GENERATING VARIETIES OF
TOPOLOGICAL GROUPS

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(Received 31st December 1971)

1. Introduction

Recently several papers on varieties of topological groups have appeared. In this note we investigate the question: if Ω is a class of topological groups, what topological groups are in the variety $V(\Omega)$ generated by Ω; that is, what topological groups can be " manufactured " from Ω using repeatedly the operations of taking subgroups, quotient groups and arbitrary cartesian products? We seek a general theorem which will be useful for investigating $V(\Omega)$ for well-known classes Ω.

Prior to this paper the only result (12) along these lines was the following. If T is the circle group, then the locally compact Hausdorff group G is in $V(T)$ if and only if it is compact and abelian.

Looking to the well-established theory of varieties of groups for " inspiration ", we prove a theorem on generating varieties which yields: If Ω is a class of compact groups, then any locally compact Hausdorff group in $V(\Omega)$ is compact. Despite this the theorem is of no great interest to us except in that it allows us to prove our main result, Theorem 2, which is peculiar to the theory of *topological* varieties. (Analogues for varieties of topological Lie algebras and varieties of linear topological spaces appear in (2) and (4), respectively.)

Our main theorem is crucial to the investigations in (13), (14), (15) and (18). We use it here to answer an open question of (17).

2. Definitions and results

If Ω is a class of (not necessarily Hausdorff) topological groups, then S denotes the class of all topological groups isomorphic to subgroups of members of Ω. Similarly we define the operators $\bar{S}$, Q, $\bar{Q}$, C, C_0 and D where they respectively denote closed subgroup, quotient group, separated quotient group, arbitrary cartesian product, countable product and finite product.

A non-empty class Ω of topological groups is said to be a *variety* if $Q\Omega \subset \Omega$, $S\Omega \subset \Omega$ and $C\Omega \subset \Omega$. The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and is denoted by $V(\Omega)$. (For an excellent treatment of varieties of *groups* see (19).)

If V is a variety, X is a topological space and F is a member of V, then F is said to be a *free topological group of V on X*, denoted by $F(X, V)$, if it has the

properties:

    (*a*) X is a subspace of F,

    (*b*) X generates F algebraically,

    (*c*) for any continuous mapping γ of X into a member of H of V, there exists a continuous homomorphism Γ of F into H such that $\Gamma \mid_X = \gamma$.

In (**11**) it was shown that $F(X, V)$ exists if and only if there is a member of V which has X as a subspace. If $F(X, V)$ exists, then it is unique (up to isomorphism).

Theorem 1. *For any non-empty class Ω of topological groups $V(\Omega) = QSC\Omega$.*

Proof. Firstly we observe the following easily verified statements: $SS\Omega = S\Omega$; $CC\Omega = C\Omega$; $QQ\Omega = Q\Omega$; $CS\Omega \subset SC\Omega$; $CQ\Omega \subset QC\Omega$; $SQ\Omega \subset QS\Omega$. Using these, we see that $QSC\Omega$ is a variety by noting that $Q[QSC\Omega] = QSC\Omega$, $S[QSC\Omega] \subset QSSC\Omega = QSC\Omega$ and $C[QSC\Omega] \subset QCSC\Omega \subset QSCC\Omega = QSC\Omega$.

Note that any variety of groups can be considered as a variety of topological groups simply by endowing each group with its indiscrete topology. Of course, as it applies to varieties of groups the above theorem is well known and our proof is a standard one (see, for example, (**7**), section 23). However, the usual proof of that case via free groups ((**19**), 15.23) does not extend. The difficulty concerns the following question:

If Ω is a class of topological groups and X is a Hausdorff space such that $F(X, V(\Omega))$ exists, does $F(X, V(\Omega))$ necessarily belong to $SC\Omega$?

The example below, provided by Gerald Edgar and Andrew Gleason, answers this question in the negative.

Example. Let Ω be the class of all totally disconnected groups and X the real line with its usual topology. Then $SC\Omega = \Omega$ and thus $F(X, V(\Omega))$, if it exists, is *not* in $SC\Omega$.

To see that $F(X, V(\Omega))$ exists, we show that $R \in V(\Omega)$, where R is the additive group of reals with its usual topology (see Corollary 2.10 of (**11**)). In fact, we show that $R \in Q\Omega$.

In the Banach space l^2 of real square-summable sequences, let G be the group of sequences which have rational coordinates. Then $G \in \Omega$. Choose any $\alpha \in l^2$, with all components non-zero. It is easily seen that the map $f: G \to R$, given by $f(a) = (a, \alpha)$ where $(\,,\,)$ is the inner product in l^2, is an open continuous homomorphism of G on to R. Thus $R \in Q\{G\} \subset Q\Omega$, as required.

Before presenting our main theorem we state a lemma, the proof of which is omitted as it is straightforward.

Lemma. *Let G be a topological group with subgroups A and B such that B is a closed normal subgroup of A. Let $\bar{A}$ and $\bar{B}$ be the closures in G of A and B respectively. Then the quotient group A/B is isomorphic to a subgroup of $\bar{A}/\bar{B}$.*

Theorem 2. *If Ω is any class of topological groups and G is a Hausdorff group in $V(\Omega)$ then $G \in SC\bar{Q}\bar{S}D\Omega$.*

Proof. We need only show that $G \in SCQSD\Omega$. (To see this reduction use the lemma together with relevant parts of the proof of Theorem 1 and the easily verified fact that any Hausdorff group in $SCQ\Omega$ is in $SC\bar{Q}\Omega$.)

By Theorem 1, $G \in \bar{Q}SC\Omega$. Therefore there exist $K_i \in \Omega$, $i \in I$ such that $K = \pi_{i \in I}K_i$ has subgroups A and B with B a closed normal subgroup of A and G isomorphic to the quotient A/B. Let Σ be the family of all finite subsets of I. For $\sigma \in \Sigma$ let $K_\sigma = \pi_{i \in \sigma}K_i$; let p_σ be the projection mapping of K onto K_σ; and let h_σ be the canonical mapping of $p_\sigma(A)$ on to the quotient $p_\sigma(A)/p_\sigma(B)$. Define a map $f: A \to \pi_{\sigma \in \Sigma}h_\sigma p_\sigma(A)$ by $f(a) = \pi_{\sigma \in \Sigma}(h_\sigma p_\sigma(a))$ for all $a \in A$; that is, f is the product of $h_\sigma p_\sigma$, $\sigma \in \Sigma$, and hence is a continuous homomorphism. For some $\tau \in \Sigma$ let $O = \pi_{i \in I}O_i$ where each O_i is a neighbourhood of the identity in K_i and in fact $O_i = K_i$ for $i \notin \tau$. We prove f is an open mapping of A onto $f(A)$ by verifying that

$$f(O \cap A) \supset (\pi_{\sigma \in \Sigma}D_\sigma) \cap f(A)$$

where $D_\sigma = h_\sigma p_\sigma(A)$ for $\sigma \neq \tau$ and $D_\tau = h_\tau(p_\tau(O) \cap p_\tau(A))$. If

$$x \in (\pi_{\sigma \in \Sigma}D_\sigma) \cap f(A)$$

then there exists $a \in A$ such that $x = f(a) \in \pi_{\sigma \in \Sigma}D_\sigma$, and so there exists $c \in O$ such that $h_\tau p_\tau(c) = h_\tau p_\tau(a)$; that is, $p_\tau(B)p_\tau(c) = p_\tau(B)p_\tau(a)$. So there exists $b \in B$ such that $p_\tau(c) = p_\tau(b)p_\tau(a) = p_\tau(ba)$. But this means that $p_\tau(ba) \in p_\tau(O)$ and therefore that $ba \in O$. This shows that $x \in f(O \cap A)$ since clearly B is in the kernel of f and so $x = f(a) = f(ba)$.

We now show that the kernel of f is in B. If $y \notin B$ then, since B is closed in A, there exists a neighbourhood $N = \pi_{i \in I}N_i$ of y in $\pi_{i \in I}K_i$ with $N \cap B = \varnothing$ and $N_i = K_i$ for all $i \notin \gamma$, where γ is suitably chosen from Σ.

Clearly $p_\gamma(N) \cap p_\gamma(B) = \varnothing$, and since $p_\gamma(y) \in p_\gamma(N)$ it follows that $p_\gamma(y) \notin p_\gamma(B)$ Since $p_\gamma(B)$ is the kernel of h_γ, this shows that $f(y) = (h_\sigma p_\sigma(y))_{\sigma \in \Sigma}$ is not the identity.

Thus we have that f is an open continuous homomorphism with kernel B, and so A/B is isomorphic to a subgroup of $\pi_{\sigma \in \Sigma}h_\sigma p_\sigma(A)$; that is, $G \in SCQSD\Omega$, as we set out to show.

Remark. In the abelian case the commutativity of the operators Q and S yields: If Ω is any class of abelian topological groups and G is a Hausdorff group in $V(\Omega)$ then $G \in SC\bar{Q}D\Omega$. We do not know whether the " abelian " condition is necessary or not.

Corollary 1. *If Ω is a class of locally compact Hausdorff groups or a class of complete metrizable groups then every Hausdorff group G in $V(\Omega)$ has a completion $\hat{G}$ and $\hat{G}$ is in $V(\Omega)$.*

Proof. If Ω consists either of locally compact Hausdorff groups or of complete metrizable groups then $\bar{Q}\bar{S}D\Omega$ contains only groups of the same kind.

Since the product of complete groups is complete it now follows from Theorem 2 that G is a subgroup of a complete group in $V(\Omega)$.

Remarks. We note that not every variety is closed under the formation of completions—for example, the variety generated by the additive group of rationals with its usual topology.

It is known ((1), Exercise 16c) that not every topological group has a completion. It therefore follows that the variety of all topological groups is not generated by the class of all locally compact Hausdorff groups. Our next corollary shows that the variety of all abelian topological groups is not generated by *its* locally compact Hausdorff groups. (Further information is obtained in (16).)

Corollary 2. *If Ω is any class of locally compact abelian groups, then any Hausdorff group in $V(\Omega)$ is maximally almost periodic.*

Proof. We abbreviate maximally almost periodic to MAP. By Theorem 2, $G \in SC\bar{Q}\bar{S}D\Omega$. Now $\bar{Q}\bar{S}D\Omega$ contains only locally compact Hausdorff abelian groups and these are MAP. To see that G is MAP we only have to note that products and subgroups of MAP groups are MAP.

Remark. It is noted in (18) that a variety generated by locally compact MAP groups can contain (locally compact) Hausdorff non-MAP groups.

Our next corollary could have been derived easily from Theorem 1.

Corollary 3. *If Ω is any class of pseudocompact Hausdorff groups, then every complete Hausdorff (in particular, locally compact Hausdorff) group G in $V(\Omega)$ is compact.*

Proof. It is shown in (3) and (20) that every member of Ω can be embedded in a compact Hausdorff group. Thus $V(\Omega) \subset V(\Gamma)$, where Γ is the class of all compact Hausdorff groups. Since G is complete, Theorem 2 implies that $G \in \bar{S}C\bar{Q}\bar{S}D\Gamma = \Gamma$.

Remark. It was shown in (6) that if X is compact Hausdorff and V is the variety of all (all abelian) topological groups, then $F(X, V)$ exists and is complete Hausdorff. Corollary 3 shows that if Ω is any class of connected compact Hausdorff groups then $F(X, V(\Omega))$ is *not* complete. (It is shown in (17) that $F(X, V(\Omega))$ exists and is Hausdorff. Therefore by Corollary 3, if it is complete then it is compact. However, by Theorem 2.8 of (11) and Theorem 2.5 (iv) of (2) $F(X, V(\Omega))$ is algebraically a free group or a free abelian group and consequently, (5), does not admit a compact Hausdorff topology.)

Corollary 4. *If Ω is a class of locally compact Hausdorff groups, then the following conditions are equivalent:*

   (i) $R \in S\Omega$
  (ii) $R \in V(\Omega)$
 (iii) *$V(\Omega)$ contains a connected non-compact locally compact Hausdorff group.*

Proof. Clearly (i) implies (ii) and (ii) implies (iii). Now suppose (iii) is true and (i) is false. Then there exists a connected non-compact locally compact Hausdorff group G in $V(\Omega)$. According to §4.13 of (**10**), G has R as a subgroup and so $R \in V(\Omega)$.

Since $R \notin S\Omega$, the component of the identity of each member of Ω is compact (again using §4.13 of (**10**)). Clearly then this is also true for each member of $\bar{S}D\Omega$. Indeed by 7.12 of (9) it is true for each member of $\bar{Q}\bar{S}D\Omega$.

By Theorem 2, $R \in V(\Omega)$ implies $R \in SC\bar{Q}\bar{S}D\Omega$; that is, R is a subgroup of a product $\pi_{i \in I}G_i$, where each $G_i \in \bar{Q}\bar{S}D\Omega$ and hence has compact component K_i of the identity. Since R is connected, its projection into G_i is contained in K_i. Hence R is a subgroup of the compact group $\pi_{i \in I}C_i$, which is impossible. Thus (iii) implies (i).

Remarks. It was proved in (**12**) that $R \notin V(T)$, $R \notin V(Z)$ and $Z \notin V(T)$, where T and Z respectively denote the circle group and the group of integers with the usual topology. We now see from Corollary 4 that $R \notin V\{Z, T\}$. Indeed, by Corollary 1, the additive group of rationals with its usual topology is not in $V\{Z, T\}$.

We can use this to answer an " open question " in (**17**). First we need a definition: A variety V is said to be a *β-variety* if for every compact Hausdorff space X, $F(X, V)$ exists and is Hausdorff. It is shown in (**17**) that any variety generated by arcwise connected Hausdorff groups is a $β$-variety. The " open question " asks if the converse is true. We answer this in the negative by proving that the $β$-variety $V\{Z, T\}$ is not generated by its connected Hausdorff groups. Indeed the corollary below says somewhat more.

Corollary 5. *Let Ω be a class of locally compact abelian groups. Then $V(\Omega)$ is generated by its connected Hausdorff groups if and only if $V(\Omega)$ equals $V(R)$ or $V(T)$.*

Proof. If G is any connected Hausdorff group in $V(\Omega)$, then $G \in SC\bar{Q}\bar{S}D\Omega$; that is, $G \in SC\{H_\alpha : \alpha \in I\}$, where each $H_\alpha \in \bar{S}\bar{Q}\bar{S}D\Omega$ and the projection of G is dense in H_α. Thus H_α is connected and locally compact. Hence $V(\Omega)$ is generated by its connected locally compact Hausdorff groups. Since every such group is a projective limit of Lie groups, we have that $V(\Omega)$ is generated by connected abelian Lie groups. These groups are of the form $R^a \times T^b$, where a and b are non-negative integers. So $V(\Omega)$ equals $V(R)$ or $V(T)$, as required.

Remark. Let Ω be a class of locally compact groups. Then, as above, $V(\Omega)$ is generated by its connected Hausdorff groups if and only if it is generated by its connected Lie groups. Corollary 7 then says that every discrete group in $V(\Omega)$ is countable. Thus if Ω is the class of all locally compact groups, $V(\Omega)$ is not generated by its connected Hausdorff groups. So we have another proof that $V(\Omega)$ is a proper subvariety of the variety of all topological groups as it was shown in (**8**) that the latter variety *is* generated by connected Hausdorff groups.

Corollary 6. *If Ω is any class of topological groups and G is a metrizable group in $V(\Omega)$ then $G \in SC_0 \bar{Q} \bar{S} D\Omega$.*

Proof. By Theorem 2 we need only show that if a metrizable group G is a subgroup of a product $\pi_{i \in I} H_i$ of topological groups H_i, then there exists a countable subset J of I such that G is isomorphic to a subgroup of $\pi_{i \in J} H_i$.

Now G has a base of neighbourhoods $N_j, j = 1, 2, \ldots$ of the identity element such that $N_j = (\pi_{i \in I} D_{ij}) \cap G$, where each D_{ij} is a neighbourhood of the identity in H_i and there is a finite subset I_j of I with $D_{ij} = H_i$ for all $i \notin I_j$. Put $J = \bigcup_{j=1}^{\infty} I_j$ and let f be the natural projection of $\pi_{i \in I} H_i$ on to $\pi_{i \in J} H_i$. It is now a routine matter to verify that $f(G)$ is isomorphic to G.

As an immediate consequence we have.

Corollary 7. *If Ω is a class of separable metric groups and G is a metrizable group in $V(\Omega)$ then G is separable.*

Acknowledgement

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