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LOCAL INTEGRAL METRICS AND DANIELL-LOOMIS INTEGRALS

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Recently Guerrero and the first author (Diaz Carrillo) proved an anologue to Daniell's extension process which works for arbitrary nonnegative linear functionals, without any continuity conditions. With the aid of Schäfke's local integral metrics we generalise this extension process and prove convergence theorems using a suitable local mean convergence, which can be traced back to Loomis.

INTRODUCTION

Recently in [3] an analogue to Daniell's extension process was given which works for arbitrary nonnegative linear functionals, without any continuity conditions. With the aid of Schäfke's local integral metrics [19] we generalise here this extension process and prove convergence theorems (which do not hold for $\overline{B}$ of [3]); this is possible using a suitable local mean convergence, which can be traced back to Loomis [13, p.179].

In Section 1 we recapitulate the Aumann-Schäfke integration theory for general integral metrics, prove an analogue of Lebesgue's convergence theorem, and introduce measurability. In Section 2, for general local integral metrics, more convergence theorems are obtained, extending results of [19] (we do not use $q((1/n)h) \to 0$ or the restrictive 2 of [19]). In Section 3 a unified treatment of Riemann-μ, abstract Riemann-Loomis, Daniell and Bourbaki integrals is given.

With Schäfke [16, 17, 18, 19], most results extend to Banach space valued functions, using $x \cap t$ of [10], some even to function-valued integral metrics of [18, 19]. Part of our results of 3.E on $L(I \mid B)$ have been announced in [7].

0. NOTATIONS

We extend the usual $+$ in $\overline{\mathbb{R}} := \{-\infty\} \cup$ reals $\mathbb{R} \cup \{\infty\}$ to $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

$$(1) \qquad \begin{aligned} &r + s := 0, \quad r \dot{+} s := \infty, \quad r \underset{\cdot}{+} s := -\infty, \text{ if } r = -s \in \{\infty, -\infty\}; \\ &r - s := r + (-s) \text{ et cetera.} \end{aligned}$$

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With $r \vee s := \max(r, s)$, $r \wedge s := \min(r, s)$, $r \cap t := (r \wedge t) \vee (-t)$ if $0 \leqslant t \in \overline{\mathbb{R}}$ one still has for arbitrary $a, b, c, d \in \overline{\mathbb{R}}$, $0 \leqslant t \in \overline{\mathbb{R}}$ the Birkhoff-inequalities

$$(2) \qquad \begin{aligned} &|a \cap t - b \cap t| \leqslant 2(|a-b| \wedge t), \quad |a \wedge c - b \wedge c| \leqslant |a-b|, \\ &|a \vee c - b \vee c| \leqslant |a-b| \end{aligned}$$

$$(3) \qquad ||a| - |b|| \leqslant |a-b| \leqslant |a-c| + |c-b|, \quad |(a+b)-(c+d)| \leqslant |a-c| + |b-d|$$

(Aumann [3] ($^*b$, $^*c$)); $+$, $\dot{+}$, $\underset{\cdot}{+}$ are commutative, $+$ is distributive with $0 \cdot (\pm\infty) := 0 =: (\pm\infty) \cdot 0$, but not associative; $\dot{+}$ is associative. On the set $\overline{\mathbb{R}}^X$ of functions $f \colon X \to \overline{\mathbb{R}}$ we define $=$, $\pm$, $\dot{+}$, $\wedge$, $\vee$, $\cap$, $\cdot$, $|\ |$, $\leqslant$ pointwise on X with $|f|(x) := |f(x)|$, $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in X$. Define

$$(4) \qquad +M := \{f \in M : 0 \leqslant f\} \quad \text{if} \quad M \subset \overline{\mathbb{R}}^X.$$

$(k_j) = (k_j)_{j \in J}$ means a net with $k_j \in \overline{\mathbb{R}}^X$, $j \in J$ where J is a set directed by $\leqslant$.

1. Integral Metrics

Standard assumptions in this and the following sections are

$$(5) \qquad X \text{ is a non-empty set}, \quad 0 \in B \subset \overline{\mathbb{R}}^X, \quad q \colon +\overline{\mathbb{R}}^X \to +\overline{\mathbb{R}} := [0, \infty].$$

For later reference and the benefit of the reader, we first collect some results mostly due to Aumann [1], Schäfke [16, 19]:

Definition 1: $q \colon +\overline{\mathbb{R}}^X \to +\overline{\mathbb{R}}$ is called an *integral metric on* X if

$$(6) \qquad q(0) = 0 \text{ and } q(a) \leqslant q(b) + q(c) \text{ if } a \leqslant b + c, \quad a, b, c \in +\overline{\mathbb{R}}^X. \ ([\mathbf{16}])$$

For an integral metric q one gets with (3)

$$(7) \qquad |q(|a|) - q(|b|)| \leqslant q(|a-b|), \quad q(c) \leqslant q(d), \quad \text{if } a, b, c, d \in \overline{\mathbb{R}}^X, \quad 0 \leqslant c \leqslant d.$$

$\left(\overline{\mathbb{R}}^X, q\right)$ is a topological space using $\{a \in \overline{\mathbb{R}}^X : q(|a - a_0|) < \varepsilon\}$ with $-$ of (1).

Definition 2: $N_q := \{k \in \overline{\mathbb{R}}^X : q(|k|) = 0\}$ (*q-nulfunctions*); if $M \subset \overline{\mathbb{R}}^X$ and q is an integral metric on X, we define

$$(8) \qquad M^q := \text{ closure of } M \text{ in } \left(\overline{\mathbb{R}}^X, q\right)$$

(*q-integrable functions*, with respect to M).

Lemma 1. *If q is an integral metric on X, $0 \in B \subset \overline{\mathbb{R}}^X$, $* \in \{+, -, \ldots, |\ |, s\cdot$ with $s \in \mathbb{R}\}$ and B is closed with respect to $*$, for example $h + k \in B$ (respectively $h \cap |k|$, respectively $|h| \in B$) if $h, k \in B$, then B^q is also closed with respect to this $*$; $B \cup N_q \subset B^q = (B^q)^q$. N_q is closed with respect to all operations from $\{\cdots\}$.*

We call $B \subset \overline{\mathbb{R}}^X$ a *function vector lattice* if it is a real linear space under pointwise $=, +, s\cdot$, such that $h \in B$ implies $|h| \in B$; then $h \wedge k$, $h \vee k \in B$ for $h, k \in B$, thus $h \cap |k| \in B$.

$I \mid B$ *Loomis system* (on X) means

(9) B is a function vector lattice, $I\colon B \to R$ is linear, $I(h) \geqslant 0$ if $h \in +B$.

(10)

qBI means q is an integral metric on X, (9) holds, I is q-continuous in 0

If only an integral metric q on X and a function vector lattice B are given, $qB0$ is true with $I \equiv 0$, all the following results hold then.

Theorem 1. (Aumann). *If qBI of (10) holds, then the set B^q of q-integrable functions is closed with respect to $\pm$, $s\cdot$ $(s \in \mathbb{R})$, $|\ |$, $\vee$, $\wedge$, $\cap$. Also $B \cup N_q \subset B^q = (B^q)^q$ and there is a unique q-continuous extension $I^q\colon B^q \to \mathbb{R}$ of $I \mid B$.*

For this I^q one has $I^q(sf) = sI^q(f)$, $I^q(f+g) = I^q(f) + I^q(g)$, $|I^q(f)| \leqslant I^q(|f|) =: \|f\|_{I^q}$, $I^q(k) \leqslant I^q(l)$ if $f, g, k, l \in B^q$, $s \in \mathbb{R}$, $k \leqslant l$; $\|\ \|_{I^q}$ is a semi-norm on B^q which is q-continuous, so $q(|k|) = 0$ implies $k \in B^q$ and $I^q(k) = 0$.

Definition 3: If (5) holds, for $k, l \in \overline{\mathbb{R}}^X$, with $k^+ := k \vee 0$:

(11)

$k = l\,(q)$ respectively $k \leqslant l\ (q)$ means $q(|k-l|) = 0$ respectively $q\Big((k-l)^+\Big) = 0$.

Lemma 2. *If (5) holds and q is an integral metric on X, then $=\ (q)$ is an equivalence relation in $\overline{\mathbb{R}}^X$; $\leqslant (q)$ is there transitive; they are further compatible with $+$ of (1) and $s\cdot$, $s \in \mathbb{R}$ respectively $+\mathbb{R}$.*

Corollary I. *If qBI of (10) holds, $f \in B^q$, $k \in \overline{\mathbb{R}}^X$ with $k = f(q)$ then $k \in B^q$ and $I^q(k) = I^q(f)$.*

Corollary II. *qBI, $f \in B^q$ imply $f_e \in B^q$, $f_{\pm\infty}$ and $f_u \in N_q$, $I^q(f) = I^q(f_e)$.*

Here $k_e(x) := k(x)$ if $k(x) \in \mathbb{R}$, $:= 0$ else; $k_u := k - k_e$, $k_\infty := k_u \vee 0$, $k_{-\infty} := k_u \wedge 0$.

Corollary III. *qBI, $f, g \in B^q$, $k \in \overline{\mathbb{R}}^X$ with $k(x) = f(x) + g(x)$ if $f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$ imply $k \in B^q$ and $I^q(k) = I^q(f) + I^q(g)$.*

Corollary IV. *qBI, $f,g \in B^q$ imply $f\dot{\pm}g$, $f \pm g \in B^q$ and $I^q(f\dot{\pm}g) = I^q(f) \pm I^q(g)$.*

Corollary V. *qBI, $f,g \in B^q$ $f \leqslant g(q)$ imply $I^q(f) \leqslant I^q(g)$.*

Corollary VI. *The extension process $I \mid B \to I^q B^q$ is iteration complete.*

This means: If qBI holds, $\widetilde{B} := B^q \cap \mathbb{R}^X$, $\widetilde{I} := I^q \mid \widetilde{B}$, then $q\widetilde{B}\widetilde{I}$ holds, $\widetilde{B}^q = B^q$ and $\widetilde{I}^q = I^q$.

In the proofs one uses $|f_e - h_n| \leqslant |f - h_n|$, $|f_u| \leqslant 2|f - h_n|$, for Corollary III $|k - (h + l)| \leqslant |f - h| + |g - l|$, and, valid for any $a, b, c, d \in \overline{\mathbb{R}}$, $t \in +\overline{\mathbb{R}}$,

$$(a - b) \vee t \leqslant (a - c) \vee t + (c - b) \vee t, \tag{12}$$

$$((a + b) - (c + d)) \vee t \leqslant ((a - c) \vee t) + ((b - d) \vee t)$$

$$a \leqslant b \dot{+} (a - b)^+. \tag{13}$$

Definition 4: With (5), $k \in \overline{\mathbb{R}}^X$: k *q-B-measurable* means that $k \cap h \in B^q$ for all $h \in +B$,

$$M_\cap(q, B) := \{k \in \overline{\mathbb{R}}^X : k\ q\text{-}B\text{-measurable}\}. \tag{14}$$

Theorem 2. *If qBI holds and $k \in \overline{\mathbb{R}}^X$, then the following three statements are equivalent:*

(a) $k \in B^q$

(b) *k is q-B-measurable and $|k| \in B^q$*

(c) *k is q-B-measurable and there is $\varphi \in B^q$ with $|k| \leqslant \varphi$ (q).*

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) by Theorem 1. For (c) $\Rightarrow$ (a), with (13) and Corollary IV we can assume $|k| \leqslant \varphi$; then $0 \leqslant \varphi$ and for $\varepsilon > 0$ there is $h \in B$ with $q(|\varphi - h|) < \varepsilon$; since

$$|\varphi - |h|| \leqslant |\varphi - h|, \quad |h| \in +B, \text{ one can assume } h \geqslant 0. \tag{15}$$

Then $|k - k \cap h| = |k \cap \varphi - k \cap h| \leqslant 2|\varphi - h|$ by (2), so $q(|k - k \cap h|) < 2\varepsilon$, or $k \in (B^q)^q = B^q$. ∎

For convergence theorems, we need an analogue to "convergence in measure":

Definition 5: For J net, $k, k_j \in \mathbb{R}^X$ for $j \in J$, (5):

$$k_j \to k\ (q, B) \quad \text{means}$$

for each $\varepsilon > 0$ and $h \in +B$ there is $j = j_{\varepsilon,h} \in J$ such that $q(|k - k_i| \wedge h) < \varepsilon$ if $j \leqslant i \in J$ (*q-B-local convergence*).

If with integral metric q, fixed B and the same J, $k_j \to k\ (q, B)$ and $l_j \to l(q, B)$, then $sk_j \to sk\ (q, B)$ and $k_j + l_j \to k + l\ (q, B)$ for $s \in \mathbb{R}$ by (8), (3).

Theorem 3. *If q is an integral metric on X, $0 \in B \subset \overline{\mathbb{R}}^X$, J net, $k_j, k \in \overline{\mathbb{R}}^X$ with $k_j \to k\ (q, B)$ and $|k_j - k| \leqslant \varphi\ (q)$ for $j \in J$ with some $\varphi \in B^q$, then $q(|k - k_j|) \to 0$.*

Proof: If $l_j := |k_j - k| \leqslant \varphi$, then as in the proof of Theorem 2, for $\varepsilon > 0$ choose $h \in +B$ with $q(|\varphi - h|) < \varepsilon$, then j with $q(|k - k_i| \wedge h) < \varepsilon$ if $i \geqslant j$. Now $l_i = l_i \wedge h + (l_i - l_i \wedge h) = l_i \wedge h + (l_i \wedge \varphi - l_i \wedge h) \leqslant l_i \wedge h + |\varphi - h|$, so $q(l_i) < 2\varepsilon$ if $i \geqslant j = j_\varepsilon$.

In the general case, $l_i \leqslant \varphi \dotplus p_i \leqslant |\varphi| + p_i$ by (15) with $p_i := (l_i - \varphi)^+ \in N_q$; then $0 \leqslant h_i := l_i - l_i \wedge p_i \leqslant |\varphi| \in B^q$ for $i \in J$ and $q(h_i \wedge h) \leqslant q(l_i \wedge h) \to 0$ if $h \in +B$, that is $h_i \to 0\ (q, B)$, so $q(h_i) \to 0$ by the above. Now $l_i = h_i + l_i \wedge p_i$, therefore $0 \leqslant q(l_i) \leqslant q(h_i) + q(p_i) = q(h_i) \to 0$. ∎

Corollary VII. (Lebesgue's convergence theorem for B^q): *If qBI holds, f_j, $\varphi \in B^q$, $k \in \overline{\mathbb{R}}^X$, $f_j \to k\ (q, B)$ and $|f_j - k| \leqslant \varphi(q)$ for $j \in J$ then $k \in B^q$, $q(|f_j - k|) \to 0$, $I^q(f_j) \to I^q(k)$.*

Proof: By Theorem 3 and 1, $k \in (B^q)^q = B^q$ and $|I^q(f_j) - I^q(k)| \leqslant I^q(|f_j - k|) \to 0$. ∎

Lemma 3. *If (5) holds and q is an integral metric on X, $g_j \in M_\cap(q, B)$, $k \in \overline{\mathbb{R}}^X$ and $g_j \to k(q, B)$ then $k \in M_\cap(q, B)$.*

Proof: $|g_j \cap h - k \cap h| \leqslant 2(|g_j - k| \wedge h)$ by (2), so $k \cap h \in (B^q)^q = B^q$ by Lemma 1. ∎

Lemma 4. *If $qB0$ of (10) holds, $f, g \in M_\cap(q, B)$, $l \in B^q$, $s \in \mathbb{R}$ and $k \in \overline{\mathbb{R}}^X$ then l, $|f|$, $f \wedge g$, $f \vee g$, $f^\pm$, sf, $f \cap |g|$, $f + l$ belong to $M_\cap(q, B)$.*

$f + g \in M_\cap(q, B)$ if there are $f_0, g_0 \in B^q$ with $f \geqslant f_0$ and $g \geqslant g_0$.

Besides $k \in M_\cap(q, B) \Leftrightarrow k \wedge h \in M_\cap(q, B)$ for $h \in +B \Leftrightarrow f^+, f^- \in M_\cap(q, B)$.

With only $f \geqslant 0$, for example, in general $f + g \notin M_\cap(q, B)$. ([10], A5.100).

Proof: $|r| \cap t = |r \cap t|$, $(-r) \cap t = -(r \cap t)$, $(sr) \cap (st) = s(r \cap t)$, $(p \wedge r) \cap t = (p \cap t) \wedge (r \cap t)$, $r^\pm = (\pm r) \vee 0$, $(p \cap |r|) \cap t = (p \cap t) \cap |r \cap t|$ if $p, r, s, t \in \overline{\mathbb{R}}$, $s \geqslant 0$, $t \geqslant 0$, (2), (3) and Theorem 1. $f + l \in M_\cap$ by Lemma 3 if $f + v \in M_\cap$ for $v \in B$; but then $(f + v) \cap h = ((f_v) \wedge h) \vee (-h) = (((f \wedge (h - v)) + v) \vee (-h)) = [(f \wedge (h - v)) \vee (-h - v)] + v$; $[\ ] \in M_\cap$ since $M_\cap$ is $\wedge$ and $\vee$-closed; also $|[\ ]| \leqslant |h| + |v| \in B$, do $[\ ] \in B^q$ by Theorem 2, then $(f + v) \cap h \in B^q$. $k \cap h = (k \wedge h) \cap h = k^+ \wedge h - k^- \wedge h$ if $h \in +B$, Theorems 1,2. If $f, g \geqslant 0$, then $(f + g) \cap h = (f \wedge h + g \wedge h) \wedge h \in B^q$; else $f + g = (f^+ + g^+) - (f^- + g^-)$, with $f^+ + g^+ \in M_\cap$, $|f^- + g^-| \leqslant |f_0| + |g_0| \in B^q$, so $-l := f^- + g^- \pi B^q$ by Theorem 2. ∎

Corollary VIII. *If qBI holds, $g_j \in M_\cap(q, B)$, $k \in \overline{\mathbb{R}}^X$, $g_j \to k\ (q, B)$ and there is $\varphi \in B^q$ with $|k| \leqslant \varphi(q)$, then $k \in B^q$.*

This variant to Corollary VII follows from Lemma 3 and Theorem 2; however, even if $g_n \in B^q$, $I^q(g_n) \to I^q(k)$ is false in general.

Also Theorem 3/Corollary VII, VIII become false if only $|f_j - k| \leqslant \varphi$ (q_B) with q_B of Section 2 (Example 2 in Günzler [**11**, see Section 2]).

Finally, $\mathfrak{L}^q := B^q/N_q$ is a vector lattice with integral I^q and norm $I^q(|\cdot|)$ (Aumann [**1**, p.445]).

2. Local integral metrics

Definition 6: If X, B, q are as in (5), then

$$q_B(k) := \sup\{q(k \wedge h);\ h \in +B\}, \text{ for } k \in +\overline{\mathbb{R}}^X. \tag{16}$$

This is a simplified version of Schäfke's definition [**19**, p.120]. Here it gives all the relevant results, under weaker assumptions.

Lemma 5. *If (5) holds and q is an integral metric on X, then q_B of (16) is also an integral metric on X, with $q_{BB} = q_B \leqslant q$; $q_B(k) = q(k)$ if $k \in +\overline{\mathbb{R}}^X$ and $k \leqslant$ some $h \in B$, so $k_j \to k$ (q, B) of Definition 5 is equivalent with $k_j \to k$ (q_B, B); if additionally $|B| \subset B$, then $q_B(k \wedge \varphi) = q(k \wedge \varphi)$ for $k \in +\overline{\mathbb{R}}^X$, $\varphi \in +B^q$.*

Proof: As in Schäfke [**19**, p.120–121], with

$$(r + s) \wedge t \leqslant r \wedge t + s \wedge t \quad \text{if } r, s, t \in +\overline{\mathbb{R}}^X. \tag{17}$$

∎

Lemma 6. *If qBI of (10) holds, then also q_BBI is true, so B^{q_B} and $I^{I_B} : B^{q_B} \to \mathbb{R}$ are well defined, with*

$$B \subset B^q \subset B^{q_B}, \qquad I \subset I^q \subset I^{q_B}. \tag{18}$$

Theorem 4. *If (5) holds, q is an integral metric on X, $k_j, k \in \overline{\mathbb{R}}^X$, then $q_B(|k_j - k|) \to 0$ if and only if*

$$k_j \to k\ (q, B) \quad \text{and} \quad (k_j) \text{ is a } q_B\text{-Cauchy-net.} \tag{19}$$

Proof: As in Schäfke [**19**, p.121–122]. ∎

Corollary IX. *If (5) holds with integral metric q, then $f \in B^{q_B}$ if and only if to f there exists a sequence (h_n) with $h_n \in B$ and*

$$h_n \to f\ (q, B), \quad q(|h_n - h_m|) \to 0 \quad \text{as } n, m \to \infty. \tag{20}$$

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Corollary X. *If (5) holds with integral metric $q, f_j \in B^{q_B}$, $k \in \overline{\mathbb{R}}^X$, $f_j \to k$ (q, B) and (f_j) q_B-Cauchy then $k \in B^{q_B}$, $q_B(|k - f_j|) \to 0$ (and $I^{q_B}(f_j) \to I^{q_B}(k)$ if qBI holds).*

This closedness-property of B^{q_B} can be looked at as a convergence theorem for B^{q_B}; by Example 2 of [11] an analogue for general B^q is false.

Corollary XI. *If qBI holds, for $f \in \overline{\mathbb{R}}^X$ the following statements are equivalent:*

(a) $f \in B^q$
(b) $f \in B^{q_B}$ *and there exists* $\varphi \in B^q$ *with* $|f| \leqslant \varphi(q)$
(c) $f \in M_\cap(q_B, B)$ *and there exists* $\varphi \in B^q$ *with* $|f| \leqslant \varphi(q)$.

So $M_\cap(q_B, B) = M_\cap(q, B)$.

Proof: (a) $\Rightarrow$ (b) with $\varphi = |f|$ and (b) $\Rightarrow$ (c) follows from Theorems 1 and 2. For (c) $\Rightarrow$ (a), $f \cap h \in B^{q_B}$ for $h \in +B$, so there are $h_n \in B$ with $q_B(|f \cap h - h_n \cap h|) \leqslant q_B(2|f - h_n|) \to 0$ by (2); since $|f \cap h - h_n \cap h| \leqslant 2h \in B$, by Lemma 5 $q(|f \cap h - h_n \cap h|) = q_B(|f \cap h - h_n \cap h|) \to 0$, thus $f \cap h \in B^q$; Theorem 2 yields $f \in B^q$. $|f \cap h| \leqslant h$ for the $M_\cap$-statement. ∎

Again, $|f| \leqslant \varphi(q)$ cannot be weakened here to $\leqslant (q_B)$; similarly $= (q)$ is stronger than $= (q_B)$. One only has, for $k, l \in \overline{\mathbb{R}}^X$, with (5):

$$k = l\,(q_B) \Leftrightarrow k = l\,(q, B) \!: \Leftrightarrow K_n := k \to l\,(q, B) \quad \text{(Definition 5)}, \tag{21}$$

similarly for $\leqslant (q_b)$.

For q_B Theorem 2 can be sharpened with:

Definition 7: (Schäfke). Under (5), q is called *B-semiadditive* if one has

$$h_n \in +B, \sup\{q\left(\sum_1^n h_m\right); n \in \mathbb{N}\} < \infty \Rightarrow q(h_n) \to 0 \text{ as } n \to \infty. \tag{22}$$

q is called *B-additive* if $h, k \in +B$ imply $q(k + h) = q(k) + q(h)$.

Obviously "$q\,B$-additive" implies "$q\,B$-semiadditive" if $+B$ is $+$ closed.

Any integral metric extension of $\| \ \|_p = \left(\int | \ |^p \, d\mu\right)^{1/p}$ from $L^p(\mu, \overline{\mathbb{R}})$ is B-semiadditive if $1 \leqslant p < \infty$, where B = step functions B_Ω or L^p, with arbitrary measure space (X, Ω, μ); only for $p = 1$ has one B-additivity; see Section 3.F.

Lemma 7. *If (5) holds, q is an integral metric on X, $|B| + |B| \subset |B| \subset B$ and q B-semiadditive then q is B^q-semiadditive and q_B B^{q_B}-semiadditive.*

This analogue to Satz 5.1 of Schäfke [19] follows as in [16, p.161–162].

THEOREM 5. *If $qB0$ of (10) holds, q is B-semiadditive, k is q-B-measurable and $q_B(|k|) < \infty$, then $k \in B^{q_B}$.*

PROOF: With Lemmas 5, 4 and Theorem 2 one can assume $k \geqslant 0$; then one can argue as in Schäfke [19, p.127], showing first: For $\varepsilon > 0$ there exists $h_\varepsilon \in +B$ with

$$(23) \qquad q(k \wedge h - k \wedge h_\varepsilon) < \varepsilon \quad \text{for all } h \text{ with } h_\varepsilon \leqslant h \in B.$$

∎

Without "B-semiadditive" or with q instead of q_B Theorem 5 becomes false though q- and q_B-measurability coincide (Corollary XI): Example 1 below.

LEMMA 8. *If (5) holds, k, $k_j \in +\overline{\mathbb{R}}^X$, $k_j \to k\ (q, B)$ then $q_B(k) \leqslant \underline{\mathrm{Lim}}\, q_B(k_j)$.*

PROOF: $q(|k \wedge h - k_j \wedge h|) \leqslant 2q(|k - k_j| \wedge h) \to 0$ by (2) if $h \in +B$; if $a < q_B(k)$ there is $h \in +B$ with $a < q(k \wedge h)$; now $q(k \wedge h) - q(k_j \wedge h) \leqslant q(|k \wedge h - k_j \wedge h|) \to 0$, so $a < q(k_j \wedge h) \leqslant q_B(k_j)$ if $j \geqslant j_{a,h}$. ∎

With this and Lemma 3 one gets the following analogue to Fatou's Lemma.

COROLLARY XII. *If qBI holds, q is B-semiadditive, $f_j \in M_\cap(q, B)$, $k \in \overline{\mathbb{R}}^X$, $f_j \to k\ (q, B)$ and $\underline{\mathrm{Lim}}\, q_B(|f_j|) < \infty$ or $q_B(|k|) < \infty$, then $k \in B^{q_B}$.*

Here the same remarks as after Theorem 5 apply; also in general $q_B(|f_j|) \not\to q_B(|k|)$.

THEOREM 6. (Monotone Convergence Theorem for B^{q_B})

Assume qBI, qB-semiadditive, $k \in \overline{\mathbb{R}}^X$, $(f_j)_{j \in J}$ net with $f_j \in B^{q_B}$, $f_j \to k\,(q, B)$, $f_i \leqslant f_j\ (q_B)$ if $i \leqslant j \leqslant J$, and $\underline{\mathrm{Lim}}_j\, q_B(|f_j - f_{j_0}|) < \infty$ for some $j_0 \in J$. Then $k \in B^{q_B}$, $q_B(|f_j - k|) \to 0$, $f_j \leqslant k\ (q_B)$, $I^{q_B}(f_j) \to I^{q_B}(k) = \sup_{j \in J} I^{q_B}(f_j)$.

EXAMPLE 1: If $X = \mathbb{N}$, $B = c_{00} := \{a \in \mathbb{R}^{\mathbb{N}} \colon a(n) = 0 \text{ for } n \geqslant \text{ some } n_a\}$, $q(k) = q_B(k) = \|k\|_\infty := \sup_{n \in \mathbb{N}} |k(n)|$, $f_n(m) := 1$ if $1 \leqslant m \leqslant n$, $:= 0$ else, $I = 0$, then one has qBI, $0 \leqslant f_{n-1} \leqslant f_n \to k := 1\ (q, B)$, $q_B(f_n) \equiv 1$, but $k \notin B^q = B^{q_B} = c_0(\mathbb{N}, \mathbb{R}) \subset \mathbb{R}^{\mathbb{N}}$, so q is not B-semiadditive.

If $q(h) < \infty$ for $h \in +B$, also $q_B \neq \infty$ on B^{q_B}, the $\underline{\lim}\, q_B(|f_j - f_{j_0}|) < \infty$ if and only if $\underline{\lim}\, q_B(|f_j|) < \infty$.

For B^q instead of B^{q_B} Theorem 6 becomes false by Example 2 of [11].

PROOF OF THEOREM 6: $g_j := f_j - f_{j_0} \to l := k - f_{j_0}\ (q, B)$ by the remark after Definition 5; Lemma 4, Corollary XII and III give $l, k \in B^{q_B}$.

Furthermore $f_i - f_j \to f_i - k(q, B)$ and $p_{ij} := (f_i - f_j)^+ \to (f_i - k)^+\ (q, B)$ as $j \to \infty$ by (2); since $p_{ij} \in +N_{q_B}$ if $i \leqslant j$, one gets $(f_i - k)^+ \in N_{q_B}$ or $f_i \leqslant k(q_B)$ for $i \in J$ by Lemma 8; Corollary V yields $I^{q_B}(f_i) \leqslant I^{q_B}(k)$.

With (13) and $p_i := p_{j_0 i}$ one has $0 \leqslant g_i \dotplus p_i$, furthermore one can show $g_i \leqslant g_j \dotplus p_{ij} \in N_{q_B}$, yielding $q_B(|g_i|) \leqslant q_B(|g_j|)$ or $\sup_{j \geqslant j_0} q_B(|g_j|) = \underline{\lim} q_B(|g_j|) := t_0 < \infty$.

(f_j) is q_B-Cauchy, if for each $\varepsilon > 0$ there is i_ε with $q_B(f_j - f_{i_\varepsilon}) \leqslant \varepsilon$ if $j \geqslant i_\varepsilon$; if this is false, there is $\varepsilon_0 > 0$ and recursively a sequence (j_m) with $j_0 \leqslant j_1 \leqslant \dots$ and $q_B(|f_{j_k} - f_{j_{k-1}}|) > \varepsilon_0$, $k \in \mathbb{N}$. Since $|f_j - f_i| \leqslant (f_j - f_i) \dotplus 2p_{ij}$, inductively $\sum_1^n |f_{j_k} - f_{j_{k-1}}| \leqslant |f_{j_n} - f_{j_0}| + l_n$ with $q_B(l_n) = 0$. The associativity of $\dotplus$ gives

$$\begin{aligned}\sum_1^{n+1} |f_{j_k} - f_{j_{k-1}}| &\leqslant (|f_{j_n} - f_{j_0}| + l_n) + ((f_{j_{n+1}} - f_{j_n}) \dotplus r_1)\\ &\leqslant ((f_{j_n} - f_{j_0}) \dotplus r_2) \dotplus ((f_{j_{n+1}} - f_{j_n}) \dotplus r_1)\\ &= [(f_{j_{n+1}} - f_{j_n}) \dotplus (f_{j_n} - f_{j_0})] \dotplus r_3\\ &\leqslant |f_{j_{n+1}} - f_{j_0}| + r_4\end{aligned}$$

with $r_1, \dots, r_4 \in N_{q_B}$, since with (3) one can show $[\cdots] \leqslant |f_{j_{n+1}} - f_{j_0}| + p_{j_0, j_n} + p_{j_n, j_{n+1}}$.

But then $q_B\left(\sum_1^n |f_{j_k} - f_{j_{k-1}}|\right) \leqslant q_B(|f_{j_n} - f_{j_0}|) \leqslant t_0 < \infty$ for $n \in \mathbb{N}$. With Lemma 7 one gets the contradiction $q_B(|f_{j_k} - f_{j_{k-1}}|) \to 0$.

Theorem 4 gives therefore $q_B(|f_j - k|) \to 0$, then $I^{q_B}(f_j) \to I^{q_B}(k)$. □

LEMMA 9. *If $0 \in |B| \subset B \subset \overline{\mathbb{R}}^X$, q is an integral metric on X, $\tilde{q} := q_B$ and $B \subset \widetilde{B} \subset B^q$, then $\tilde{q}_B = \tilde{q}_{\widetilde{B}} = \tilde{q}_B \leqslant q$ and all are integral metrics, $B^q \subset B^{q_{\widetilde{B}}} \subset \widetilde{B}^{q_{\widetilde{B}}} \subset B^q = \widetilde{B}^q$.*

This follows from Lemma 5, (7), (2) and the definitions.

EXAMPLE 2: $X = \mathbb{N}$, $B = c_{00}$ as in Example 1, $q := 0$ on c_{00}, $q := 1$ else, q is even B-additive, $\widetilde{B} := B^{q_B} \cap \mathbb{R}^X$; here all five "$\subset$" and the two "$\leqslant$" in Lemma 9 are strict.

Especially, the extension process $I \mid B \to I^{q_B} \mid B^{q_B}$ is also iteration complete (see Corollary VI):

COROLLARY XIII. *If qBI holds and $\widetilde{B} = B^{q_B} \cap \mathbb{R}^X$, $\widetilde{I} := I^{q_B} \mid \widetilde{B}$, $\tilde{q} := q_B$, then $q\widetilde{B}\widetilde{I}$ and $\tilde{q}\widetilde{B}\widetilde{I}$ hold, $B^{q_B} = B^{\tilde{q}} = B^{\tilde{q}_{\widetilde{B}}} = \widetilde{B}^{\tilde{q}} = \widetilde{B}^{q_B} = \widetilde{B}^{\tilde{q}_{\widetilde{B}}}$, $I^{q_B} = \widetilde{I}^{q_B} = \widetilde{I}^{\tilde{q}_{\widetilde{B}}}$.*

DEFINITION 8: For $M \subset \overline{\mathbb{R}}^X$, $q : +\overline{\mathbb{R}}^X \to +\overline{\mathbb{R}}$: $C_\infty(q, M)$ means

$$q(h - h \wedge n) \to 0 \text{ as } n \to \infty, \text{ for each } h \in +M.$$

M is called *Stonean* if $h \wedge 1 \in M$ for each $h \in +M$.

With this, B^{q_B} is closed with respect to improper integration:

Corollary XIV. *Assume $qB0$ of (10), $C_\infty(q, B)$, q B-semiadditive and $k \in \overline{\mathbb{R}}^X$; if $k \cap n$ is q_B-B-measurable for each $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}}(|k \cap n|) < \infty$, then $k \in B^{q_B}$ and $q_B(|k - k \cap n|) \to 0$.*

Proof: If $h \in +B$, $k \cap (h \wedge n) = (k \cap n) \cap h \in B^q$ by Corollary XI and $q(|k \cap h - k \cap (h \wedge n)|) \leqslant 2q(h - h \wedge n) \leqslant q(|k| \cap h - |k| \cap (h \wedge n)) + q(|k| \cap (h \wedge n)) \leqslant 2q(h - h \wedge n) + q_B(|k| \wedge n) \leqslant 1 + \sup_m q_B(|k \cap m|)$ for some $n = n(h)$, Theorem 5 yields $k \in B^{q_B}$.

For $\varepsilon > 0$ there is $h \in B$ with $q_B(|k - h|) < \varepsilon$, then $q_B(|k - k \cap n|) \leqslant q_B(|k - h|) + q_B(|h - h \cap n|) + q_B(|h \cap n - k \cap n|) \leqslant \varepsilon + q_B(|h| - |h| \wedge n) + 2\varepsilon = 3\varepsilon + q(|h| - |h| \wedge n)$ by Lemma 5, so $\leqslant 4\varepsilon$ for $n >$ some n_ε. ∎

Especially we have $k \cap n \to k(q, B)$; for this "q B-semiadditive" and "$k \cap n$ q_B-B-measurable" are superfluous, provided $k \in B^{q_B}$.

"$k \cap n$ q_B-B-measurable" is also necessary for Stonean B, since then B^p is Stonean for an integral metric p, implying $f \cap t \in B^p$ if $f \in B^p$, $t \in +\mathbb{R}$.

Again all assumptions are independent and essential, Corollary XIV also becomes false if one replaces q_B by q.

For Stonean B the C_∞-continuity condition can be weakened in Corollary XIV:

Lemma 10. *If $M \subset +\overline{\mathbb{R}}^X$ with $h \wedge n$ and $h - h \wedge n \in M$ for $h \in M$, $n \in \mathbb{N}$, q is an M-semiadditive integral metric with $q(h \wedge n) \to q(h)$ as $n \to \infty$ if $h \in M$, then $q(h - h \wedge n) \to 0$ as $n \to \infty$ for $h \in M$ with $q(h) < \infty$.*

Proof: If $q(h - h \wedge n) > \varepsilon_0 > 0$ for $n \in \mathbb{N}$, there are $n_1 < n_2 < \cdots \in \mathbb{N}$ with $q(h \wedge n_{j+1} - h \wedge n_j) > \varepsilon_0$ for $j \in \mathbb{N}$, since $(h - h \wedge n) \wedge m = h \wedge (n + m) - h \wedge n$; with $kj := h \wedge n_{j+1} - h \wedge n_j \in M$ one has $q\left(\sum_1^m k_j\right) = q(h \wedge n_{m+1} - h \wedge n_1) \leqslant q(h)$, yielding the contradiction $q(k_j) \to 0$. ($q(h) < \infty$ is essential here.) ∎

3. Applications and examples

Definition 9: For $\emptyset \neq M \subset P \subset \overline{\mathbb{R}}^X$, $T\colon P \to \overline{\mathbb{R}}$:

$$T^M(k) := \operatorname{Inf}\{T(h)\colon k \leqslant h \in M\} \text{ for } k \in \overline{\mathbb{R}}^X, \text{ with } \inf \emptyset = \infty.$$

Lemma 11. *If $0 \in M = M \dotplus M \subset \overline{\mathbb{R}}^X$, $T\colon M \to \overline{\mathbb{R}}$ with*

$$T(h) \leqslant T(k) \dotplus (l) \text{ if } h \leqslant k \dotplus l \text{ and } h, k, l \in M, \tag{24}$$

then $T^M\colon \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is well defined, $= T$ on M and satisfies (24) with $h, k, l \in \overline{\mathbb{R}}^X$; $(T^M)^M = T^M$ on $\overline{\mathbb{R}}^X$. (Here $\dotplus$ cannot be replaced by $+$.).

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A, PROPER RIEMANN INTEGRALS.

(25)
$$\mu \mid \Omega \text{ means } \Omega \text{ is a semiring of sets from } X,\ \mu: \Omega \to [0, \infty) \text{ is additive on } \Omega,\ B_\Omega := \text{ step functions } S(\Omega, \mathbb{R}),\ I_\mu := \int ..d\mu \text{ on } B_\Omega\ [\mathbf{10}, \text{p.10–13}].$$

With $q = I_\mu^- := (I_\mu)^{B_\Omega}$ of Definition 9, Aumann [**1**, p.447–448] respectively Lemma 11 give

(26) $I_\mu^- B_\Omega I_\mu$ of (10) holds, I_μ^- is B_Ω-additive and positive-homogeneous;

(27)
$$R_e^1(\mu, \mathbb{R}) := (B_\Omega)^{I_\mu^-},\ I_\mu: R_e^1(\mu, \mathbb{R}) \to \mathbb{R} \text{ (abstract proper Riemann-}\mu\text{-integral) are well defined, } I_\mu := (I_\mu)^{I_\mu^-} \text{ of Theorem 1.}$$

$X = \mathbb{R}$, $\Omega = \{[a, b) : -\infty < a \leqslant b < \infty\}$, $\mu([a, b)) = b - a$ give the classical proper Riemann integrable functions, as for $\mathbb{R}^n$ [**10**, p.216].

B, ABSTRACT RIEMANN-μ-INTEGRATION. With (25), $I \mid B = I_\mu \mid B_\Omega$ and the B_Ω-additive integral metric $q = I_\mu^-$ of A, for

$$R_1(\mu, \overline{\mathbb{R}}) := B^{q_B},\ \int ..d\mu := (I_\mu)^{q_B} \tag{28}$$

all results of Sections 1-2 are applicable, B_Ω is Stonean $C_\infty(I_\mu^-, B_\Omega)$ holds, $R_e^1(\mu, \mathbb{R}) \subset R_1(\mu, \overline{\mathbb{R}}) \cap \mathbb{R}^X = R_1(\mu, \mathbb{R})$ of [**10**, p.70–144] $= L(X, \Omega, \mu, \mathbb{R})$ of Dunford-Schwartz [**8**, p.112] if $X \in \Omega$. $k_j \to k\ (q, B) \Leftrightarrow k_j \to k$ μ-locally, $M_\cap(q, B) = MR[\mu, \overline{\mathbb{R}}]$ [**11**, Lemma 9], [**10**, p.142]. Our convergence theorems are still generalisations of those for $R_1(\mu, \mathbb{R})$, Corollary XIV is new even then.

C, LOOMIS COMPLETIONS. Given $I \mid B$ with (9), and $q := I^- / + \overline{\mathbb{R}}^X$ one has:

$$R_1 := R_1(I \mid B) := B^{q_B},\quad J := I^{q_B} : R_1 \to \mathbb{R}, \text{ where } I^- := I^B \text{ of Definition 9 and } (I^-)_B \text{ of Definition 5, are } B\text{-additive.}$$

$R_1(B/I) \supset R_e^1 =$ two-sided completion R of Loomis [**13**, p.170], $R_1 =$ one-sided completion U of Loomis [**13**, p.178], $= I$-integrable functions of [**14**] = closure of B in $\overline{\mathbb{R}}^X$ with respect to the distance $d(k, l) := (I^-)_B(|k - l|)$. A and B are special cases. This extension process $I \mid B \to J \mid R_1(I \mid B)$ is by [**14**, p.115–119] iteration complete, that is $R_1(I \mid B) = R_1\left(J \mid \widetilde{B}\right)$ with $\widetilde{B} := R_1(I \mid B) \cap \mathbb{R}^X$.

D, FINITELY-ADDITIVE DANIELL EXTENSION. For $I \mid B$ with (9) it is introduced in [3], for $k \in \overline{\mathbb{R}}^X$,

$$(30) \qquad I^+(k) := \sup\{I(h)\colon h \leqslant k,\, h \in B\},$$
$$B^+ := \{k \in \overline{\mathbb{R}}^X\colon\ -\infty < I^+(k), \quad k = \sup\{h\colon k \geqslant h \in B\}\},$$
$$B_+ := \{f \in B^+\colon I^+(f+g) \leqslant I^+(f) + I^+(g) \text{ for } g \in B^+\},$$
$$B \subset B_{(+)} := \{f \in B_+\colon I^+(f) < \infty\} \subset B_+ \subset B^+,$$

(31) $\qquad \overline{I} := \left(I^+\right)^{B_+}$ (Definition 9), a B-additive integral metric q on $+\overline{\mathbb{R}}^X$, with qBI,

(32) $\qquad \overline{B} := B^q =$ summable functions B_0 of [3], $I^q = \overline{I}$ on $\overline{B}$;

the convergence theorems of Section 1 are those of [11]; by [14, Theorem 6.4] one has

$$(33) \qquad R_e^1(I \mid B) \subset \overline{B}, \quad R_1(I \mid B) \subset \overline{B} + \{f \in R_1(I \mid B)\colon R_1 - I(|f|) = 0\}.$$

In the classical case $R_1 = L^1 = L_1 = \overline{B}$ by (43)–(44), but in general Section 2 is not applicable ([11, Example 2], $R_1 \not\subset \overline{B}$, $\overline{B} \not\subset R_1$.

E, LOCALISATION OF THE DANIELL-ANALOGUE.

$$(34) \qquad L := L(I \mid B) := B^{q_B} \qquad J := I^{q_B}\colon L \to \mathbb{R}, \qquad q = \overline{I} \mid +\overline{\mathbb{R}}^X$$

with B, I, $\overline{I}$ of D allow an application of all results of Sections 1–2; (18) and $\overline{I} \leqslant I^B$ imply

$$(35) \qquad \overline{B} \subset L, \quad R_e^1 \subset R_1(I \mid B) \subset L, \text{ with coinciding integrals,}$$

all $\subset$ are strict [11, Exercise 2.3], and we have a generalisation of [3]. With $C_\infty(B, I)$ of [11] one can show $1 \in \overline{B} \Rightarrow L = \overline{B}$; $1 \in L \Rightarrow L = \overline{B} + N_{\overline{I}_B}$; with C_∞: $B_{(+)} \subset R_1 \Leftrightarrow R_1 \subset \overline{B} \Leftrightarrow R_1 = L$. See also (43)–(44).

One has always the iteration closures of Theorem 1 and Corollary XII, where $\widetilde{B} := L(I/B) \cap \mathbb{R}^X$ and $\widetilde{I} := J/\widetilde{B}$.

Additional properties of $L(B/I)$ will be treated elsewhere, see also [7].

F, L^p-SPACES. For $q := \overline{\mathbb{R}}^X \to +\overline{\mathbb{R}}$, $0 < p$ real $< \infty$, with $k^p(t) := (k(t))^p$, $0^p := 0$, $\infty^p := \infty$

$$(36) \qquad q_p(k) := \left(q(k^p)\right)^{1/p} \text{ if } p > 1, \quad := q(k^p) \text{ if } 0 < p \leqslant 1, \quad k \in +\overline{\mathbb{R}}^X.$$

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Lemma 12. *If $q\colon \overline{\mathbb{R}}^X \to +\overline{\mathbb{R}}$ is an integral metric with $q(2k) = 2q(k)$, $0 < p < \infty$, then q_p is also an integral metric on $+\overline{\mathbb{R}}^X$, positive-homogeneous if $p \geqslant 1$.*

Proof: $2q(k) \leqslant q(2k)$ implies $q(tk) = tq(k)$, $0 < t < \infty$; $|s+t|^p \leqslant s^p + t^p$ if $p \leqslant 1$. If $p > 1$, q_p satisfies Minkowski's inequality for finitely-valued k, l by Bourbaki [5, p.12]; if $k, l \in +\overline{\mathbb{R}}^X$ with finite q_p, $(q_p(k+l))^p \leqslant q(2^p(k^p + l^p)) < \infty$ and $mq_p(k_\infty) = q_p(mk_\infty) \leqslant q_p(k) < \infty$, $q_p(k_\infty) = 0$ (see Corollary II); therefore $q_p(k+l) \leqslant q_p((k+l)_e) + 0 \leqslant q_p(k_e + l_e) \leqslant q_p(k_e) + q_p(l_e) \leqslant q_p(k) + q_p(l)$. ▯

So for integral norms := positive-homogeneous integral metrics q, Section 1-2 (and Hölder) hold for B^{q_p} and $B^{q_p, B}$, especially for

(37)
$$R_p(I \mid B) := B^r, \quad r = \left(\left(I^B\right)_B\right)_p, \quad L_p(I \mid B) := B^s, \quad s := \left(\overline{I}_B\right)_p, \quad 0 < p < \infty.$$

If q is a B-additive integral norm, $0 < p < \infty$, $(+B)^p = +B$, then q_p is B-semiadditive (for example $B = B_\Omega$ or $C_0(X, \mathbb{R})$); if additionally $q(h - h \wedge n) \to 0$ and $q(h \wedge 1/h) \to 0$, $n \to \infty$, $h \in +B$, then $M_\cap(q_q, B) = M_\cap(q, B)$, $k_j \to k$ $(q_p, B) \Leftrightarrow k_j \to k$ (q, B),

$$B^{q_p} = \{f \in M_\cap(q, B)\colon q_p(|f|) < \infty\}. \tag{38}$$

Example: $\|\ \|_p = (\|\ \|_1)_p$ is a $+\overline{\mathbb{R}}^X$-semiadditive integral metric, $0 < p < \infty$, $\|k\|_1 = \sum_{x \in X} k(x)$. One can also show that $q \circ \beta_p$ is an integral metric on $+\overline{\mathbb{R}}^X$ for $0 < p < \infty$ if q is, $\beta_p(k) := (k^p/(1 + k^p))^{1/p}$, though β_1 is not homogeneous.

G, Schäfke completions. If one has $qB0$ and q σ-subadditive ($k, k_n \in +\overline{\mathbb{R}}^X$, $k \leqslant \sum_1^\infty k_n \Rightarrow q(k) \leqslant \sum_1^\infty q(k_n)$ starke Integralnorm in [16]), then B^q of Theorem 1 has by Schäfke [16, 19] additional properties: For $P \subset X$, $q(1P) = 0 \Leftrightarrow q(\infty P) = 0 \Leftrightarrow$ there is $k \in N_q$ with $P \subset \{x \in X\colon k(x) \neq 0\} \Leftrightarrow P$ q-nullsets; P_n q-nullsets $\Rightarrow \bigcup_1^\infty P_n$ q-nullset; (B^q, q) is complete (Stone's axiom is not needed); q_B is also σ-subadditive, with Corollary II one gets

$$B^{q_B} = B^q + N_{q_B} := \{g + l : g \in B^q, l \in N_{q_B}\}. \tag{39}$$

q is called weakly B-semiadditive [16] if $h_n, h \in +B$ with $\sum_1^\infty h_n \leqslant h$ and $q(h) < \infty$ implies $q(h_n) \to 0$.

Lemma 13. *If $qB0$ holds, $q \neq \infty$ on $+B$, q is σ-subaditive and weakly B-semiadditive, $f_n \in M_\cap(q, B)$, $k \in \overline{\mathbb{R}}^X$ and $f_n \to k$ q_B-almost everywhere, then*

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$k \in M_\cap(q, B)$ if additionally the $f_B \in B^{q_B}$ and $k - f_n \to 0$ q_B-almost everywhere, then $f_n \to k$ (q, B).

PROOF: With Lemma 4, Corollary II and the above one can assume $0 \leqslant f_n$ and $f_n \to k$, respectively $k - f_n \to 0$ on X. If $h \in +B$, $l_n := f_n \wedge h$, then $p_n := h - \vee_n^\infty l_j \in B^q$ and $h - k \wedge h = \vee_1^\infty p + n \in B^q$ by Hilfssatz 1.4.4 of Schäfke [**16**], so $k \in M_\cap(q, B)$. For the second assertion, $\beta_n := |k - f_n| \wedge h \in B^q$ by Lemma 4 and Theorem 2, $\beta_n \to 0$; [**16**], Satz 1.4.6 gives $q(\beta_n) \to 0$. (See also Liubicich [**12**]). □

If only $f_n \to k$ q_B-almost everywhere, but $|f_n| \leqslant g_0 \in B^{q_B}$ for $n \in \mathbb{N}$, or if $g_0 \leqslant f_n \leqslant f_{n+1}$, $\sup q_B(f_n - g_0) < \infty$ and q is B-semiadditive, one gets immediately $k - f_n \to 0$ q_B-almost everywhere, then the usual L^1-convergence theorems follow from those of Sections 1 and 2. (For nets they and Lemma 17 are false.)

H, DANIELL INTEGRALS. If $I \mid B$ with (9) is σ-continuous, that is $I(h_n) \to 0$ whenever $h_n \in +B$ $h_n \geqslant h_{n+1} \to 0$ pointwise on X. Then Aumann [**1**, p.448–449]

$$I^\sigma(k) := \inf\Big\{\sum_1^\infty I(h_n) \colon k \leqslant \sum_1^\infty h_n,\ h_n \in +B\Big\}, \quad k \in +\overline{\mathbb{R}}^X, \tag{40}$$

defined a σ-subadditive B-additive integral norm with $q(k_n) \to q(k)$ for any $k_n, k \in +\overline{\mathbb{R}}^X$ with $k_n \uparrow k$, $q = I^\sigma$ or I_B^σ, (= upper S-norm of Bichteler [**2**]); so all results of Sections 1-3 and G hold for $I_B^\sigma := (I^\sigma)_B$)

$$L^1 := B^1, \quad L_1 := B^{I_B^\sigma}, \quad I := I_D := I^{I_B^\sigma}, \quad \|f\|_1 := I_D(\|f\|) \tag{41}$$

$I_D \mid L^1$ = usual Daniell extension of $I \mid B$, [**6**, **1**, **15**, **9**].

As an analogue to $\overline{B} = L^1 \cap \overline{B} + \overline{B}_N$ of [**11**] we can only show

$$\begin{aligned} &\mu\ \sigma\text{-additive on } \delta\text{-ring } \Omega \Rightarrow L_1 = R_1 \Rightarrow L^1 = \overline{B} \cap R_1 + L_{1,N} \\ &\Rightarrow L^1 \subset \overline{B} + L_{1,N} \Rightarrow L = \overline{B} + L_N \Rightarrow L = L^1 \cap L + L_N, \end{aligned} \tag{42}$$

with $M_N :=$ {null functions of M}. So at least for measure spaces (X, Ω, μ) and $I \mid B = I_\mu \mid B_\mu$ one has

$$L^1 \subset L = \overline{B} \cap L^1 + L_N = L \cap L^1 + L_N, \tag{43}$$

even then $L_1 \subsetneq \overline{B}$ is possible ([**11**]).

As an analogue to the first $\Rightarrow$ of (40) one has $L_1 = R_1$ for σ-continuous $I \mid B$ satisfying $\sum_1^\infty h_n \in B$ whenever $h_n \in +B$ with $\sum_1^\infty h_n \leqslant h \in B$.

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I, Bourbaki integrals. If $I \mid B$ with (9) is τ-continuous ($I(h_j) \to 0$ if (h_j) net from B with $h_j \downarrow 0$) then $B^+ = B_+$, $I^+ = \overline{I}$ on B^+, $I^\tau := (I^+)^{B^+} = \overline{I} \leqslant I^\sigma$ defines a B-additive integral norm on $+\overline{\mathbb{R}}^X$ with $I^\tau BI$ [**3**, **11**], so $L^\tau := B^{I^\tau}$ and $L_\tau := B^{I^\tau_B}$ are with their integrals well defined, L^τ = Bourbaki-integrable functions (Pfeffer [**15**], Floret [**9**, p.338]; see also $\| \ \|_4$ of Schäfke [**16**]) one has

$$L^1 \subset L^\tau = \overline{B} = L^1 + \overline{B}_N \subset L_\tau = L = L^1 + L_N = L_1 + L_N. \tag{44}$$

Examples: $B = C_0(X, \mathbb{R})$, X Hausdorff, or $I_\mu \mid B_\Omega$, μ = Lebesgue measure, Ω = intervals $\subset X$ open $\subset \mathbb{R}^n$ (here $L^1 = L$ by [**11**]); even then the integral is not τ-continuous on L^1. There are $I_\mu \mid B_\Omega$ with $1X \in B^+_\Omega$ and all "$\subset$" in (44) are strict.

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