK \arrow[rr, "Pn"] & & PK \end{tikzcd}$$

is commutative. If we represent E as $\{(b, \lambda) | b \in B, \lambda \in LK, e_\infty \lambda = f(b)\}$, then $s((b, \lambda), (b', \lambda')) = (m(b, b'), Pn(\lambda, \lambda') + F(b, b'))$ where $+$ denotes the usual join of two paths. We shall say that the multiplication s is obtained from the multiplier F.

As usual we say m and m' are *equivalent* multiplications on X if and only if m is homotopic to m' as a based map $X \times X \to X$, i.e. if and only if $\mathrm{id}: (X, m) \to (X, m')$ is an H-map.

Let (B, m), (B, m'), (K, n) and (K, n') be H-spaces, and $(f, F): (B, m) \to (K, n)$ be an H-map. If m is equivalent to m' and n is equivalent to n', we have two H-maps

$$(\mathrm{id}, F_1): (B, m') \to (B, m), \text{ and}$$
$$(\mathrm{id}, F_2): (K, n) \to (K, n').$$

Define $F': B \times B \to PK$ by the path sum $F_2 + F + f \circ F_1$. Clearly $(f, F'): (B, m') \to (K, n')$ is an H-map.

THEOREM 1.4. *Let E be as in Theorem 1.3. The multiplications on E obtained from the multipliers F and F' are equivalent.*

Proof. We can construct a homotopy for the multiplications on E obtained from F and F' as follows: define $H: E \times E \times I \to E$ by

$$H((b, \lambda), (b', \lambda'), s) = (F_1(b, b')(s), \omega(s) + \tau(s) + F(b, b') + \mu(s)),$$

where $\omega(s) \in L(K)$,

$$\omega(s)(t) = F_2(\lambda(t), \lambda'(t))(1 - s),$$

$\tau(s) \in P(K)$,

$$\begin{aligned} \tau(s)(t) &= F_2(f(b), f(b'))(1 - s + t) \quad \text{if } 1 - s + t \leqq 1 \\ &= F_2(f(b), f(b'))(1) \quad \text{if } 1 - s + t \geqq 1, \end{aligned}$$

and $\mu(s) \in P(K)$,

$$\begin{aligned} \mu(s)(t) &= f \circ F_1(b, b')(1 - s) \quad \text{if } t \leqq 1 - s \\ &= f \circ F_1(b, b')(t) \quad \text{if } t \geqq 1 - s. \end{aligned}$$

It is easy to verify that H has the required properties.

THEOREM 1.5. *Let E be as in Theorem 1.3. If there exists a multiplication s on E such that π is multiplicative, then f is an H-map provided B is $(p - 1)$ connected and that for some $q \geqq p$, $\pi_i(K) = 0$ for $i \leqq q$ and $i > p + q$.*

THEOREM 1.6. *Let E be as in Theorem 1.3. If E admits a multiplication s', then f is an H-map with respect to some multiplication on B provided B is $(p - 1)$ connected and $\pi_i(B) = 0$ for $i > p + q$ where $q \geqq p$ and $\pi_i(K) = 0$ for $i \leqq q$ and $i > p + q$.*

THEOREM 1.7. *A space E admits a multiplication if and only if each stage E_q of the Postnikov system for E admits a multiplication $m_q: E_q \times E_q \rightarrow E_q$ such that*

(1) *the projection $E_{q+1} \rightarrow E_q$ is multiplicative, and*
(2) *the k-invariant $k_{q+1} \in H^{q+2}(E_q; \pi_{q+1}(E))$ is primitive with respect to m_q.*

These theorems suggest a relation between multiplications and H-maps and the question arises as to whether any multiplication on the total space can be obtained from some multiplier as in Theorem 1.3. The following theorem gives an affirmative answer under certain conditions.

THEOREM 1.8. *Let (B, m) and (K, n) be H-spaces where $K = K(G, l + 1)$. Let $\Omega K \rightarrow E \rightarrow B$ be an induced fibering from $\Omega K \rightarrow LK \rightarrow K$ by an H-map (f, F). Then any multiplication s' on E for which $\pi: E \rightarrow B$ is a multiplicative map is equivalent to a multiplication obtained from some multiplier of f.*

Proof. Let s denote the multiplication obtained from F. Let

$$s((b_1, \lambda_1), (b_2, \lambda_2)) = (b, \lambda) \quad \text{and} \quad s'((b_1, \lambda_1), (b_2, \lambda_2)) = (b', \lambda')$$

where (b_i, λ_i) are elements in

$$E = \{(b, \lambda) | b \in B, \lambda \in LB, e_\infty \lambda = f(b)\}.$$

Because p is multiplicative, $b' = b$. Let us define $h': E \times E \rightarrow \Omega K$ by $h'((b_1, \lambda_1), (b_2, \lambda_2)) = \lambda' + \lambda^{-1}$ where $\lambda^{-1}(t) = \lambda(r - t)$ for some r such that

$\lambda(r') = \lambda(r)$ if $r' > r$. Since $\lambda^{-1} + \lambda$ can be shrunk to $e_\infty\lambda$, we have $h' + s \simeq s'$, fiberwise. Because s and s' agree on $E \vee E$, $h'|E \vee E$ is homotopic to the trivial map. By the homotopy extension property, there exists a map $D:I \times E \times E \to \Omega K$ such that $D|\{0\} \times E \times E = h'$, $D|\{1\} \times E \vee E = *$. Let $h = D|\{1\} \times E \times E$. Then h represents a class in $[E \times E, E \vee E; \Omega K, *]$. Since $\pi_*:H^k(B;G) \to H^k(E;k)$ is an isomorphism for $k < l$, we have:

$$\begin{array}{ccc}
H^l(E \wedge E;G) & \xrightarrow[\simeq]{(\pi \times \pi)_*} & H^l(B \wedge B;G) \\
\uparrow \simeq & & \uparrow \simeq \\
H^l(E \times E, E \vee E;G) & \xrightarrow[\simeq]{(\pi \times \pi)_*} & H^l(B \times B, B \vee B;G) \\
\uparrow \simeq & & \uparrow \simeq \\
[E \times E, E \vee E; \Omega K, *] & \xrightarrow[\simeq]{\pi \times \pi} & [B \times B, B \vee B; \Omega K, *].
\end{array}$$

Thus there exists a map $g:B \times B \to \Omega K$ such that $g|B \vee B = *$ and $h \simeq g \circ (\pi \times \pi)$. Define G to be the composite in the diagram

$$B \times B \xrightarrow{\text{diagonal}} \begin{array}{ccc} B \times B & \xrightarrow{g} & \Omega K \\ \times & & \times \\ B \times B & \xrightarrow[F]{} & PK \end{array} \xrightarrow{Pn} PK,$$

where Pn is the multiplication on PK induced from n. The G is a multiplier of f.

Now we show that the multiplication s'' obtained from the multiplier G is equivalent to s'. We set $b_1b_2 = m(b_1, b_2)$ and $\lambda_1\lambda_2 = Pn(\lambda_1, \lambda_2)$. Then

$$\begin{aligned}
s''((b_1, \lambda_1), (b_2, \lambda_2)) &= (b_1b_2, \lambda_1\lambda_2 + G(b_1, b_2)) \\
&= (b_1b_2, \lambda_1\lambda_2 + Pn(g(b_1, b_2), F(b_1, b_2))) \\
&\simeq (b_1b_2, \lambda_1\lambda_2 + f(b_1)f(b_2)g(b_1, b_2) + F(b_1, b_2)) \\
&\simeq (b_1b_2, g(b_1, b_2) + \lambda_1\lambda_2 + F(b_1, b_2)) \\
&\simeq (b_1b_2, h'((b_1, \lambda_1), (b_2, \lambda_2)) + \lambda_1\lambda_2 + F(b_1, b_2)) \\
&\simeq (h' + s)((b_1, \lambda_1), (b_2, \lambda_2)),
\end{aligned}$$

where the first two homotopies are obtained by sliding (exactly as in the proof that the loop space of an H-space is homotopy commutative) and the third homotopy comes from $g \circ (\pi \times \pi) \simeq h \simeq h'$. Therefore $s'' \simeq h' + s$, fiberwise. But we know that $s' \simeq h' + s$, fiberwise, and so $s' \simeq s''$, fiberwise. By using a retractile argument [**2**], we can show that s and s' are equivalent as multiplications on E.

COROLLARY 1.9. *Let*

$$\Omega K \to E \xrightarrow{\pi} B$$

be the fibering induced from $\Omega K \to LK \to K = K(G, l+1)$ *by a map* $f: B \to K$. *Assume* B *is* $(q-1)$ *connected,* $q \leqq l$, *that* $\pi_i(B) = 0$ *if* $i > q + l$, *and that* E *admits a multiplication* s. *Then, with respect to some multiplication* m *on* B, f *is an* H*-map and* s *is equivalent to a multiplication on* E *obtained from certain multipliers of* f *with respect to the multiplication* m *on* B.

Proof. Combine Theorems 1.5, 1.6, 1.8 and the proof of 1.6.

Corollary 1.9 says that in order to classify the equivalent multiplications on E, we need only consider the multiplications on E which make π a multiplicative map with respect to some multiplication on B. Let $H(E, m)$ denote all equivalence classes of multiplications on E such that π is multiplicative with respect to at least one multiplication on E in the equivalence class and the multiplication m on B. (Note: π need not be multiplicative with respect to every multiplication in an equivalence class in $H(E, m)$.) From Theorem 1.4, $H(E, m) = H(E, m')$ if m and m' are equivalent multiplications on B.

Again, Corollary 1.9 suggests that a study of multipliers will give some information about classification problems for the multiplications on E. We note here that we shall consider a specific map from B to K rather than a homotopy class of maps, since we want to work with the precise structure of the multiplications obtained from multipliers of the map. Of course, the results obtained from working with a map can be extended to the homotopy class containing this map.

Definition 1.10. Let (f, F) and (f, G) be two H-maps from (X_1, m_1) to (X_2, m_2). If there exists a homotopy $D: I \times X_1 \times X_1 \to PX_2$ such that

$$D|\{0\} \times X_1 \times X_1 = F,\ D|\{1\} \times X_1 \times X_1 = G,\ D(t, x, *) = D(t, *, x) = F(*, x) = x,\ e_0 D = e_0 F \quad \text{and} \quad e_\infty D = e_\infty F,$$

then we say that F is H-homotopic to G.

It is easy to see that this is an equivalence relation among all multipliers of the map f and the multiplications m_1 and m_2 on X_1 and X_2 respectively.

Let $M_m(f, n)$ be the H-homotopy classes of multipliers of $f: (B, m) \to (K, n)$. Using an argument similar to that in Theorem 1.4, it is easy to see that there is a one-to-one correspondence between $M_m(f, n)$ and $M_{m'}(f, n')$ provided m and m' are equivalent and n and n' are equivalent. In fact, in a certain sense (see Theorem 1.11), $M_m(f, n)$ depends only on the equivalence classes of m and n.

THEOREM 1.11. *Let* $K(G, l) = \Omega K \to E \to B$ *be the fibering induced from* $\Omega K \to LK \to K = K(G, l+1)$ *by an* H*-map* $f: (B, m) \to (K, n)$. *Then the map* $\Phi: M_m(f, n) \to H(E, m)$ *which sends* $\{F\}$ *to the equivalence class of multi-*

plications on E obtained from F, for any $\{F\} \in M_m(f, n)$, *is surjective. Moreover if m and n are equivalent to m′ and n′ as multiplications on B and K, respectively, then*

$$\mathrm{Im}(\Phi: M_m(f, n) \to H(E, m)) = \mathrm{Im}(\Phi: M_{m'}(f, n') \to H(E, m') = H(E, m)).$$

Proof. Definition 1.10 insures that the correspondence Φ is defined. Theorem 1.8 establishes that Φ is surjective. Finally, the last part of the theorem follows from Theorem 1.4.

In the light of Theorem 1.11 and the fact that an Eilenberg-Maclane space admits a unique (up to homotopy) multiplication, we shall write $M_m(f, n)$ by $M_m(f)$ when K is an Eilenberg-Maclane space.

2. $M_m(f)$ and $H^l(B \wedge B, G)$. In this section we want to establish a bijection between the set $M_m(f)$, where $f: (B, m) \to K(G, l + 1)$ is an H-map and G is an abelian group, and $H^l(B \wedge B; G)$. First we shall look at a special case, namely when $f = *$ is the trivial map. Then using certain correspondences between $M_m(*)$ and $M_m(f)$ we shall compute $M_m(f)$ for any H-map f.

LEMMA 2.1. *Let* (B, m) *be an H-space and* $*: B \to K(G, l + 1)$. *Then* $M_m(*) = H^l(B \wedge B; G) = [B \wedge B; K(G, l)]$.

Proof. For any multiplier $F: B \times B \to PK(G, l + 1)$ of the trivial map $*$, it follows by the definition of the multiplier that $e_0F = * = e_\infty F$ and $e_tF(*, x) = e_tF(x, *) = *$ for any real number t and $x \in B$. Hence the multiplier F can be considered as a map from $B \wedge B$ into $\Omega K(G, l + 1)$. By the definition of H-homotopic there is an obvious one-one correspondence between $M_m(*)$ and $[B \times B, B \vee B; K(G, l), *] = H^l(B \wedge B; G)$.

Now let us turn to the general case. We denote $K(G, l + 1)$ by K. Let $(f, F): (B, m) \to (K, n)$ be an H-map with the fixed multiplier F. Define a map $(F+)_\#$ from $M_m(*)$ into $M_m(f)$ as follows: Given any multiplier J of $*: (B, m) \to (K, n)$, we define $F + J$ to be the composite of the following diagram

$$F + J: B \times B \xrightarrow{\Delta} \begin{matrix} B \times B \xrightarrow{F} PK \\ \times \qquad\qquad \times \\ B \times B \xrightarrow[J]{} \Omega K \end{matrix} \xrightarrow{Pn} PK,$$

where Δ is the diagonal map and Pn is the multiplication on PK induced from the multiplication n.

It is obvious that $F + J$ is a multiplier of f. Let $(F+)_\#(\{J\})$ be the class containing $F + J$. It is easy to check that $(F+)_\#$ is a well-defined map from $M_m(*)$ into $M_m(f)$.

https://doi.org/10.4153/CJM-1972-107-4 Published online by Cambridge University Press

From [**2**, Theorem 1.1], there exists a map $q:K \times K \to K$ such that the composite

$$K \times K \xrightarrow{\Delta} \begin{matrix} K \times K \xrightarrow{n} K \\ \times \qquad\qquad \times \\ K \times K \xrightarrow[q]{} K \end{matrix} \longrightarrow K$$

is homotopic to the trivial map $*$. The homotopy yields a path $\omega(y_1, y_2)$ in K from $*$ to $n(n(y_1, y_2), q(y_1, y_2))$ where $y_1, y_2 \in K$. Moreover since $K(G, l + 1) = K$ is an abelian group when G is an abelian group [**4**], we can choose the multiplication n on K to be strictly associative and commutative. Therefore the path $\omega(y_1, y_2)$ also joins $*$ and $n(q(y_1, y_2), n(y_1, y_2))$. As before (see the end of § 1), the result obtained on $M_m(f)$ from this multiplication can be extended to the homotopy class containing this multiplication.

Now define $(F-)_{\#}:M_m(f) \to M_m(*)$ as follows. For any multiplier Q of f define

$$(F\Theta Q)(x_1, x_2) = \omega(f(x_1), f(x_2)) + n(Q(x_1, x_2) + F(x_1, x_2)^{-1}, \\ q(f(x_1), f(x_2))) + \omega(f(x_1), f(x_2))^{-1}.$$

Because

$$(F\Theta Q)(*, x) = \omega(*, f(x)) + \omega(*, f(x))^{-1}$$

and

$$(F\Theta Q)(x, *) = \omega(f(x), *) + \omega(f(x), *)^{-1},$$

then, by the homotopy extension property, $F\Theta Q$ can be deformed to a multiplier Q' of $*$. Moreover Q' is unique up to H-homotopy. If Q_1 and Q_2 are H-homotopic multipliers of f, then $F\Theta Q_1 \simeq F\Theta Q_2$, canonically. The homotopy extension property then shows that Q_1' is homotopic to Q_2'. Therefore the term $(F-)_{\#}\{Q\}$ which denotes the class containing Q', gives a map from $M_m(f)$ to $M_m(*)$.

In the following discussion we abbreviate $\omega(f(x_1), f(x_2))$, $J(x_1, x_2)$, $F(x_1, x_2)$, $Q(x_1, x_2)$ and $q(f(x_1), f(x_2))$ by ω, J, F, Q and q respectively. By the "sliding" argument, we have, for any multiplier J of $*$:

$$\begin{aligned} F\Theta(F + J)(x_1, x_2) &= \omega + n(F + J + F^{-1}, q) + \omega^{-1} \\ {}^1\simeq\ & \omega + n(n(J, n(f(x_1), f(x_2))) + F + F^{-1}, q) + \omega^{-1} \\ {}^2\simeq\ & \omega + n(n(J, n(f(x_1), f(x_2))), q) + \omega^{-1} \\ {}^3\simeq\ & J + \omega + \omega^{-1} \\ {}^4\simeq\ & J \end{aligned}$$

where ^{1} is obtained by the definition of $F + J$ and "sliding", ^{2} and ^{4} are obtained by "shrinking" and ^{3} is obtained by "sliding." The above homotopies ^{1}, ^{2}, ^{3} and ^{4} leave the end points of the path fixed. Therefore $(F-)_{\#}(F+)_{\#} = \text{id}$.

Now we show that $(F+)_{\#}(F-)_{\#} = \mathrm{id}$. Given any multiplier Q of f, consider

$$\begin{aligned}
(F + Q')(x_1, x_2) &= Pn(F, Q') \\
{}^1\simeq{}& n(\omega + n(Q + F^{-1}, q) + \omega^{-1}, n(f(x_1), f(x_2))) + F \\
={}& n(\omega, n(f(x_1), f(x_2))) + n(n(Q + F^{-1}, q), n(f(x_1), f(x_2))) \\
&\qquad + n(\omega^{-1}, n(f(x_1), f(x_2))) + F \\
{}^2\simeq{}& n(\omega, n(f(x_1), f(x_2))) + n(Q + F^{-1}, n(q, n(f(x_1), f(x_2)))) \\
&\qquad + n(\omega^{-1}, n(f(x_1), f(x_2))) + F \\
{}^3\simeq{}& Q + F^{-1} + F \\
{}^4\simeq{}& Q,
\end{aligned}$$

where ^{1} is obtained by the definition of Q' and by "sliding", ^{2} is obtained by "associativity" and ^{3} is the homotopy $D: I \times B \times B \to PK$ defined by

$$\begin{aligned}
D_0(x_1, x_2) &= n(\omega, n(f(x_1), f(x_2))) + n(Q + F^{-1}, n(q, n(f(x_1), f(x_2)))) \\
&\qquad + n(\omega^{-1}, n(f(x_1), f(x_2))) \\
D_1(x_1, x_2) &= Q + F^{-1} \\
D_t(x_1, x_2) &= n(\omega_t, n(f(x_1), f(x_2))) + n(Q + F^{-1}, \omega(1 - t)) \\
&\qquad + n(\omega_t^{-1}, n(f(x_1), f(x_2)))
\end{aligned}$$

where ω_t is defined as $\omega_t(s) = \omega(1 - ts)$. By strict commutativity of n on $K(G, l + 1)$ the usual path join "+" in $D_t(x_1, x_2)$ is meaningful. Again, each of the above homotopies ^{1}, ^{2}, ^{3} and ^{4} leave the end points of the path fixed. Therefore $(F+)_{\#}(F-)_{\#} = \mathrm{id}$. Hence $(F+)_{\#}$ is a bijection. So, from Lemma 2.1 we have the following result:

THEOREM 2.2. *Let $(f, F): (B, m) \to K(G, l + 1)$ be an H-map. Then there exists a one-to-one correspondence between $M_m(f)$ and $H^l(B \wedge B; G)$.*

3. Classification of the multiplications on the total space of a fibering. In the rest of this paper, let $\Omega K \to E \to B$ be the induced fibering from $\Omega K \to LK \to K = K(G, l + 1)$ by the map $f: B \to K$, where G is an abelian group. We assume B is $(q - 1)$ connected where $q \leqq l$ and $\pi_i(B) = 0$ if $i > q + l$. We denote by $H(E)$ the set of equivalence classes of multiplications on E. Then from Corollary 1.9 and Theorem 1.4 we have:

THEOREM 3.1. *Let L be an arbitrary representation of the set $\{\alpha \in H(B) | f$ is an H-map with respect to $\alpha\}$. Then $H(E) = \bigcup_{m \in L} H(E, m)$ as a set, provided that B is $(q - 1)$ connected where $q \geqq l$, and $\pi_i(B) = 0$ if $i > q + l$.*

We now consider how the multiplications on E change when we change the multiplications on B. We denote by π and i the fiber map and the inclusion of the fibering $\Omega K \to E \to B$ respectively.

LEMMA 3.2. *Consider any two multiplications s_1 and s_2 on E and m_1 and m_2 on B such that $\pi: (E, s_i) \to (B, m_i)$ is a multiplicative map for $i = 1, 2$. If s_1 and s_2 are equivalent, then m_1 and m_2 are equivalent provided B is $(q - 1)$ connected, $q \leqq l$ and $\pi_i(B) = 0$ if $i > q + l$.*

Proof. Since $\pi_i(K) = 0$ if $i \leqq l$, we can construct a cross section $\chi: B^l \to E$, where B^l is the lth skeleton of B. It follows that $\pi \circ s_1 \circ (\chi \times \chi) = m_1|B^l \times B^l$ and $\pi \circ s_2 \circ (\chi \times \chi) = m_2|B^l \times B^l$. Since s_1 and s_2 are equivalent, so are $m_1|B^l \times B^l$ and $m_2|B^l \times B^l$. The obstructions to m_1 and m_2 provide a single element in

$$H^i(B \wedge B, B^l \wedge B^l \pi_i(B))$$

which vanishes since $\pi_i(B) = 0$ if $i > q + l$.

This lemma shows that the union in Theorem 3.1 is a disjoint union. Therefore we reduce the problem of computing $H(E)$ to computing $H(E, m)$. In the light of Theorem 1.11, we can study $M_m(f)$ instead of $H(E, m)$. Now we wish to see when two different equivalence classes in $M_m(f)$ give the same multiplication equivalence class on E.

Let s be any multiplication on E. Define $s_\#: [E \wedge E; E] \to H(E)$ to be the map which sends $\{q\}$ in $[E \wedge E; E]$ to the class containing the composite

$$\begin{tikzcd} & E \wedge E \arrow[r, "q"] & F & \\ E \times E \arrow[ur] \arrow[rr] & & E & \end{tikzcd} \quad \begin{matrix} F \\ \times \\ E \end{matrix} \xrightarrow{s} E\,.$$

Since E is in the CW category and s induces a loop structure on $[E \times E, E]$ and $[E \wedge E, E]$, we have the following version of Copeland's Theorem [**1**]:

THEOREM 3.3. *$s_\#: [E \wedge E; E] \to H(E)$ is a bijection.*

Let us consider the following diagram

$$\begin{tikzcd} {[E \wedge E, \Omega B]} \arrow[r, "(\Omega f)_\#"] & {[E \wedge E, \Omega K]} \arrow[r, "i_\#"] & {[E \wedge E, E]} \\ {[B \wedge B, \Omega B]} \arrow[u, "1"] \arrow[r, "(\Omega f)_\#"] & {[B \wedge B, \Omega K]} \arrow[u, "2", "(\pi \wedge \pi)_\#"'] \arrow[r, "i_\#"] & {[B \wedge B, E]} \arrow[u, "(\pi \wedge \pi)_\#"'] \end{tikzcd}$$

where each row is exact. The map 2 is an isomorphism (by dimension reasons since K is an Eilenberg-Maclane space). An argument similar to the one in the proof of Lemma 3.2, shows that if B is $(q - 1)$ connected where $q \leqq l$ and $\pi_i(B) = 0$ when $i > q + l$, then the map 1 is onto. Therefore under this condition we have

$$\operatorname{Im}(\Omega f_\#: [E \wedge E, \Omega B] \to [E \wedge E, \Omega K]) \\ = (\pi \wedge \pi)_\#(\operatorname{Im}(\Omega f: [B \wedge B, \Omega B] \to [B \wedge B, \Omega K])).$$

LEMMA 3.4. *Let $G_1 = F + q_1$ and $G_2 = F + q_2$ be two multipliers of the H-map $(f, F): (B, m) \to (K, n)$, where q_1 and q_2 map $B \wedge B$ into ΩK (cf. Theorem 2.2). Let s_1, s_2 and s be the multiplications on E obtained from G_1, G_2*

and F respectively. If s_1 and s_2 are equivalent multiplications on E and (B, m) is an H-space, then

$$\{q_2\}\{q_1\}^{-1} \in \mathrm{Im}[(\Omega f)_{\#}:[B \wedge B, \Omega B] \to [B \wedge B, \Omega K]].$$

Proof. Recall the definition of the multiplication s_j obtained from the multiplier G_j (see Theorem 1.3), where $s_j = s(i(\pi \wedge \pi)q_j, s)$. Then if $\{s_1\} = \{s_2\}$, it follows that

$$\{s(i(\pi \wedge \pi)q_1, s)\} = s_{\#}\{i(\pi \wedge \pi)q_1\} = \{s(i(\pi \wedge \pi)q_2, s)\} = s_{\#}\{i(\pi \wedge \pi)q_2\}.$$

So by Theorem 3.3, $i_{\#}(\pi \wedge \pi)_{\#}\{q_1\} = i_{\#}(\pi \wedge \pi)_{\#}\{q_2\}$. By the exactness of the rows in the above diagram and the fact that the map 1 is an epimorphism and the map 2 is an isomorphism we have

$$i_{\#}(\pi \wedge \pi)_{\#}\{q_1\} = i_{\#}(\pi \wedge \pi)_{\#}\{q_2\} \text{ if and only if } i_{\#}(\pi \wedge \pi)_{\#}(\{q_1\}\{q_2\}^{-1}) = * \text{ in } [E \wedge E, E],$$

i.e. if and only if $\{q_1\}\{q_2\}^{-1} \in \mathrm{Im}[(\Omega f)_{\#}:[B \wedge B, \Omega B] \to [B \wedge B, \Omega K]]$.

Let R be the relation in $M_m(f)$ defined as follows: for any $\{F\}, \{G\}$ in $M_m(f)$, $\{F\}$ is R-related to $\{G\}$ if and only if $\Phi\{F\} = \Phi\{G\}$. It is obvious that R is an equivalence relation.

THEOREM 3.5. *The map Φ (cf. Theorem 1.11) induces a bijection from*

$$M_m^{(f)}/R = [B \wedge B, \Omega K]/\mathrm{Im}[(\Omega f)_{\#}:[B \wedge B, \Omega B] \to [B \wedge B, \Omega K]]$$

onto $H(E, m)$, provided B is $(q-1)$ connected, $q \leqq l$, and $\pi_i(B) = 0$ if $i > q + l$.

Proof. Combine Theorem 2.2, Theorem 1.11, and Lemma 3.4.

THEOREM 3.6. *There is a one-to-one correspondence between $H(E)$ and $\bigcup_{m \in L} A_m$ where $A_m = H^l(B \wedge B, G)/\mathrm{Im}[(\Omega f)_{\#}:[B \wedge B, \Omega B] \to [B \wedge B, \Omega K]]$ and L is an arbitrary representation of the set $\{\alpha \in H(B) | f$ is an H-map with respect to $\alpha\}$, provided B is $(q-1)$ connected where $q \leqq l$, and $\pi_i(B) = 0$ if $i > q + l$.*

Proof. Theorem 3.5 holds for any multiplication m on B so long as $f:(B, m) \to K = K(G, l + 1)$ is an H-map. Therefore the result follows from Theorem 3.1 and Theorem 3.5.

By using Theorem 3.6 inductively on the Postnikov decomposition of a space E with finitely many non-vanishing homotopy groups, we can reduce the problem of computing $H(E)$ to computing quotients of certain cohomology groups and to determining the primitive elements.

4. Special cases. As we saw in Theorem 3.6, in order to compute $H(E)$ we need to know $H^l(B \wedge B, G)$, $\mathrm{Im}[\Omega f_{\#}:[B \wedge B, \Omega B] \to [B \wedge B, \Omega K]]$ and L

(knowledge of L is equivalent to the knowledge of which multiplications on B make $f(c)$ primitive, where c is the fundamental class of ΩK). In general these three sets are not easy to determine. Here we shall give two cases in which

$$\mathrm{Im}[\Omega f_{\#}:[B \wedge B; \Omega B] \to [B \wedge B; \Omega K]] = 0$$

and these cases indicate that our results agree with known results in this area.

1. If B is an Eilenberg-Maclane space, Theorem 3.6 gives the same result as Copeland [**1**].

2. If B is $(q-1)$ connected and $\pi_i(B) = 0$ if $i > 2q$, then

$$\mathrm{Im}[\Omega f_{\#}:[B \wedge B; \Omega B] \to [B \wedge B; \Omega K]] = 0.$$

In fact an easy obstruction argument shows that $[B \wedge B; \Omega B] = 0$. Thus Theorem 3.6 agrees with McCarty's result [**3**].

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