



MONOID ACTIONS AND ULTRAFILTER METHODS IN RAMSEY THEORY

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Abstract

First, we prove a theorem on dynamics of actions of monoids by endomorphisms of semigroups. Second, we introduce algebraic structures suitable for formalizing infinitary Ramsey statements and prove a theorem that such statements are implied by the existence of appropriate homomorphisms between the algebraic structures. We make a connection between the two themes above, which allows us to prove some general Ramsey theorems for sequences. We give a new proof of the Furstenberg–Katznelson Ramsey theorem; in fact, we obtain a version of this theorem that is stronger than the original one. We answer in the negative a question of Lupini on possible extensions of Gowers’ Ramsey theorem.

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1. Introduction

The main point of the paper is studying actions of monoids and establishing a relationship between monoid actions and Ramsey theory. Further, we make a connection with partial orders that have earlier proved important in set theoretic considerations.

In Section 2, we study the dynamics of actions of monoids by continuous endomorphisms on compact right topological semigroups. We outline now the notions relevant to this study and its outcome: (1) a partial order $\mathbb{Y}(M)$; (2) a class of monoids; and (3) the theorem on dynamics of monoid actions.

(1) We associate with each monoid M a partial order $\mathbb{Y}(M)$ on which M acts in an order preserving manner. We define first the order $\mathbb{X}(M)$ consisting of all principal right ideals in M , that is, sets of the form aM for $a \in M$, with the order relation $\leq_{\mathbb{X}(M)}$ being inclusion. This order is considered in the representation theory of monoids as in [14]. The monoid M acts on $\mathbb{X}(M)$ by left translations. We then let $\mathbb{Y}(M)$ consist of all nonempty linearly ordered by $\leq_{\mathbb{X}(M)}$ subsets of $\mathbb{X}(M)$. We order $\mathbb{Y}(M)$ by end-extension, that is, we let $x \leq_{\mathbb{Y}(M)} y$ if x is included in y and all elements of $y \setminus x$ are larger with respect to $\leq_{\mathbb{X}(M)}$ than all elements of x . The construction of the partial order $\mathbb{Y}(M)$ from the partial order $\mathbb{X}(M)$ is a special case of a set theoretic construction going back to Kurepa [9]. An order preserving action of the monoid M on $\mathbb{Y}(M)$ is induced in the natural way from its action on $\mathbb{X}(M)$.

(2) We introduce a class of monoids we call almost R-trivial, which contains the well known class of R-trivial monoids (see [14]) and all the monoids of interest

to us. In a monoid M , by the **R-class** of $a \in M$ we understand, as in [14], the equivalence class of a with respect to the equivalence relation that makes two elements equivalent if the principal right ideals generated by the two elements coincide, that is, b_1 and b_2 are equivalent if $b_1M = b_2M$. We call a monoid M **almost R-trivial** if for each element b whose R -class has strictly more than one element we have $ab = b$ for each $a \in M$. (A monoid is R -trivial if the R -class of each element contains only that element.) In Section 2.3, we provide the relevant examples of almost R -trivial monoids.

(3) In Theorem 2.4, which is the main theorem of Section 2, we show that each action of a finite almost R -trivial monoid by continuous endomorphisms on a compact right topological semigroup contains, in a precise sense, the action of M on $\mathbb{Y}(M)$; in fact, it contains a natural action of M by endomorphisms on the semigroup $\langle \mathbb{Y}(M) \rangle$ generated canonically by $\mathbb{Y}(M)$, Corollary 2.7. This result was inspired by Ramsey theoretic considerations, but it may also be of independent dynamical interest.

In Section 3, we introduce new algebraic structures, we call **function arrays**, that are appropriate for formalizing various Ramsey statements concerning sequences. We isolate the notions of **basic sequence** and **tame coloring**. In Theorem 3.1, the main theorem of this section, we show that finding a basic sequence on which a given coloring is tame follows from the existence of an appropriate homomorphism. This theorem reduces proving a Ramsey statement to establishing an algebraic property. We introduce a natural notion of tensor product of the algebraic structures studied in this section, which makes it possible to strengthen the conclusion of Theorem 3.1.

In Section 4, we connect the previous two sections with each other and explore Ramsey theoretic issues. In Corollary 4.1, we show that the main result of Section 2 yields a homomorphism required for the main result of Section 3. This corollary has various Ramsey theoretic consequences. For example, we introduce a notion of **Ramsey monoid** and prove that, among finite almost R -trivial monoids M , being Ramsey is equivalent to linearity of the order $\mathbb{X}(M)$. We use this result to show that an extension of Gowers' Ramsey theorem [4] inquired for by Lupini [10] is false. As other consequences, we obtain some earlier Ramsey results by associating with each of them a finite almost R -trivial monoid. For example, we show the Furstenberg–Katznelson Ramsey theorem for located words, which is stronger than the original version of the theorem from [3]. Our proof is also different from the one in [3].

We state here one Ramsey theoretic result from Section 4, which has Furstenberg–Katznelson's and Gowers' theorems, [3, 4], as special instances; see Section 4.3. Let M be a monoid. By a **located word over M** we understand a function from a finite nonempty subset of \mathbb{N} to M . For two such words w_1 and

w_2 , we write $w_1 < w_2$ if the largest element of the domain of w_1 is smaller than the smallest element of the domain of w_2 . In such a case, we write $w_1 w_2$ for the located word that is the function whose graph is the union of the graphs of w_1 and w_2 . For a located word w and $a \in M$, we write $a(w)$ for the located word that results from multiplying on the left each value of w by a . Given a finite coloring of all located words, we are interested in producing a sequence $w_0 < w_1 < \dots$ of located words, for which we control the color of

$$a_0(w_{n_0}) \cdots a_k(w_{n_k}),$$

for arbitrary $a_0, \dots, a_k \in M$ and $n_0 < \dots < n_k$. The control over the color is exerted using the partial order $\mathbb{Y}(M)$ introduced above. With each partial order (P, \leq_P) , one naturally associates a semigroup $\langle P \rangle$, with its binary operation denoted by \vee , that is the semigroup generated freely by the elements of P subject to the relations

$$p \vee q = q \vee p = q \quad \text{if } p \leq_P q. \tag{1.1}$$

We consider the semigroup $\langle \mathbb{Y}(M) \rangle$ produced from the partial order $\mathbb{Y}(M)$ in this manner. We now have the following statement, which is proved as Corollary 4.3.

Let M be almost R -trivial and finite. Fix a finite subset F of the semigroup $\langle \mathbb{Y}(M) \rangle$ and a maximal element \mathbf{y} of the partial order $\mathbb{Y}(M)$. For each coloring with finitely many colors of all located words over M , there exists a sequence

$$w_0 < w_1 < w_2 < \dots$$

of located words such that the color of

$$a_0(w_{n_0}) \cdots a_k(w_{n_k}),$$

for $a_0, \dots, a_k \in M$ and $n_0 < \dots < n_k$, depends only on the element

$$a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y})$$

of $\langle \mathbb{Y}(M) \rangle$ provided that $a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y}) \in F$.

One can view $a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y})$ as the ‘type’ of $a_0(w_{n_0}) \cdots a_k(w_{n_k})$ and the theorem as asserting that the color of $a_0(w_{n_0}) \cdots a_k(w_{n_k})$ depends only on its type. In general, the element $a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y})$ contains much less information than the located word $a_0(w_{n_0}) \cdots a_k(w_{n_k})$, due partly to the disappearance of w_{n_0}, \dots, w_{n_k} and partly to the influence of relations (1.1).

We comment now on our view of the place of the present work within Ramsey theory. A large portion of Ramsey Theory can be parametrized by a triple (a, b, c) , where a, b, c are natural numbers or ∞ and $a \leq b \leq c$. (We exclude here, for example, a very important part of Ramsey theory called structural Ramsey

theory, for which a general approach is advanced in [6].) The simplest Ramsey theorems are those associated with $a \leq b < \infty = c$. (For example, for each finite coloring of all a -element subsets of an infinite set C , there exists a b -element subset of C such that all of its a -element subsets get the same color.) These simplest theorems are strengthened in two directions.

Direction 1: $a \leq b \leq c < \infty$. This is the domain of Finite Ramsey Theory. (For example, for each finite coloring of all a -element subsets of a c -element set C , there exists a b -element subset of C such that all of its a -element subsets get the same color.) Appropriate structures for this part of the theory are described in [13].

Direction 2: $a = b = c = \infty$. This is the domain of Infinite Dimensional Ramsey Theory. (For example, for each finite Borel coloring of all infinite element subsets of an infinite countable set C , there exists an infinite subset of C such that all of its infinite subsets get the same color.) Appropriate structures for this theory were developed in [16] and a general Ramsey theorem (GRT) for them was proved there.

The frameworks in 1 and 2 are quite different in particulars, but, roughly speaking, the GRTs in both cases have the same form:

GRT: Pigeonhole principle implies Ramsey statement.

Such GRT, reduces proving concrete Ramsey statements to proving appropriate pigeonhole principles. In 1, pigeonhole principles are either easy to check directly or, more frequently, they are reformulations of Ramsey statements proved earlier using GRT with the aid of easier pigeonhole principles. So it is a self-propelling system. In 2, pigeonhole principles cannot be obtained this way and they require separate proofs. (The vague reason for this is that the pigeonhole principles here correspond to the case $b = c = \infty$ and $a = \text{potential } \infty$.)

This paper can be viewed as providing appropriate structures and general theorems that handle proofs of pigeonhole principles in 2. These structures are quite different from those in 1 and 2.

The concurrently written interesting paper [11] also touches on the theme of ultrafilter methods in Ramsey theory. This work and ours are independent from each other.

2. Monoid actions on semigroups

The theme of this section is purely dynamical. We study actions of finite monoids on compact right topological semigroups by continuous endomorphisms. We isolate the class of almost R -trivial monoids that extends the well studied class of R -trivial monoids. We prove in Theorem 2.4 that each action of an almost R -trivial finite monoid M on a compact right topological semigroup by continuous

endomorphisms contains, in a precise sense, a finite action defined only in terms of M . This finite action is an action of M on a partial order $\mathbb{Y}(M)$ introduced in Section 2.1. An important to us reformulation of Theorem 2.4 is done in Corollary 2.7.

2.1. Monoid actions on partial orders. A **monoid** is a semigroup with a distinguished element that is a left and right identity. By convention, if a monoid acts on a set, the identity element acts as the identity function.

Let M be a monoid. By an **M -partial order** we understand a set X equipped with an action of M and with a partial order \leq_X such that if $x \leq_X y$, then $ax \leq_X ay$, for $x, y \in X$ and $a \in M$. Let X and Y be M -partial orders. A function $f: X \rightarrow Y$ is an **epimorphism** if f is onto, f is M -equivariant, and \leq_Y is the image under f of \leq_X . We say that an M -partial order X is **strong** if, for all $y \in X$ and $a \in M$,

$$\{ax \in X : x \leq_X y\} = \{x \in X : x \leq_X ay\}.$$

For a monoid M , consider M acting on itself by multiplication on the left. Set

$$\mathbb{X}(M) = \{aM : a \in M\} \tag{2.1}$$

with the order relation being inclusion. Then, $\mathbb{X}(M)$ is an M -partial order. We actually have more.

LEMMA 2.1. *Let M be a monoid. Then $\mathbb{X}(M)$ is a strong M -partial order.*

Proof. We need to see that if $cM \subseteq abM$, then there is c' such that $c'M \subseteq bM$ and $ac'M = cM$. Since $cM \subseteq abM$, we have $c \in abM$, so $c = abd$ for some $d \in M$. Let $c' = bd$. It is easy to check that this c' works. \square

For each finite partial order X , let

$$\text{Fr}(X) = \{x \subseteq X : x \neq \emptyset \text{ and } x \text{ is linearly ordered by } \leq_X\}. \tag{2.2}$$

The order relation on $\text{Fr}(X)$ is defined by letting for $x, y \in \text{Fr}(X)$,

$$x \leq_{\text{Fr}(X)} y \iff x \subseteq y \text{ and } i <_X j \text{ for all } i \in x \text{ and } j \in y \setminus x.$$

Observe that $\text{Fr}(X)$ is a **forest**, that is, it is a partial order in which the set of predecessors of each element is linearly ordered. (We take this sentence as our definition of the notion of forest.) As pointed out by Todorcevic, the operation Fr is a finite version of certain constructions from infinite combinatorics of partial orders [9, 15].

Let X be an M -partial order. For $x \in \text{Fr}(X)$ and $a \in M$, let

$$ax = \{ai : i \in x\}.$$

Clearly, $ax \in \text{Fr}(X)$ and $M \times \text{Fr}(X) \ni (a, x) \rightarrow ax \in \text{Fr}(X)$ is an action of M on $\text{Fr}(X)$.

The following lemma is easy to verify.

LEMMA 2.2. *Let M be a monoid, and let X be a finite M -partial order.*

- (i) $\text{Fr}(X)$ with the action defined above is a strong M -partial order.
- (ii) The function $\pi : \text{Fr}(X) \rightarrow X$ given by $\pi(x) = \max x$ is an epimorphism between the two M -partial orders.

For a finite monoid M , set

$$\mathbb{Y}(M) = \text{Fr}(\mathbb{X}(M)). \quad (2.3)$$

By Lemma 2.2, $\mathbb{Y}(M)$ is a strong M -partial order.

2.2. Compact right topological semigroups. We recall here some basic notions concerning right topological semigroups.

Let U be a semigroup. As usual, let

$$E(U)$$

be the set of all idempotents of U . There is a natural transitive, antisymmetric relation \leq^U on U defined by

$$u \leq^U v \iff uv = vu = u.$$

This relation is reflexive on the set $E(U)$. So \leq^U is a partial order on $E(U)$.

A semigroup equipped with a topology is called **right topological** if, for each $u \in U$, the function

$$U \ni x \rightarrow xu \in U$$

is continuous.

In the proposition below, we collect facts about idempotents in compact semigroups needed here. They are proved in [16, Lemma 2.1, Lemma 2.3 and Corollary 2.4, Lemma 2.11].

PROPOSITION 2.3. *Let U be a compact right topological semigroup.*

- (i) $E(U)$ is nonempty.
- (ii) For each $v \in E(U)$ there exists a minimal with respect to \leq^U element $u \in E(U)$ with $u \leq^U v$.
- (iii) For each minimal with respect to \leq^U element $u \in E(U)$ and each right ideal $J \subseteq U$, there exists $v \in J \cap E(U)$ with $uv = u$.

If U is equipped with a compact topology, that may not interact with multiplication in any way, then there exists the smallest under inclusion compact two-sided ideal of U , see [5]. So, for a compact right topological semigroup U , let

$$I(U)$$

stand for the smallest compact two-sided ideal with respect to the compact topology on U .

2.3. Almost R-trivial monoids. Two elements a, b of a monoid M are called **R-equivalent** if $aM = bM$. Of course, by an **R-class** of $a \in M$ we understand the set of all elements of M that are R-equivalent to a . A monoid M is called **R-trivial** if each R-class has exactly one element, that is, if for all $a, b \in M$, $aM = bM$ implies $a = b$. This notion with an equivalent definition was introduced in [12]. For the role of R-trivial monoids in the representation theory of monoids, see [14, Chapter 2].

Note that if M is R-trivial, then the partial order $\mathbb{X}(M)$ can be identified with M taken with the partial order $a \leq_M b$ if and only if $a \in bM$. We call a monoid M **almost R-trivial** if, for each $b \in M$ whose R-class has more than one element, we have $ab = b$ for all $a \in M$.

We present now examples of almost R-trivial monoids relevant in Ramsey theory.

Examples. 1. Let $n \in \mathbb{N}$, $n > 0$. Let

$$G_n$$

be $\{0, \dots, n - 1\}$ with multiplication defined by

$$i \cdot j = \min(i + j, n - 1).$$

We set $1_{G_n} = 0$.

The monoid G_n is R-trivial since, for each $i \in G_n$, we have $iG_n = \{i, \dots, n - 1\}$.

The monoid G_n is associated with Gowers' Ramsey theorem [4], see also [16].

2. Fix $n \in \mathbb{N}$, $n > 0$. Let

$$I_n$$

be the set of all nondecreasing functions that map n onto some $k \leq n$. These are precisely the nondecreasing functions $f: n \rightarrow n$ such that $f(0) = 0$ and $f(i+1) \leq f(i) + 1$ for all $i < n-1$. The multiplication operation is composition and 1 is the identity function from n to n .

The monoid I_n is R-trivial. To see this, let $f, g \in I_n$ be such that $f \in gI_n$ and $g \in fI_n$, that is, $f = g \circ h_1$ and $g = f \circ h_2$, for some $h_1, h_2 \in I_n$. It follows from these equations that $f(i) \leq g(i)$, for all $1 \leq i \leq n$, and $g(i) \leq f(i)$, for all $1 \leq i \leq n$. Thus, $f = g$.

The monoid I_n is associated with Lupini's Ramsey theorem [10].

3. Fix two disjoint sets A, B , and let 1 not be an element of $A \cup B$. Let

$$J(A, B)$$

be $\{1\} \cup A \cup B$. Define multiplication on $J(A, B)$ by letting, for each $c \in A \cup B$,

$$\begin{aligned} c \cdot a &= c, & \text{if } a \in A; \\ c \cdot b &= b, & \text{if } b \in B. \end{aligned}$$

Of course, we define $1 \cdot c = c \cdot 1 = c$ for all $c \in J(A, B)$. We leave it to the reader to check that so defined multiplication is associative.

The monoid $J(A, B)$ is almost R-trivial. Indeed, a quick check gives, for $a \in A$ and $b \in B$,

$$aJ(A, B) = \{a\} \cup B, \quad bJ(A, B) = B, \quad 1J(A, B) = J(A, B).$$

Thus, the only elements of $J(A, B)$, whose R-classes can possibly have size bigger than one, are elements of B . But for all $c \in J(A, B)$ and $b \in B$, we have $cb = b$. It follows that $J(A, B)$ is almost R-trivial (and not R-trivial if the cardinality of B is strictly bigger than one).

The monoid $J(\emptyset, B)$ for a one element set B is associated with Hindman's theorem (see [16]) and for arbitrary finite B with the infinitary Hales–Jewett theorem, see [16]. For arbitrary finite A and B , $J(A, B)$ is associated with the Furstenberg–Katznelson theorem [3].

2.4. The theorem on monoid actions. *In the results of this section, we adopt the following conventions:*

- U is a compact right topological semigroup;
- M is a finite monoid acting on U by continuous endomorphisms.

The following theorem is the main result of this section.

THEOREM 2.4. *Assume M is almost R -trivial. There exists a function $g: \mathbb{Y}(M) \rightarrow E(U)$ such that*

- (i) g is M -equivariant;
- (ii) g is order reversing with respect to $\leq_{\mathbb{Y}(M)}$ and \leq^U ;
- (iii) g maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$.

Moreover, if $\mathbb{X}(M)$ has at most two elements, then g maps maximal elements of $\mathbb{Y}(M)$ to minimal elements of $E(U)$.

We will need the following lemma. Its proof borrows some ideas from [10].

LEMMA 2.5. *Let F be a strong M -partial order that is a forest. Assume that $f: F \rightarrow U$ is M -equivariant. Then there exists $g: F \rightarrow E(U)$ such that*

- (i) g is M -equivariant;
- (ii) g is order reversing with respect to \leq_F and \leq^U ;
- (iii) $g^{-1}(I(U))$ contains $f^{-1}(I(U))$.

Proof. Let $A \subseteq F$ be downward closed. Assume we have a function $g_A: F \rightarrow E(U)$ such that (i) and (iii) hold and additionally, for all $i, j \in A$,

$$(\star) \text{ if } i <_F j, \text{ then } g_A(j)g_A(i) = g_A(j).$$

Note the condition that the values of g_A are in $E(U)$, so they are idempotents. Let $B \subseteq F$ be such that $A \subseteq B$ and all the immediate predecessors of elements of B are in A . We claim that there exists $g_B: F \rightarrow E(U)$ fulfilling (i), (iii), (\star) for all $i, j \in B$, and $g_B \upharpoonright A = g_A \upharpoonright A$.

First, define $g'_B: F \rightarrow U$ by letting, for $j \in F$,

$$g'_B(j) = g_A(i_k)g_A(i_{k-1}) \cdots g_A(i_1),$$

where $i_1 <_F \cdots <_F i_k$ lists the set $\{i \in F: i \leq_F j\}$ in the increasing order.

We check that g'_B fulfills (i), (iii), and (\star) for $i, j \in B$. Point (i) holds since for each $a \in M$ we have

$$\begin{aligned} a(g'_B(j)) &= a(g_A(i_k))a(g_A(i_{k-1})) \cdots a(g_A(i_1)) \\ &= g_A(a(i_k))g_A(a(i_{k-1})) \cdots g_A(a(i_1)) = g'_B(a(j)), \end{aligned} \quad (2.4)$$

with the second equality holding since the function g_A is M -equivariant and the third one holding by idempotence of the values of g_A and the fact that F is an M -forest. Point (iii) holds since $I(U)$ is a right ideal and the function g_A fulfills (iii). To check (\star) for $i, j \in B$ with $i <_F j$, let

$$i_1 <_F \cdots <_F i_k <_F \cdots <_F i_l$$

list all the predecessors of j in the increasing order so that $i_k = i$ and, of course, $i_l = j$. Then, since $j \in B$, we have $i_1, \dots, i_k \in A$ and, therefore, we get

$$g'_B(j) = g_A(i_1) \cdots g_A(i_k) = g_A(j) \cdots g_A(i).$$

By the same computation carried out for $i = j \in A$, we see

$$g'_B(i) = g_A(i), \quad \text{for } i \in A. \tag{2.5}$$

It follows that

$$g'_B(j)g'_B(i) = g_A(j) \cdots g_A(i)g_A(i) = g_A(j) \cdots g_A(i) = g'_B(j).$$

This equality shows that (\star) holds for $i, j \in B$. Finally, note that (2.5) implies that $g'_B \upharpoonright A = g_A \upharpoonright A$. Thus, g'_B has all the desired properties.

To construct g_B from g'_B , consider U^F with coordinatewise multiplication and the product topology. This is a right topological semigroup. Define $H \subseteq U^F$ to consist of all $x \in U^F$ such that

- (α) the function $F \ni i \rightarrow x_i \in U$ fulfills (i), (iii), and (\star) for $i, j \in B$ and
- (β) $x_i = g_A(i)$ for all $i \in A$.

First we observe that H is a subsemigroup of U^F . Condition (i) is clearly closed under multiplication. Condition (iii) is closed under multiplication since $I(U)$ is a two-sided ideal. Condition (\star) is closed under multiplication in the presence of (β) since, for $x, y \in H$ and $i, j \in B$ with $i <_F j$, we have $i \in A$ and, therefore,

$$x_j y_j x_i y_i = x_j y_j g_A(i) g_A(i) = x_j y_j y_i y_i = x_j y_j.$$

This verification shows that (α) is closed under multiplication in the presence of (β). Condition (β) is closed under multiplication since $g_A(i)$ is an idempotent.

Next note that H is compact since all conditions defining H are clearly topologically closed with a possible exception of (\star) for $i, j \in B$ with $i <_F j$. Note that in this case $i \in A$. Since $x \in U^F$ and $i \in A$, we have $x_i = g_A(i)$, condition (\star) translates to $x_j g_A(i) = x_j$ for $i \in A$ and $j \in B$ with $i <_F j$. This condition is closed since U is right topological. Finally note that H is nonempty

since g'_B is its element. By Ellis' Lemma [16, Lemma 2.1], H contains an idempotent. Let $g_B \in H$ be such an idempotent. It has all the required properties.

The above procedure describes the passage from g_A to g_B if all immediate predecessors of elements of B are in A .

We now define $g_\emptyset: F \rightarrow E(U)$ that fulfills (i), (iii), and (\star) for $A = \emptyset$, with the last condition holding vacuously. Note that f has all the properties required of g_\emptyset except its values may not be in $E(U)$. To remedy this shortcoming, consider again the compact right topological semigroup U^F with coordinatewise multiplication and the product topology. Define $H \subseteq U^F$ to consist of all $x \in U^F$ such that the function $F \ni i \rightarrow x_i \in U$ fulfills (i) and (iii) (and, vacuously, (\star) for $i, j \in \emptyset$). Then H is nonempty since $f \in H$. As above, we check that H is a compact subsemigroup of U^F . Let g_\emptyset be an idempotent in H . Clearly, g_\emptyset has the required properties.

Starting with g_\emptyset and recursively using the procedure of going from g_A to g_B described above, we produce $g_F: F \rightarrow E(U)$ fulfilling (i), (iii) and (\star) for all $i, j \in F$.

Now define, for $j \in F$,

$$g(j) = g_F(j_1) \cdots g_F(j_l), \tag{2.6}$$

where $j_1 <_F \cdots <_F j_l = j$ list all elements of the set $\{i \in F : i \leq_F j\}$. This g is as required by the conclusion of the lemma. Keeping in mind that all values of g_F are idempotents, we see that point (i) for g holds by the calculation as in (2.4). Point (iii) for g is clear since it holds for $g_F, j_l = j$ in formula (2.6), and $I(U)$ is a two-sided, so left, ideal. To see point (ii) for g , we use an argument similar to one applied earlier in the proof. To do this, fix $i \leq_F j$ in F , and let

$$i_1 <_F \cdots <_F i_k = i \quad \text{and} \quad j_1 <_F \cdots <_F j_l = j$$

list elements of F that are $\leq_F i$ and $\leq_F j$, respectively. Note that $i_1 = j_1, \dots, i_k = j_k$. Using (\star) for g_F and idempotency of $g_F(i_k) = g_F(j_k)$, we see that

$$\begin{aligned} g(i)g(j) &= g_F(i_1) \cdots g_F(i_k)g_F(j_1) \cdots g_F(j_l) \\ &= g_F(i_1) \cdots g_F(i_k)g_F(j_k) \cdots g_F(j_l) = g(j), \end{aligned}$$

while using only (\star) for g_F if $i <_F j$ and (\star) and idempotency of $g_F(j)$ if $i = j$, we get

$$\begin{aligned} g(j)g(i) &= g_F(j_1) \cdots g_F(j_l)g_F(i_1) \cdots g_F(i_k) \\ &= g_F(j_1) \cdots g_F(j_l) = g(j). \end{aligned}$$

Thus, point (ii) is verified for g . Note that point (ii) implies that values of g are idempotent, that is, they are elements of $E(U)$. Therefore, we checked that $g: F \rightarrow E(U)$ and (i)–(iii) hold for g . \square

LEMMA 2.6. Assume that $ab = b$, for all $a, b \in M$ with $b \neq 1_M$. Then there exists minimal $u_1 \in E(U)$ such that

- (i) $u_1 \in I(U)$
- (ii) $a(u_1) = b(u_1)$, for all $a, b \in M$ with $a \neq 1_M \neq b$.

Proof. Observe that, for $a, b \in M \setminus \{1_M\}$, since $ba = a$, we have

$$a(U) = ba(U) = b(a(U)) \subseteq b(U).$$

By symmetry, we see that $a(U) = b(U)$. Let T be the common value of the images of U under the elements of $M \setminus \{1_M\}$. Clearly T is a compact subsemigroup of U . Note that

$$a(u) = u, \quad \text{for } u \in T, a \in M. \quad (2.7)$$

Let

$$u_0 \in T$$

be a minimal with respect to \leq^T idempotent.

Let $u_1 \in U$ be a minimal idempotent in U with $u_1 \leq^U u_0$. Since u_1 is minimal, we have

$$u_1 \in I(U). \quad (2.8)$$

We show that

$$a(u_1) = u_0, \quad \text{for all } a \in M \setminus \{1_M\}. \quad (2.9)$$

Indeed, since $u_1 \leq^U u_0$ and $u_0 \in T$, by (2.7), we get

$$a(u_1) \leq^U a(u_0) = u_0.$$

Thus, $a(u_1) \leq^T u_0$ and $a(u_1) \in T$. Since u_0 is minimal in T , we get $a(u_1) = u_0$.

Equations (2.8) and (2.9) show that u_1 is as required. \square

Proof of Theorem 2.4. Let

$$B = \{b \in M : ab = b \text{ for all } a \in M\}.$$

Note that $M' = \{1_M\} \cup B$ is a monoid fulfilling the assumption of Lemma 2.6. Let $u_1 \in U$ be an element as in the conclusion of Lemma 2.6.

Define a function $h: M \rightarrow U$ by $h(a) = a(u_1)$. Note that h is M -equivariant if M is taken with left multiplication action. Observe the following two implications:

- (a) if $a_1 \in M \setminus B$, $a_2 \in M$, and $a_1 M = a_2 M$, then $a_1 = a_2$;
- (b) if $a \in M$ and $b \in B$, then $bM \subseteq aM$.

Point (a) follows from M being almost R-trivial. Point (b) is a consequence of $b = ab \in aM$. Let $\rho: M \rightarrow \mathbb{X}(M)$ be the equivariant surjection $\rho(a) = aM$. Note that by (a) and (b), ρ is injective on $M \setminus B$, all points in B are mapped to a single point of $\mathbb{X}(M)$ that is the smallest point of this partial order, and no point of $M \setminus B$ is mapped to this smallest point. It now follows from the properties of u_1 listed in Lemma 2.6 that h factors through ρ giving a function $h': \mathbb{X}(M) \rightarrow U$ with $h' \circ \rho = h$. Since ρ and h are M -equivariant, so is h' . Let $\pi: \mathbb{Y}(M) \rightarrow \mathbb{X}(M)$ be the M -equivariant function given by Lemma 2.2(ii). Then $f: \mathbb{Y}(M) \rightarrow U$, defined by $f = h' \circ \pi$, is M -equivariant. Furthermore, since $u_1 \in I(U)$ gives $h(1_M) \in I(U)$, and hence $h'(1_M M) \in I(U)$, we see that the maximal elements of $\mathbb{Y}(M)$ are mapped by f to $I(U)$. Note that if $\mathbb{X}(M)$ has at most two elements, then we can let $g = f$. Then $h'(1_M M)$ is an idempotent minimal with respect to \leq^U , and g maps all maximal elements of $\mathbb{Y}(M)$ to $h'(1_M M)$. Without any restrictions on the size of $\mathbb{X}(M)$, Lemma 2.5 can be applied to f giving a function g as required by points (i)–(iii). \square

2.5. Semigroups from partial orders and a restatement of the theorem.

For a partial order P , let

$$\langle P \rangle$$

be the semigroup, whose binary operation is denoted by \vee , generated freely by elements of P modulo the relations

$$p \vee q = q \vee p = q, \quad \text{for } p, q \in P \text{ with } p \leq_P q. \quad (2.10)$$

That is, each element of $\langle P \rangle$ can be uniquely written as $p_0 \vee \cdots \vee p_n$ for some $n \in \mathbb{N}$ and with p_i and p_{i+1} being incomparable with respect to \leq_P , for all $0 \leq i < n$. Note that if P is linear, then $\langle P \rangle = P$.

Observe that if M is a monoid and P is an M -partial order, then the action of M on P naturally induces an action of M on $\langle P \rangle$ by endomorphisms, namely, for $a \in M$ and $p_0 \vee \cdots \vee p_n \in \langle P \rangle$ with $p_0, \dots, p_n \in P$, we let

$$a(p_0 \vee \cdots \vee p_n) = a(p_0) \vee \cdots \vee a(p_n). \quad (2.11)$$

It is easy to see that the right-hand side of the above equality is well defined and that formula (2.11) defines an endomorphism of $\langle P \rangle$ and, in fact, an action of M on $\langle P \rangle$.

A moment of thought convinces one that the function from Theorem 2.4 extends to a homomorphism from $\langle \mathbb{Y}(M) \rangle$ to U —condition (ii) of Theorem 2.4 and the fact that the function in that theorem takes values in $E(U)$ are responsible for this. Therefore, we get the following corollary, which we state with the conventions of Section 2.4.

COROLLARY 2.7. *Assume M is almost R -trivial. There exists an M -equivariant homomorphism of semigroups $g: \langle \mathbb{Y}(M) \rangle \rightarrow U$ that maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$. Additionally, if $\mathbb{X}(M)$ has at most two elements, then g maps maximal elements of $\mathbb{Y}(M)$ to minimal idempotents in U .*

3. Infinitary Ramsey theorems

The goal of this section is Ramsey theoretic. We introduce structures, we call function arrays, that generalize the partial semigroup setting of [2]. One important feature of these structures is their closure under naturally defined tensor product. For function arrays, we introduce the notion of basic sequence. Basic sequences appear in Ramsey statements whose aim is to control the behavior of a coloring on them. We introduce a new general notion of such control, which is akin to finding upper bounds on Ramsey degrees, but whose nature is algebraic. (For a definition and applications of Ramsey degrees, see, for example, [7].) The main result then is Theorem 3.1, which gives control over a coloring on a basic sequence from the existence of an appropriate homomorphism. Thus, proving Ramsey statements is reduced to finding homomorphisms. Furthermore, as mentioned above, we introduce a natural notion of tensor product for function arrays that allows us to propagate the existence of homomorphisms and, therefore, to propagate Ramsey statements.

3.1. Function arrays and total function arrays. Here we recall the notion of partial semigroup and, more importantly, we introduce our main Ramsey theoretic structures: function arrays, total function arrays and homomorphisms between them.

As in [2] and [16], a **partial semigroup** is a set S with a function (operation) from a subset of $S \times S$ to S such that for all $r, s, t \in S$ if one of the two products $(rs)t, r(st)$ is defined, then so is the other and $(rs)t = r(st)$. Note that a semigroup is a partial semigroup whose binary operation is total.

Now, let Λ be a nonempty set. Let S be a partial semigroup and let X be a set. By a **function array over S indexed by Λ and based on X** we understand an assignment to each $\lambda \in \Lambda$ of a partial function, which we also call λ , from X to S with the property that for all $s_0, \dots, s_k \in S$ there exists $x \in X$ such that, for each $\lambda \in \Lambda$,

$$s_0\lambda(x), \dots, s_k\lambda(x) \text{ are all defined.} \quad (3.1)$$

So the domain of each $\lambda \in \Lambda$ is a subset, possibly proper, of X ; condition (3.1) means, in particular, that x is in the domain of each $\lambda \in \Lambda$. We call a function array as above **total** if S is a semigroup and the domain each $\lambda \in \Lambda$ is equal to X .

We call a function array **point based** if X consist of one point, which we usually denote by \bullet ; so $X = \{\bullet\}$ in this case. Note that, by condition (3.1), the domain of each λ is equal to $\{\bullet\}$. A point based function array can be, therefore, identified with a function $\Lambda \rightarrow S$ given by

$$\Lambda \ni \lambda \rightarrow \lambda(\bullet) \in S.$$

Moreover, if S is a semigroup, then a point based function array is automatically total.

We give now some constructions that will be used in Section 4. Let S be a partial semigroup. As usual, a function $h: S \rightarrow S$ is an **endomorphism** if for all $s_1, s_2 \in S$ with s_1s_2 defined, $h(s_1)h(s_2)$ is defined and $h(s_1s_2) = h(s_1)h(s_2)$. Let M be a monoid. An action of M on S is called an **endomorphism action of M on S** if, for each $a \in M$, the function $s \rightarrow a(s)$ is an endomorphism of S and, for all $s_1, \dots, s_n \in S$ and each $a \in M$, there is $t \in S$ such that $s_1a(t), \dots, s_na(t)$ are defined. Obviously, we will identify such an action with the function $\alpha: M \times S \rightarrow S$ given by $\alpha(a, s) = a(s)$.

An endomorphism action α of a monoid M on a semigroup S gives rise to two types of function arrays, both of which are over S and indexed by M but are based on different sets. The first of these function arrays is based on S and is defined as follows. Let

$$S(\alpha) \tag{3.2}$$

be the function array over S indexed by M and based on S that is obtained by interpreting each $a \in M$ as the function from S to S given by the action, that is,

$$S \ni s \rightarrow \alpha(a, s) \in S.$$

The second function array arising from α is point based and needs a specification of an element of S . So fix $\mathbf{s} \in S$. Let

$$S(\alpha)_{\mathbf{s}} \tag{3.3}$$

be the point based function array over S indexed by M obtained by interpreting each $a \in M$ as the function on $\{\bullet\}$ given by

$$a(\bullet) = \alpha(a, \mathbf{s}).$$

Function arrays used in this paper will be of the above form or will be obtained from such by the tensor product operation defined in Section 3.5.

3.2. Basic sequences and tame colorings. Assume we have function arrays \mathcal{S} and \mathcal{A} both indexed by Λ , but with \mathcal{S} being over a partial semigroup S and based

on X , while \mathcal{A} being over a semigroup A and point based. To make notation clearer, we use \vee for the binary operation on A . We write

$$\Lambda(\bullet) = \{\lambda(\bullet) : \lambda \in \Lambda\} \subseteq A.$$

A sequence (x_n) of elements of X is called **basic in \mathcal{S}** if for all $n_0 < \dots < n_l$ and $\lambda_0, \dots, \lambda_l \in \Lambda$ the product

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1}) \cdots \lambda_l(x_{n_l}) \quad (3.4)$$

is defined in \mathcal{S} .

We say that a coloring of \mathcal{S} is **\mathcal{A} -tame on (x_n)** , where (x_n) is a basic sequence in \mathcal{S} , if the color of the elements of the form (3.4) with the additional condition

$$\lambda_k(\bullet) \vee \cdots \vee \lambda_l(\bullet) \in \Lambda(\bullet), \quad \text{for each } k \leq l, \quad (3.5)$$

depends only on the element

$$\lambda_0(\bullet) \vee \cdots \vee \lambda_l(\bullet) \in A.$$

So, subject to condition (3.5), the color of the product (3.4) is entirely controlled by $\lambda_0(\bullet) \vee \cdots \vee \lambda_l(\bullet)$, which, in general, contains much less information than (3.4).

3.3. Total function arrays from function arrays. There is a canonical way of associating a total function array to each function array, which generalizes the operation γS of compactification of a directed partial semigroup S from [2]. The definition of γS , the semigroup structure and compact topology on it should be recalled here, for example, from [16, p.31]; in particular, we use the symbol $*$ to denote the semigroup operation on γS , that is, the product of ultrafilters.

Let \mathcal{S} be a function array over a partial semigroup S , indexed by Λ and based on X . Note that (3.1) implies that S is directed as defined in [16, p.30]. It follows that γS is defined. Let γX be the set of all ultrafilters \mathcal{U} on X such that for each $s \in S$ and $\lambda \in \Lambda$

$$\{x \in X : s\lambda(x) \text{ is defined}\} \in \mathcal{U}.$$

It is clear from condition (3.1) that γX is nonempty. It is also easy to verify that γX is compact with the usual Čech–Stone topology on ultrafilters. Each λ extends to a function, again called λ , from γX to γS by the usual formula, for $\mathcal{U} \in \gamma X$,

$$A \in \lambda(\mathcal{U}) \quad \text{iff} \quad \lambda^{-1}(A) \in \mathcal{U}.$$

It is easy to see that the image of each λ is, indeed, included in γS and each function $\lambda: \gamma X \rightarrow \gamma S$ is continuous. Since γS is a semigroup, we get a total function array over γS , indexed by Λ and based on γX . We denote this total function array by $\gamma \mathcal{S}$. The topologies on γS and γX will play a role later on.

3.4. The Ramsey theorem. The following natural notion of homomorphism will be crucial in stating Theorem 3.1. Assume we have total function arrays \mathcal{A} and \mathcal{B} both indexed by Λ , with \mathcal{A} being over A and based on X and \mathcal{B} being over B and based on Y . A **homomorphism from \mathcal{A} to \mathcal{B}** is a pair of functions (f, g) such that $f: X \rightarrow Y$, $g: A \rightarrow B$, g is a homomorphism of semigroups, and, for each $x \in X$ and $\lambda \in \Lambda$, we have

$$\lambda(f(x)) = g(\lambda(x)).$$

The following theorem is the main result of Section 3. The bottom line of it is that a homomorphism from a point based function array \mathcal{A} gives rise to basic sequences on which colorings are \mathcal{A} -tame.

THEOREM 3.1. *Let \mathcal{A} and \mathcal{S} be function arrays both indexed by a finite set Λ , with \mathcal{A} being point based and over a semigroup and \mathcal{S} being over a partial semigroup S . Let $(f, g): \mathcal{A} \rightarrow \gamma\mathcal{S}$ be a homomorphism. Then for each $D \in f(\bullet)$ and each finite coloring of S , there exists a basic sequence (x_n) of elements of D on which the coloring is \mathcal{A} -tame.*

Proof. Consistently with our conventions, \vee denotes the semigroup operation in the semigroup over which \mathcal{A} is defined. Let \mathcal{S} be based on a set X .

Set $\mathcal{U} = f(\bullet)$. Observe that if $\lambda(\bullet) = \lambda'(\bullet)$, then $\lambda(\mathcal{U}) = \lambda'(\mathcal{U})$ since

$$\lambda(f(\bullet)) = g(\lambda(\bullet)) = g(\lambda'(\bullet)) = \lambda'(f(\bullet)).$$

This observation allows us to define for $\sigma \in \Lambda(\bullet)$,

$$\sigma(\mathcal{U}) = \lambda(\mathcal{U})$$

for some, or, equivalently, each, $\lambda \in \Lambda$ with $\lambda(\bullet) = \sigma$. Observe further that for $\sigma \in \Lambda(\bullet)$ we have

$$g(\sigma) = \sigma(\mathcal{U}). \quad (3.6)$$

Indeed, fix $\lambda \in \Lambda$ with $\sigma = \lambda(\bullet)$. Then we have

$$\sigma(\mathcal{U}) = \lambda(f(\bullet)) = g(\lambda(\bullet)) = g(\sigma).$$

For $P \subseteq X$ and $\sigma \in \Lambda(\bullet)$, set

$$\sigma(P) = \bigcap \{\lambda(P) : \lambda(\bullet) = \sigma\}.$$

Note that if $P \in \mathcal{U}$ and $\lambda \in \Lambda$, then $\lambda(P) \in \lambda(\mathcal{U})$ since $P \subseteq \lambda^{-1}(\lambda(P))$. So, for λ with $\lambda(\bullet) = \sigma$, we have $\lambda(P) \in \sigma(\mathcal{U})$, and, therefore, by finiteness of Λ , we get $\sigma(P) \in \sigma(\mathcal{U})$.

Consider a finite coloring of S . Let $P \in \mathcal{U}$ be such that the coloring is constant on $\sigma(P)$ for each $\sigma \in \Lambda(\bullet)$, using the obvious observation that $\sigma(P) \subseteq \sigma(P')$ if $P \subseteq P'$.

Now, we produce $x_n \in X$ and $P_n \subseteq X$ so that

- (i) $x_n \in D$, $P_n \subseteq P$;
- (ii) $\lambda_1(x_{m_1})\lambda_2(u) \in (\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_{m_1})$, for all $m_1 < n$, all $u \in P_n$, and all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet)$;
- (iii) the set of u fulfilling the following condition belongs to \mathcal{U} : $\lambda_1(x_m)\lambda_2(u) \in (\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_m)$, for all $m \leq n$ and all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet)$;
- (iv) $\lambda(x_m) \in \lambda(\bullet)(P_m)$, for all $m \leq n$ and all $\lambda \in \Lambda$.

Note that in points (ii) and (iii) above the condition $\lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet)$ ensures that $(\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_{m_1})$ and $(\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_m)$ are defined.

Assume we have x_m, P_m for $m < n$ as above. We produce x_n and P_n so that points (i)–(iv) above hold. Define P_n by letting

$$P_n = P \cap \bigcap_{m < n} C_m,$$

where C_m consists of those $u \in X$ for which, for all $\lambda_1, \lambda_2 \in \Lambda$,

$$\text{if } \lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet), \text{ then } \lambda_1(x_m)\lambda_2(u) \in (\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_m).$$

For $n = 0$, by convention, we set $\bigcap_{m < n} C_m = S$. Observe that (ii) holds for n . Our inductive assumption (iii) implies that $C_m \in \mathcal{U}$. Thus, the definition of P_n gives that $P_0 = P \in \mathcal{U}$ and, for $n > 0$, $P_n \in \mathcal{U}$.

Using (3.6) in the last equality, we have that, for all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet)$,

$$\begin{aligned} \lambda_1(\mathcal{U}) * \lambda_2(\mathcal{U}) &= \lambda_1(f(\bullet)) * \lambda_2(f(\bullet)) = g(\lambda_1(\bullet)) * g(\lambda_2(\bullet)) \\ &= g(\lambda_1(\bullet) \vee \lambda_2(\bullet)) = (\lambda_1(\bullet) \vee \lambda_2(\bullet))(\mathcal{U}). \end{aligned} \quad (3.7)$$

Separately, we note that $P_n \in \mathcal{U}$ and therefore, for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \vee \lambda_2(\bullet) \in \Lambda(\bullet)$,

$$\begin{aligned} (\lambda_1(\bullet) \vee \lambda_2(\bullet))(P_n) &\in (\lambda_1(\bullet) \vee \lambda_2(\bullet))(\mathcal{U}) \quad \text{and} \\ \lambda_1(\bullet)(P_n) &\in \lambda_1(\bullet)(\mathcal{U}) = \lambda_1(\mathcal{U}). \end{aligned} \quad (3.8)$$

It follows from (3.7) and (3.8) that we can pick x_n for which (iii) and (iv) hold. Since $D \in \mathcal{U}$, we can also arrange that $x_n \in D$. So (i) is also taken care of.

Now, it suffices to show that the sequence (x_n) constructed above is as needed. The entries of (x_n) come from D by (i). By induction on l , we show that for all $m_0 < m_1 < \dots < m_l$ and all $\lambda_0, \lambda_1, \dots, \lambda_l \in \Lambda$, we have

$$\lambda_0(x_{m_0})\lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}) \in (\lambda_0(\bullet) \vee \lambda_1(\bullet) \cdots \vee \lambda_l(\bullet))(P_{m_0}), \quad (3.9)$$

provided that $\lambda_k(\bullet) \vee \dots \vee \lambda_l(\bullet) \in \Lambda(\bullet)$ for all $k \leq l$. This claim will establish the theorem since $P_{m_0} \subseteq P$ by (i).

The case $l = 0$ of (3.9) is (iv). We check the inductive step for (3.9) using point (ii). Let $l > 0$. Fix $m_0 < m_1 < \dots < m_l$ and $\lambda_0, \lambda_1, \dots, \lambda_l \in \Lambda$. By our inductive assumption, we have

$$\lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}) \in (\lambda_1(\bullet) \vee \dots \vee \lambda_l(\bullet))(P_{m_1}). \quad (3.10)$$

Let $\lambda \in \Lambda$ be such that

$$\lambda(\bullet) = \lambda_1(\bullet) \vee \dots \vee \lambda_l(\bullet). \quad (3.11)$$

Since

$$(\lambda_1(\bullet) \vee \dots \vee \lambda_l(\bullet))(P_{m_1}) \subseteq \lambda(P_{m_1}),$$

by (3.10), there exists $y \in P_{m_1}$ such that

$$\lambda(y) = \lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}). \quad (3.12)$$

Since $m_0 < m_1$ and since $y \in P_{m_1}$, from (ii) with $n = m_1$, we get

$$\lambda_0(x_{m_0})\lambda(y) \in (\lambda_0(\bullet) \vee \lambda(\bullet))(P_{m_0}). \quad (3.13)$$

Note that (ii) can be applied here as $\lambda_0(\bullet) \vee \lambda(\bullet) \in \Lambda(\bullet)$ as

$$\lambda_0(\bullet) \vee \lambda(\bullet) = \lambda_0(\bullet) \vee \lambda_1(\bullet) \vee \dots \vee \lambda_l(\bullet).$$

Now (3.9) follows from (3.13) together with (3.11) and (3.12). \square

In the proof above, at stage n , x_n is chosen arbitrarily from sets belonging to $f(\bullet)$. It follows that if $f(\bullet)$ is assumed to be nonprincipal, then the sequence (x_n) can be chosen to be injective.

3.5. Tensor product of function arrays. We introduce and apply a natural notion of tensor product for function arrays.

Let A_0, A_1 be finite sets. Let

$$A_0 \star A_1 = A_0 \cup A_1 \cup (A_0 \times A_1),$$

where the union is taken to be disjoint. Fix a partial semigroup S . Let \mathcal{S}_i , $i = 0, 1$, be function arrays over S indexed by Λ_i and based on X_i , respectively. Define

$$\mathcal{S}_0 \otimes \mathcal{S}_1$$

to be the function array over S indexed by $\Lambda_0 \star \Lambda_1$ and based on $X_0 \times X_1$ defined as follows. With $\lambda_0 \in \Lambda_0$, $\lambda_1 \in \Lambda_1$, and $(\lambda_0, \lambda_1) \in \Lambda_0 \times \Lambda_1$, we associate partial functions from $X_0 \times X_1$ to S by letting

$$\lambda_0(x_0, x_1) = \lambda_0(x_0), \quad \lambda_1(x_0, x_1) = \lambda_1(x_1), \quad (\lambda_0, \lambda_1)(x_0, x_1) = \lambda_0(x_0)\lambda_1(x_1),$$

where the product on the right-hand side of the last equality is computed in S and the left-hand side is declared to be defined whenever this product exists. To check that the object $\mathcal{S}_0 \otimes \mathcal{S}_1$ defined above is indeed a function array one needs to see condition (3.1). To see this fix $s_0, \dots, s_m \in S$. Pick $x_0 \in X_0$ with $s_i \lambda(x_0)$ defined for all $i \leq m$ and $\lambda \in \Lambda_0$. Now pick $x_1 \in X_1$ with $s_i \lambda'(x_1)$ and $(s_i \lambda(x_0)) \lambda'(x_1)$ are defined for all $i \leq m$, $\lambda \in \Lambda_0$ and $\lambda' \in \Lambda_1$. Then clearly $s_i \vec{\lambda}(x_0, x_1)$ are defined for all $i \leq m$ and $\vec{\lambda} \in \Lambda_0 \star \Lambda_1$, as required.

It is clear that the operation of tensor product is associative; if each \mathcal{S}_i , $i < n$, is a function array over S indexed by Λ_i , then $\bigotimes_{i < n} \mathcal{S}_i$ is a function array over S indexed by $\Lambda_0 \star \dots \star \Lambda_{n-1}$, which consists of all sequences $(\lambda_0, \dots, \lambda_m)$, where $\lambda_i \in \Lambda_{j_i}$ for some $j_0 < \dots < j_m < n$. Note that if each \mathcal{S}_i is point based, then so is the tensor product.

PROPOSITION 3.2. *Let S be a partial semigroup. Let \mathcal{S}_i , $i = 0, 1$, be function arrays over S based on Λ_i , respectively. Then there is a homomorphism*

$$\gamma \mathcal{S}_0 \otimes \gamma \mathcal{S}_1 \rightarrow \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1).$$

Proof. Let \mathcal{S}_0 be based on X_0 and \mathcal{S}_1 on X_1 . Then $\gamma \mathcal{S}_0 \otimes \gamma \mathcal{S}_1$ is based on $\gamma X_0 \times \gamma X_1$, while $\gamma(\mathcal{S}_0 \otimes \mathcal{S}_1)$ on $\gamma(X_0 \times X_1)$. Consider the natural map $\gamma X_0 \times \gamma X_1 \rightarrow \gamma(X_0 \times X_1)$ given by

$$(\mathcal{U}, \mathcal{V}) \rightarrow \mathcal{U} \times \mathcal{V},$$

where, for $C \subseteq X_0 \times X_1$,

$$C \in \mathcal{U} \times \mathcal{V} \iff \{x_0 \in X_0 : \{x_1 \in X_1 : (x_0, x_1) \in C\} \in \mathcal{V}\} \in \mathcal{U}.$$

Then

$$(f, \text{id}_{\gamma S}) : \gamma \mathcal{S}_0 \otimes \gamma \mathcal{S}_1 \rightarrow \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1),$$

where $f(\mathcal{U}, \mathcal{V}) = \mathcal{U} \times \mathcal{V}$, is the desired homomorphism. □

PROPOSITION 3.3. Fix semigroups A and B . For $i = 0, 1$, let \mathcal{A}_i and \mathcal{B}_i be total function arrays over A and B , respectively, indexed by Λ_i . Let $(f_i, g): \mathcal{A}_i \rightarrow \mathcal{B}_i$ be homomorphisms. Then

$$(f_0 \times f_1, g): \mathcal{A}_0 \otimes \mathcal{A}_1 \rightarrow \mathcal{B}_0 \otimes \mathcal{B}_1$$

is a homomorphism.

Proof. Let \mathcal{A}_i be based on a set X_i for $i = 0, 1$. For $(x_0, x_1) \in X_0 \times X_1$ and $\vec{\lambda} \in \Lambda_0 * \Lambda_1 = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1)$, we need to check

$$g(\vec{\lambda}(x_0, x_1)) = \vec{\lambda}((f_0 \times f_1)(x_0, x_1)).$$

We do it only for $\vec{\lambda} \in \Lambda_0 \times \Lambda_1$, the case $\vec{\lambda} \in \Lambda_0 \cup \Lambda_1$ being essentially identical. So for $\vec{\lambda} \in \Lambda_0 \times \Lambda_1$, we have

$$\begin{aligned} g(\vec{\lambda}(x_0, x_1)) &= g(\lambda_0(x_0)\lambda_1(x_1)) = g(\lambda_0(x_0))g(\lambda_1(x_1)) \\ &= \lambda_0(f_0(x_0))\lambda_1(f_1(x_1)) = \vec{\lambda}((f_0 \times f_1)(x_0, x_1)), \end{aligned}$$

where the second equality holds since g is a homomorphism of semigroups and the third equality holds since each (f_i, g) is a homomorphism from \mathcal{A}_i to \mathcal{B}_i . \square

3.6. Propagation of homomorphisms. The first application of tensor product has to do with relaxing condition (3.5). This is done in condition (3.14).

Let \mathcal{A} be a point based function array over a semigroup A and indexed by Λ . As before, we denote by \vee the binary operation on A . Let F be a subset of A , let \mathcal{S} be a function array over a partial semigroup S indexed by Λ , and let (x_n) be a basic sequence in \mathcal{S} . A coloring of S is said to be F - \mathcal{A} -tame on (x_n) if the color of elements of the form (3.4) with the additional condition

$$\lambda_k(\bullet) \vee \cdots \vee \lambda_l(\bullet) \in F, \quad \text{for each } k \leq l, \quad (3.14)$$

depends only on the element $\lambda_0(\bullet) \vee \cdots \vee \lambda_l(\bullet)$ of A . The notion of F - \mathcal{A} -tameness becomes \mathcal{A} -tameness if $F = \Lambda(\bullet)$. In applications, it will be desirable to take F strictly larger than $\Lambda(\bullet)$ thus making F - \mathcal{A} -tameness strictly stronger.

The following corollary is a strengthening of Theorem 3.1, but it follows from that theorem via the tensor product construction.

COROLLARY 3.4. Let \mathcal{A} and \mathcal{S} be function arrays both indexed by a finite set Λ , with \mathcal{A} being point based and over a semigroup A and \mathcal{S} being over a partial semigroup S . Let $(f, g): \mathcal{A} \rightarrow \gamma\mathcal{S}$ be a homomorphism. Then for each $D \in f(\bullet)$, each finite coloring of S , and each finite set $F \subseteq A$, there exists a basic sequence (x_n) of elements of D on which the coloring is F - \mathcal{A} -tame.

Proof. For a natural number $r > 0$, by $\Lambda_{<r}$ we denote the set of all sequences $\vec{\lambda} = (\lambda_0, \dots, \lambda_m)$ of elements of Λ with $m < r$. We associate with each such $\vec{\lambda}$ an element $\vec{\lambda}(\bullet)$ of A by letting

$$\vec{\lambda}(\bullet) = \lambda_0(\bullet) \vee \dots \vee \lambda_m(\bullet).$$

Since F is finite, there exists r such that

$$F \cap \left\{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \bigcup_{r' \geq 1} \Lambda_{<r'} \right\} \subseteq \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<r} \}.$$

Thus, fixing this r , it suffices to show the corollary for $F = \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<r} \}$. We consider the function arrays $\mathcal{A}^{\otimes r}$ and $(\gamma\mathcal{S})^{\otimes r}$ indexed by Λ^{*r} . Note that $\mathcal{A}^{\otimes r}$ is based on a set consisting of only the r -tuple $(\bullet, \dots, \bullet)$ and that

$$\Lambda^{*r}(\bullet, \dots, \bullet) = \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<r} \}.$$

Further, by Proposition 3.3, there exists a homomorphism from $\mathcal{A}^{\otimes r}$ to $(\gamma\mathcal{S})^{\otimes r}$, which is equal to (f^r, g) , where f^r stands for the r -fold product $f \times \dots \times f$. Note also that $D \times X^{r-1} \in f^r(\bullet)$. Since, by Proposition 3.2, there is a homomorphism from $(\gamma\mathcal{S})^{\otimes r}$ to $\gamma(\mathcal{S}^{\otimes r})$, we are done by Theorem 3.1 by composing the two homomorphisms. \square

We have one more corollary of Theorem 3.1 and the tensor product construction. It concerns double sequences. Let \mathcal{S} be a function array over S indexed by Λ and based on X . A double sequence (x_n, y_n) of elements of X will be called **basic** if the single sequence

$$x_0, y_0, x_1, y_1, x_2, y_2, \dots$$

is basic. Having a basic sequence (x_n, y_n) , we will be interested in controlling the color on elements of the form

$$\lambda_0(x_{m_0})\lambda'_0(y_{n_0})\lambda_1(x_{m_1})\lambda'_1(y_{n_1})\lambda_2(x_{m_2})\lambda'_2(y_{n_2}) \cdots \lambda_l(x_{m_l})\lambda'_l(y_{n_l})$$

and

$$(3.15)$$

$$\lambda_0(x_{m_0})\lambda'_0(y_{n_0})\lambda_1(x_{m_1})\lambda'_1(y_{n_1})\lambda_2(x_{m_2})\lambda'_2(y_{n_2}) \cdots \lambda_l(x_{m_l})$$

for $m_0 \leq n_0 < m_1 \leq n_1 < \dots < m_l \leq n_l$ and $\lambda_0, \lambda'_0, \dots, \lambda_l, \lambda'_l \in \Lambda$.

Let \mathcal{A} be a point based total function array over a semigroup A indexed by Λ . For $\lambda, \lambda' \in \Lambda$, we say that λ and λ' are **conjugate** if $\lambda(\bullet) = \lambda'(\bullet)$. For a set $F \subseteq A$, we say that a coloring of S is **conjugate F - \mathcal{A} -tame on** (x_n, y_n) if the color of the elements of the form (3.15) such that for each $k \leq l$

$$\lambda_k \quad \text{and} \quad \lambda'_k \quad \text{are conjugate and} \quad \lambda_k(\bullet) \vee \dots \vee \lambda_l(\bullet) \in F$$

depends only on

$$\lambda_0(\bullet) \vee \dots \vee \lambda_l(\bullet) \in A.$$

COROLLARY 3.5. Let \mathcal{A} and \mathcal{S} be function arrays both indexed by a finite set Λ , with \mathcal{A} being point based and over a semigroup A and \mathcal{S} being based on X and over a partial semigroup S . Let $(f, g): \mathcal{A} \rightarrow \gamma\mathcal{S}$ be a homomorphism. Let $v \in \gamma X$ be such that for each $\lambda \in \Lambda$

$$\lambda(f(\bullet))\lambda(v) = \lambda(f(\bullet)). \quad (3.16)$$

Then, for each finite subset F of A , $D \in f(\bullet)$ and $E \in v$ and each finite coloring of S , there exists a basic sequence (x_n, y_n) , with $x_n \in D$ and $y_n \in E$, on which the coloring is conjugate F - \mathcal{A} -tame.

Proof. Let \mathcal{A} be based on $\{\bullet\}$. Let

$$\Lambda' = \Lambda \cup \{(\lambda_0, \lambda_1) \in \Lambda \times \Lambda : \lambda_0(\bullet) = \lambda_1(\bullet)\}.$$

Fix a point \bullet' . Let \mathcal{A}' be the point based function array over A and indexed by Λ' that is based on the point (\bullet, \bullet') and such that for $\lambda, (\lambda_0, \lambda_1) \in \Lambda'$

$$\lambda(\bullet, \bullet') = \lambda(\bullet) \quad \text{and} \quad (\lambda_0, \lambda_1)(\bullet, \bullet') = \lambda_0(\bullet).$$

The index set of $\gamma(\mathcal{S} \otimes \mathcal{S})$ and $\gamma\mathcal{S} \otimes \gamma\mathcal{S}$ is $\Lambda * \Lambda = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1)$, where $\Lambda_0 = \Lambda_1 = \Lambda$ and the union is disjoint. We view Λ' as included in $\Lambda * \Lambda$ with the copy of Λ in Λ' identified with Λ_0 in $\Lambda * \Lambda$. This inclusion allows us to consider $\gamma(\mathcal{S} \otimes \mathcal{S})$ and $\gamma\mathcal{S} \otimes \gamma\mathcal{S}$ with the index set restricted to Λ' , which we do below. Clearly, any homomorphism $\gamma\mathcal{S} \otimes \gamma\mathcal{S} \rightarrow \gamma(\mathcal{S} \otimes \mathcal{S})$, when the two function arrays are indexed by $\Lambda * \Lambda$, is also a homomorphism when they are indexed by Λ' .

Now, to obtain the conclusion of the corollary, by Corollary 3.4, it suffices to produce a homomorphism $\mathcal{A}' \rightarrow \gamma(\mathcal{S} \otimes \mathcal{S})$ such that $D \times E$ is in the image of (\bullet, \bullet') . Let $f': \{\bullet'\} \rightarrow \gamma X$ be given by $f'(\bullet') = v$. Since, by Proposition 3.2, there is a homomorphism

$$(\rho, \pi): \gamma\mathcal{S} \otimes \gamma\mathcal{S} \rightarrow \gamma(\mathcal{S} \otimes \mathcal{S})$$

and $(D, E) \in \rho \circ (f \times f')(\bullet, \bullet')$, it suffices to show that

$$(f \times f', g): \mathcal{A}' \rightarrow \gamma\mathcal{S} \otimes \gamma\mathcal{S}$$

is a homomorphism. This amounts to showing that if $\lambda, \lambda_0, \lambda_1 \in \Lambda$ and $\lambda_0(\bullet) = \lambda_1(\bullet)$, then we have the following two equalities

$$\begin{aligned} \lambda((f \times f')(\bullet, \bullet')) &= g(\lambda(\bullet, \bullet')), \\ (\lambda_0, \lambda_1)((f \times f')(\bullet, \bullet')) &= g((\lambda_0, \lambda_1)(\bullet, \bullet')). \end{aligned} \quad (3.17)$$

We verify only the second equality, the first one being easier. To start, we note that

$$\lambda_0(f(\bullet)) = g(\lambda_0(\bullet)) = g(\lambda_1(\bullet)) = \lambda_1(f(\bullet)).$$

Using this equality and (3.16), we check the second equality in (3.17) by a direct computation as follows

$$\begin{aligned} (\lambda_0, \lambda_1)((f \times f')(\bullet, \bullet')) &= (\lambda_0, \lambda_1)(f(\bullet), f'(\bullet')) = (\lambda_0, \lambda_1)(f(\bullet), v) \\ &= \lambda_0(f(\bullet))\lambda_1(v) = \lambda_1(f(\bullet))\lambda_1(v) \\ &= \lambda_1(f(\bullet)) = \lambda_0(f(\bullet)) = g(\lambda_0(\bullet)) \\ &= g((\lambda_0, \lambda_1)(\bullet, \bullet')). \end{aligned} \quad \square$$

4. Monoid actions and infinitary Ramsey theorems

In this section, M will be a finite monoid.

We connect here the dynamical result of Section 2 with the algebraic/Ramsey theoretic result of Section 3. This connection is made possible by Corollary 4.1, which translates Theorem 2.4 (via Corollary 2.7) into a statement about the existence of a homomorphism needed for applications of Theorem 3.1. Ramsey theoretic consequences of Corollary 4.1 are investigated later in the section, in particular, a general Ramsey theoretic result, Corollary 4.3, is derived from it. We also introduce the notion of Ramsey monoid and give a characterization of those among almost R-trivial monoids. We use this characterization to determine which among the monoids I_n from Section 2.3 are Ramsey. This result implies an answer to Lupini's question [10] on possible extensions of Gowers' theorem. We derive some concrete Ramsey results from our general considerations. For example, we obtain the Furstenberg–Katznelson Ramsey theorem for located words.

4.1. Connecting Theorems 2.4 and 3.1. In order to apply Theorem 3.1, through Corollaries 3.4 and 3.5, one needs to produce appropriate homomorphisms between function arrays. We show how Theorem 2.4, through Corollary 2.7, gives rise to exactly such homomorphisms. This is done in Corollary 4.1. All the Ramsey theorems are results of combining Corollaries 3.4 and 3.5 with Corollary 4.1.

Let S be a partial semigroup. For $A \subseteq S$, we say that S is **A -directed** if for all $x_1, \dots, x_n \in S$ there exists $x \in A$ such that x_1x, \dots, x_nx are all defined. So S is directed as defined in [16] if it is S -directed. We say that $I \subseteq S$ is a **two-sided ideal in S** if it is nonempty and, for $x, y \in S$ for which xy is defined, $xy \in I$ if $x \in I$ or $y \in I$.

Recall the definitions of the function arrays from (3.2) and (3.3) in Section 3.1. Recall also from (2.11) the endomorphism action of M on the semigroup $\langle \mathbb{Y}(M) \rangle$ generated by $\mathbb{Y}(M)$ as in Section 2.5. Denoting this natural action by β and taking $y_0 \in \mathbb{Y}(M)$, we form the point based total function array $\langle \mathbb{Y}(M) \rangle(\beta)_{y_0}$. For the sake of simplicity, we denote it by

$$\langle \mathbb{Y}(M) \rangle_{y_0}.$$

The following corollary will be seen to be a consequence of Corollary 2.7.

COROLLARY 4.1. *Assume M is almost R -trivial. Let $\mathbf{y} \in \mathbb{Y}(M)$ be a maximal element, and let α be an endomorphism action of M on a partial semigroup S . Let $I \subseteq S$ be a two-sided ideal such that S is I -directed.*

(i) *There exists a homomorphism $(f, g): \langle \mathbb{Y}(M) \rangle_{\mathbf{y}} \rightarrow \gamma(S(\alpha))$ with*

$$I \in f(\bullet).$$

(ii) *Assume that M is the monoid $J(\emptyset, B)$ for a finite set B as defined in Section 2.3. If $E \subseteq S$ is a right ideal such that S is E -directed, then there exists a homomorphism $(f, g): \langle \mathbb{Y}(M) \rangle_{\mathbf{y}} \rightarrow \gamma(S(\alpha))$ and an ultrafilter $\mathcal{V} \in \gamma(S)$ such that*

$$I \in f(\bullet)$$

and, additionally,

$$E \in \mathcal{V} \quad \text{and} \quad f(\bullet) * \mathcal{V} = f(\bullet).$$

To prove Corollary 4.1, we will need the following lemma, whose proof is standard.

LEMMA 4.2. *Let S be a partial semigroup.*

(i) *Let I be a two-sided ideal in S such that S is I -directed. Then $\{\mathcal{U} \in \gamma S: I \in \mathcal{U}\}$ is a compact two-sided ideal in γS .*

(ii) *Let E be a right ideal in S such that S is E -directed. Then $\{\mathcal{U} \in \gamma S: E \in \mathcal{U}\}$ is a compact right ideal in γS .*

Proof. We give an argument only for (i); the argument for (ii) being similar.

For $x \in S$, let $S/x = \{y \in S: xy \text{ is defined}\}$.

Let $H = \{\mathcal{U} \in \gamma S: I \in \mathcal{U}\}$. Then, by definition, H is clopen. It is nonempty since, by I -directedness of S , the family $\{I\} \cup \{S/x: x \in S\}$ of subsets of S has the

finite intersection property, so it is contained in an ultrafilter, which is necessarily an element of H .

We check that $I \in \mathcal{U} * \mathcal{V}$ if $I \in \mathcal{U}$ or $I \in \mathcal{V}$. Assume first that $I \in \mathcal{U}$. For $x \in I$, $S/x \subseteq \{y: xy \in I\}$, therefore, since $S/x \in \mathcal{V}$, for each $x \in I$, we have $\{y: xy \in I\} \in \mathcal{V}$. So $I \subseteq \{x: \{y: xy \in I\} \in \mathcal{V}\}$. Since $I \in \mathcal{U}$, we get

$$\{x: \{y: xy \in I\} \in \mathcal{V}\} \in \mathcal{U}$$

which means $I \in \mathcal{U} * \mathcal{V}$. Assume now $I \in \mathcal{V}$. For each $x \in S$, we have $I \cap (S/x) \subseteq \{y: xy \in I\}$. Therefore, since $I, S/x \in \mathcal{V}$, we have $\{y: xy \in I\} \in \mathcal{V}$ for each $x \in S$. So

$$\{x: \{y: xy \in I\} \in \mathcal{V}\} = S \in \mathcal{U},$$

which means $I \in \mathcal{U} * \mathcal{V}$. □

Proof of Corollary 4.1. (i) We denote by γI the compact two-sided ideal $\{\mathcal{U} \in \gamma S: I \in \mathcal{U}\}$ from Lemma 4.2.

Observe that the action α naturally induces an action of M by continuous endomorphisms on γS . We call this resulting action $\gamma\alpha$. By Corollary 2.7, there exists a homomorphism $g: \langle \mathbb{Y}(M) \rangle \rightarrow \gamma S$ such that all maximal elements of $\mathbb{Y}(M)$ are mapped to $I(S)$. In particular, $g(\mathbf{y}) \in I(\gamma S)$. Since, by Lemma 4.2, γI is a compact two-sided ideal, we have $I(\gamma S) \subseteq \gamma I$. Thus, $g(\mathbf{y}) \in \gamma I$, that is,

$$I \in g(\mathbf{y}).$$

Note now that if we let $f(\bullet) = g(\mathbf{y})$, then $(f, g): \langle \mathbb{Y}(M) \rangle_{\mathbf{y}} \rightarrow (\gamma S)(\gamma\alpha)$ is a homomorphism. A quick check of definitions gives $(\gamma S)(\gamma\alpha) = \gamma(S(\alpha))$. Thus, (f, g) is as required.

(ii) We proceed constructing g and f as in point (i) above. So we have $I \in f(\bullet)$. We note that $\mathbb{X}(M)$ consists of two points: the R-class of 1_M and the R-class consisting of all elements of B . Thus, by the last sentence of Corollary 2.7, $f(\bullet)$ is a minimal idempotent in $\gamma(S)$. Now consider $J = \{\mathcal{U} \in \gamma S: E \in \mathcal{U}\}$, which by Lemma 4.2 is a compact right ideal in $\gamma(S)$. By Proposition 2.3(iii), there exists an idempotent $\mathcal{V} \in J$ with $f(\bullet) * \mathcal{V} = f(\bullet)$. Since $\mathcal{V} \in J$, we have $E \in \mathcal{V}$, which completes the proof. □

4.2. Ramsey theorems from monoids. Given a sequence (X_n) of sets, let

$$\langle (X_n) \rangle$$

consist of all finite sequences $x_1 \cdots x_k$ for $k \in \mathbb{N}$, for which there exist $m_1 < \cdots < m_k$ such that $x_i \in X_{m_i}$. We make this set into a partial semigroup by declaring the

product $(x_1 \cdots x_k)(y_1 \cdots y_l)$ of two such sequences defined if there exist $m_1 < \cdots < m_k < n_1 < \cdots < n_l$ such that $x_i \in X_{m_i}$ and $y_i \in X_{n_i}$ and then letting the product be equal to the concatenation $x_1 \cdots x_k y_1 \cdots y_l$ of the two sequences. Two special cases of this construction are usually considered.

(1) We fix a set X , and let $X_n = X$ for each n . In this case, $\langle\langle X_n \rangle\rangle$ is a semigroup, not just a partial semigroup, of all **words in X** with concatenation as the semigroup operation.

(2) We fix a set X , and let $X_n = \{n\} \times X$ for each n . In this case, $\langle\langle X_n \rangle\rangle$ is isomorphic to the partial semigroup of **located words in X** , that is, all partial functions from \mathbb{N} to X with finite domains, where two such functions f and g have their product defined if all elements of the domain of f are smaller than all elements of the domain of g , and the product is set to be the function whose graph is the union of the graphs of f and g . The identification

$$\langle\langle X_n \rangle\rangle \ni (m_1, x_1) \cdots (m_k, x_k) \rightarrow f$$

where the domain of f is $\{m_1, \dots, m_k\}$ and $f(m_i) = x_i$, establishes a canonical isomorphism between $\langle\langle X_n \rangle\rangle$ and the partial semigroup of located words.

Since making the assumptions as in (1) or (2) causes no simplification in our arguments, we work with the general definition of $\langle\langle X_n \rangle\rangle$ as above. We only note that a Ramsey statement (like the ones given later in this paper) formulated for located words as in (2) is stronger than the analogous statement formulated for words as in (1). One derives the latter from the former by applying the map associating with a located word f defined on a set $\{m_1, \dots, m_k\}$ the word $f(m_1) \cdots f(m_k)$.

A **pointed M -set** is a set X equipped with an action of M and a distinguished point x such that $Mx = X$. Let (X_n) be a sequence of pointed M -sets. The monoid M acts on $\langle\langle X_n \rangle\rangle$ in the natural manner:

$$a(x_1 \cdots x_k) = a(x_1) \cdots a(x_k), \quad \text{for } a \in M \quad \text{and} \quad x_1 \cdots x_k \in \langle\langle X_n \rangle\rangle.$$

Note that since $x_i \in X_{m_i}$ implies that $a(x_i) \in X_{m_i}$, the action above of each element of M is defined on each element of $\langle\langle X_n \rangle\rangle$. It is clear that this is an endomorphism action.

A sequence (w_i) of elements of $\langle\langle X_n \rangle\rangle$ is called **basic in $\langle\langle X_n \rangle\rangle$** if for all $i_1 < \cdots < i_k$, the product $w_{i_1} \cdots w_{i_k}$ is defined. We note that if each X_n is a pointed M -set for a monoid M and the sequence (w_i) is basic, then for $i_1 < \cdots < i_k$ and $a_1, \dots, a_k \in M$ the product $a_1(w_{i_1}) \cdots a_k(w_{i_k})$ is defined. For this reason no confusion will arise from using the word basic to describe certain sequences in $\langle\langle X_n \rangle\rangle$ and certain sequences in base sets of function arrays over $\langle\langle X_n \rangle\rangle$ as in Section 3.2.

We will register the following general result that follows from Corollary 3.4 and Corollary 4.1(i). This result allows us to control the color of words as in (4.1) by their ‘type’ as in (4.2).

COROLLARY 4.3. *Assume M is almost R -trivial. Let F be a finite subset of the semigroup $\langle \mathbb{Y}(M) \rangle$, whose operation is denoted by \vee . Let $\mathbf{y} \in \mathbb{Y}(M)$ be a maximal element of the forest $\mathbb{Y}(M)$. Given pointed M -sets X_n , $n \in \mathbb{N}$, for each finite coloring of $\langle (X_n) \rangle$, there exists a basic sequence (w_i) in $\langle (X_n) \rangle$ such that*

(i) *for each i , w_i contains the distinguished point of some X_n and*

(ii) *for each $i_0 < \dots < i_k$ and $a_0, \dots, a_k \in M$, the color of*

$$a_0(w_{i_0}) \cdots a_k(w_{i_k}) \tag{4.1}$$

depends only on

$$a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y}) \tag{4.2}$$

provided $a_0(\mathbf{y}) \vee \cdots \vee a_k(\mathbf{y}) \in F$.

Proof. We regard $S = \langle (X_n) \rangle$ as a partial semigroup with concatenation as a partial semigroup operation and with the natural action α of M . This leads to the function array $S(\alpha)$. Let I be the subset of $\langle (X_n) \rangle$ consisting of all words that contain a distinguished element of some X_n . It is clear that I is a two-sided ideal and that $\langle (X_n) \rangle$ is I -directed. By Corollary 4.1(i), there exists a homomorphism $(f, g): \langle \mathbb{Y}(M) \rangle_{\mathbf{y}} \rightarrow \gamma S(\alpha)$ with $I \in f(\bullet)$.

It is evident from the definition of $\langle \mathbb{Y}(M) \rangle$ that we can find a finite set $F' \subseteq \langle \mathbb{Y}(M) \rangle$ such that if $z_0, \dots, z_l \in \mathbb{Y}(M)$ and $z_0 \vee \cdots \vee z_l \in F$, then $z_k \vee \cdots \vee z_l \in F'$ for each $0 \leq k \leq l$. Now, from the existence of the homomorphism (f, g) , by Corollary 3.4, we get the existence of a basic sequence (w_i) in I such that, for $i_0 < \dots < i_l$ and $a_0, \dots, a_l \in M$, the color of $a_0(w_{i_0}) \cdots a_l(w_{i_l})$ depends only on $a_0(\bullet) \vee \cdots \vee a_l(\bullet)$ as long as $a_k(\bullet) \vee \cdots \vee a_l(\bullet) \in F'$, for each $0 \leq k \leq l$. Since this last condition is implied by $a_0(\bullet) \vee \cdots \vee a_l(\bullet) \in F$ and since for each $a \in M$, $a(\bullet) = a(\mathbf{y})$, we are done. \square

We also have the following result analogous to Corollary 4.3 for the monoid $J(\emptyset, B)$, as defined in Section 2.3, that follows from Corollary 3.4 and Corollary 4.1(ii).

COROLLARY 4.4. *Consider the monoid $J(\emptyset, B)$ for a finite set B . Let X_n , $n \in \mathbb{N}$ be pointed $J(\emptyset, B)$ -sets, and let E be a right ideal such that $\langle (X_n) \rangle$ is E -directed. For each finite coloring of $\langle (X_n) \rangle$, there exist a basic double sequence (v_i, v'_i) in*

$\langle\langle X_n \rangle\rangle$, with x occurring in each v_i and with $v'_i \in E$ for each i , such that the coloring is fixed on sequences of the form

$$b_0(v_{i_0})b'_0(v'_{j_0})b_1(v_{i_1})b'_1(v'_{j_1}) \cdots b_n(v_{i_n}) \quad (4.3)$$

for $b_0, b'_0, b_1, b'_1, \dots, b_n \in B$ and $i_0 \leq j_0 < i_1 \leq j_1 < \cdots < i_n$.

Proof. We start with analyzing $\mathbb{X}(M)$, $\mathbb{Y}(M)$, and $\langle\mathbb{Y}(M)\rangle$. The partial order $\mathbb{X}(M)$ consists of two elements: the R-class of 1_M , which we again denote by 1_M , and the common R-class of all $b \in B$, which we denote by \mathbf{b} . Clearly $\mathbf{b} \leq_{\mathbb{X}(M)} 1_M$. Thus, $\mathbb{Y}(M)$ has a unique maximal element $\{\mathbf{b}, 1_M\}$, which we denote by \mathbf{y} . Note that for each $b \in B$

$$b(\mathbf{y}) = \{\mathbf{b}\}.$$

Two conclusions important to us follow from this equality. First, computing in $\langle\mathbb{Y}(M)\rangle$, for $b_0, \dots, b_n \in B$, we get

$$b_0(\mathbf{y}) \vee \cdots \vee b_n(\mathbf{y}) = \{\mathbf{b}\}, \quad (4.4)$$

that is, the product $b_0(\mathbf{y}) \vee \cdots \vee b_n(\mathbf{y})$ does not depend on b_0, \dots, b_n . Second, in the point based function array $\langle\mathbb{Y}(M)\rangle_{\mathbf{y}}$ over M , for $b_0, b_1 \in B$, we have

$$b_0(\bullet) = b_0(\mathbf{y}) = b_1(\mathbf{y}) = b_1(\bullet), \quad (4.5)$$

that is, all elements of B are conjugate to each other (with the notion of conjugate as in Corollary 3.5).

Let I be the subset of $\langle\langle X_n \rangle\rangle$ consisting of all words that contain a distinguished element of some X_n . The set I is a two-sided ideal and that $\langle\langle X_n \rangle\rangle$ is I -directed. By Corollary 4.1(ii), there exist a homomorphism

$$(f, g): \langle\mathbb{Y}(M)\rangle_{\mathbf{y}} \rightarrow \gamma(\langle\langle X_n(B) \rangle\rangle)(\alpha)$$

and an ultrafilter $\mathcal{V} \in \gamma(\langle\langle X_n(B) \rangle\rangle)$ such that

$$I \in f(\bullet), \quad E \in \mathcal{V} \quad \text{and} \quad f(\bullet) * \mathcal{V} = f(\bullet).$$

Let F consist of one point $\{\mathbf{b}\} \in \langle\mathbb{Y}(M)\rangle$. So F is a finite subset of $\langle\mathbb{Y}(M)\rangle$. Given a finite coloring of $\langle\mathbb{Y}(M)\rangle$, Corollary 3.5 and the existence of the homomorphism (f, g) above imply that there exists a basic sequence (v_i, v'_i) in $\langle\langle X_n(B) \rangle\rangle$ with $v'_i \in E$ that is conjugate F - $\langle\mathbb{Y}(M)\rangle$ -tame. In view of the definition of F , (4.4), and (4.5) the coloring is fixed on sequences of the form (4.3). \square

Let (X_n) be a sequence of pointed M -sets for a finite monoid M . We say that (X_n) has the **Ramsey property** if for each finite coloring of $\langle\langle X_n \rangle\rangle$ there exists a basic sequence (w_i) in $\langle\langle X_n \rangle\rangle$ such that

- each w_i contains the distinguished element of X_n as an entry;
- all words of the form

$$a_0(w_{i_0}) \cdots a_l(w_{i_l}),$$

where $l \in \mathbb{N}$, $a_i \in M$ with at least one $a_i = 1_M$, are assigned the same color.

A monoid M is called **Ramsey** if each sequence of pointed M -sets has the Ramsey property.

We deduce from Corollary 4.3 the following result characterizing Ramsey monoids.

COROLLARY 4.5. (i) *If M is almost R -trivial and the partial order $\mathbb{X}(M)$ is linear, then M is Ramsey.*

(ii) *If $\mathbb{X}(M)$ is not linear, then the sequence of pointed M -sets $X_n = \mathbb{X}(M)$, with the canonical action of M and with the R -class of 1_M as the distinguished point, does not have the Ramsey property.*

Thus, if M is almost R -trivial, then M is Ramsey if and only if the partial order $\mathbb{X}(M)$ is linear.

Proof. (i) Fix a sequence of pointed M -sets (X_n) . We need to show that it has the Ramsey property. One checks easily that linearity of $\mathbb{X}(M)$ implies that there exists an order preserving M -equivariant embedding of $\mathbb{X}(M)$ to $\mathbb{Y}(M)$ mapping the top element of $\mathbb{X}(M)$ to a maximal element of $\mathbb{Y}(M)$ —map the R -class of a to the set of all predecessors of the class of a in $\mathbb{X}(M)$. We identify $\mathbb{X}(M)$ with its image in $\mathbb{Y}(M)$. Note that, by linearity of $\mathbb{X}(M)$, $\mathbb{X}(M) = \langle \mathbb{X}(M) \rangle$, so $\mathbb{X}(M)$ is a subsemigroup of $\langle \mathbb{Y}(M) \rangle$. Let y_0 be the top element of $\mathbb{X}(M)$, which is the R -class of $[1_M]$. Since $[a] \vee [1_M] = [1_M] \vee [a] = [1_M]$, for the R -class $[a]$ of each $a \in M$, it follows immediately from Corollary 4.3 that (X_n) has the Ramsey property. Since (X_n) was arbitrary, we get the conclusion of (i).

(ii) Let X_n , $n \in \mathbb{N}$, be the pointed M -sets described in the statement of (ii). Let $a, b \in M$ be two elements whose R -classes $[a]$ and $[b]$ are incomparable in $\mathbb{X}(M)$. Then $a \not\leq bM$ and $b \not\leq aM$, which implies that

$$[a] \notin b\mathbb{X}(M) \quad \text{and} \quad [b] \notin a\mathbb{X}(M). \tag{4.6}$$

We color $w \in \langle (X_n) \rangle$ with color 0 if $[a]$ occurs in w and its first occurrence precedes all the occurrences of $[b]$, if there are any. Otherwise, we color w with color 1. Let (w_i) be a basic sequence in $\langle (X_n) \rangle$ with the R -class $[1_M]$ of 1_M occurring in each w_i . Then, in $a(w_0)w_1$, $[a]$ occurs in $a(w_0)$ and, by (4.6), $[b]$ does not occur in $a(w_0)$. It follows that $a(w_0)w_1$ is assigned color 0. For similar reasons, $b(w_0)w_1$ is assigned color 1. Thus, the Ramsey property fails for (X_n) . \square

Using the characterization from Corollary 4.5, a more concrete characterization of Ramsey monoids among almost R-trivial monoids was recently given in [8]. It turns out that almost R-trivial monoids are rarely Ramsey. Note, however, that, by Corollary 4.3, one obtains Ramsey theorems even from non-Ramsey monoids; see the Furstenberg–Katznelson theorem in Section 4.3.

4.3. Some concrete applications. We will present detailed arguments for showing that

- (1) the Furstenberg–Katznelson theorem for located words and
- (2) the Hales–Jewett theorem for located left-variable words

follow from combining Corollaries 3.4 and 3.5 with Corollary 4.1, in case of (1) through Corollary 4.3. We also indicate how (3) Gowers’ theorem and (4) Lupini’s theorem are special cases of Corollary 4.3. We provide more details in the cases (1) and (2) since (1) appears to be new and the derivation of (2) appears to be the most subtle one involving an application of Corollary 3.5. We emphasize however that all these derivations essentially amount to translations and specifications of our general results.

1. Furstenberg–Katznelson’s theorem for located words. We state here the Furstenberg–Katznelson theorem for located words. The original version from [3] is stated in terms of words and, as explained in the beginning of Section 4.2, is implied by our statement. We refer the reader to [3] for the original version. What follows in this point is, in essence, a translation of a particular case of Corollary 4.3 to the language used to state the Furstenberg–Katznelson theorem.

We fix an element x , called a variable. For a set C with $x \notin C$, let

$$X_n(C) = \{n\} \times (C \cup \{x\}). \quad (4.7)$$

Fix now two finite disjoint sets A, B with $x \notin A \cup B$. If $w \in \langle X_n(A \cup B) \rangle$, x occurs in w , and $c \in A \cup B \cup \{x\}$, then

$$w[c] \quad (4.8)$$

is an element of $\langle X_n(A \cup B) \rangle$ obtained from w by replacing each occurrence of x by c .

A **reduced string in A** is a sequence $a_0 \cdots a_k$, possibly empty, such that $a_i \neq a_{i+1}$ for all $i < k$. With a sequence $c_0 \cdots c_k$ with entries in $A \cup B$, we associate a reduced string $\overline{c_0 \cdots c_k}$ in A as follows. We delete all entries coming from B thereby forming a sequence $c'_0 \cdots c'_{k'}$ for some $k' \leq k$. In this sequence, we replace each run of each element of A by a single occurrence of that element forming a sequence $c''_0 \cdots c''_{k''}$ with $k'' \leq k'$. This sequence is $\overline{c''_0 \cdots c''_{k''}}$.

Here is the statement of the Furstenberg–Katznelson theorem for located words.

Let F be a finite set of reduced strings in A . Color $\langle X_n(A \cup B) \rangle$ with finitely many colors. There exists a basic sequence (w_i) in $\langle X_n(B) \rangle$ such that x occurs in each w_i and, for each $n_0 < \dots < n_k$ and $c_0, \dots, c_k \in A \cup B$, the color of

$$w_{n_0}[c_0] \cdots w_{n_k}[c_k]$$

depends only on $\overline{c_0 \cdots c_k}$ provided $\overline{c_0 \cdots c_k} \in F$.

This theorem is obtained by considering the monoid $J(A, B)$ from Section 2.3. For brevity's sake, set

$$M = J(A, B).$$

Forgetting about the Ramsey statement for a moment, we make some computations in $\mathbb{Y}(M)$.

Observe that all elements of B are in the same R-class, which we denote by \mathbf{b} , the R-class of each element of A consists only of this element only, and the R-class of 1_M consists only of 1_M . So, with some abuse of notation, we can write

$$\mathbb{X}(M) = \{\mathbf{b}, 1_M\} \cup A.$$

We have that, for each $a \in A$,

$$\mathbf{b} \leq_{\mathbb{X}(M)} a \leq_{\mathbb{X}(M)} 1_M$$

and elements of A are incomparable with each other with respect to $\leq_{\mathbb{X}(M)}$. The action of M on $\mathbb{X}(M)$ is induced by the action of M on itself by left multiplication.

Pick $a_0 \in A$. Note that the sets

$$\{\mathbf{b}\}, \{\mathbf{b}, a\}, \quad \text{for } a \in A, \quad \text{and} \quad \{\mathbf{b}, a_0, 1_M\}$$

are in $\mathbb{Y}(M)$, and we write $\mathbf{b}, a, 1_0$ for these elements, respectively. We notice that

$$\mathbf{b} \leq_{\mathbb{Y}(M)} a, \quad \text{for all } a \in A, \quad \text{and} \quad \mathbf{b}, a_0 \leq_{\mathbb{Y}(M)} 1_0, \quad (4.9)$$

and $\leq_{\mathbb{Y}(M)}$ does not relate any other two of the above elements. Furthermore, 1_0 is a maximal element of $\mathbb{Y}(M)$. The action of M on these elements is induced by the left multiplication action of M on itself, so

$$a(1_0) = a \quad \text{and} \quad b(1_0) = \mathbf{b}, \quad \text{for } a \in A, b \in B.$$

Using relations (4.9), we observe that, for $c_0, \dots, c_k \in \{\mathbf{b}\} \cup A$, the product

$$c_0 \vee \cdots \vee c_k$$

in the semigroup of $\langle \mathbb{Y}(M) \rangle$ is equal to \mathbf{b} if $c_i = \mathbf{b}$, for each $i \leq k$, or is obtained from $c_0 \vee \cdots \vee c_k$ by removing all occurrences of \mathbf{b} and shortening a run of

each $a \in A$ to one occurrence of a , if $c_i \in A$, for some $i \leq k$. Thus, the map assigning to a sequence $c_0 \cdots c_k$ of elements of $A \cup B$ the element $c_0 \vee \cdots \vee c_k$ of $\mathbb{Y}(M)$ factors through the map $c_0 \cdots c_k \rightarrow \overline{c_0 \cdots c_k}$ giving an injective map $\overline{c_0 \cdots c_k} \rightarrow c_0 \vee \cdots \vee c_k$.

We note that M acts on $X_n(A \cup B)$ as follows. We identify $X_n(A \cup B)$ with M by identifying (n, x) with 1_M and (n, c) with c for $c \in A \cup B$. Since M acts on M by left multiplication, this identification gives an action of M on $X_n(A \cup B)$. We make (n, x) the distinguished element, thereby turning $X_n(A \cup B)$ into a pointed M -set.

We apply Corollary 4.3 to this sequence of pointed M -sets. In the statement of the theorem, we take $\mathbf{y} = 1_0$ and, for the finite subset $\langle \mathbb{Y}(M) \rangle$, we take

$$\{c_0 \vee \cdots \vee c_k : c_0 \cdots c_k \in F\}.$$

Now, an application of Corollary 4.3 gives a basic sequence (w'_i) in $\langle (X_n(A \cup B)) \rangle$. Let $w_i \in \langle X_n(B) \rangle$ be got from w'_i by replacing each value taken in A by x . By the discussion above, the sequence (w_i) is as required.

2. The Hales–Jewett theorem for left-variable words. The Hales–Jewett theorem for located words is just the Furstenberg–Katznelson theorem for located words with $A = \emptyset$. We state now and prove the Hales–Jewett theorem for located left-variable words as in [16, Theorem 2.37]. We use the notation as in (4.7) and (4.8). We call $w \in \langle (X_n(C)) \rangle$ **left-variable** if the first entry in the sequence w is of the form (n, x) for some $n \in \mathbb{N}$.

Let B be a finite set. For each finite coloring of $\langle (X_n(B)) \rangle$, there exists a basic sequence (w_i) in $\langle (X_n(B)) \rangle$ such that x does not occur in w_0 , each w_i with $i \geq 1$ is left-variable, and the color of the sequences

$$w_0 w_{n_0}[b_0] \cdots w_{n_k}[b_k], \tag{4.10}$$

with $0 < n_0 < \cdots < n_k$ and $b_0, \dots, b_k \in B$, is fixed.

We will use the monoid $M = J(\emptyset, B)$ and apply Corollary 4.4. Note that $M = B \cup \{1_M\}$ is in a bijective correspondence with $X_n(B)$ mapping each $b \in B$ to (n, b) and 1_M to (n, x) . We transfer the left multiplication action of M to $X_n(B)$ and make (n, x) the distinguished element of $X_n(B)$. Thus, $X_n(B)$ is a pointed M -set. We consider the semigroup $\langle (X_n(B)) \rangle$ with the induced endomorphism action of M . Let $E \subseteq \langle (X_n(B)) \rangle$ consist of all left-variable elements. Note that E is a right ideal in $\langle (X_n(B)) \rangle$ and $\langle (X_n(B)) \rangle$ is E -directed. Now Corollary 4.4 produces a basic double sequence (v_i, v'_i) in $\langle (X_n) \rangle$, with x occurring in each v_i and with $v'_i \in E$ for each i . Fix $b \in B$, and let

$$w_0 = v_0[b] \quad \text{and} \quad w_i = v'_{i-1}v_i \quad \text{for } i \geq 1.$$

Clearly x does not occur in w_0 and each w_i is left-variable for $i \geq 1$. The coloring is fixed on sequences of the form (4.10) since it is fixed on ones of the form (4.3).

3. Gowers' theorem. The monoid G_k is defined in Section 2.3. Gowers' Ramsey theorem from [4] (see [16, Theorem 2.22]) is obtained by applying Corollary 4.3 to $X_n = G_k$ with the left multiplication action and with the distinguished element 1_{G_k} . We note that $\mathbb{X}(G_k)$ is linear, and we apply Corollary 4.3 as in the proof of Corollary 4.5(i).

4. Lupini's theorem. Lupini's Ramsey theorem from [10] is an infinitary version of a Ramsey theorem found by Bartošova and Kwiatkowska in [1]. To prove it we consider the monoid I_k defined in Section 2.3. We take for $X_n = \{0, \dots, k-1\}$ with the natural action of I_k and the distinguished element $k-1$. The result is obtained by applying Lemma 2.5 and Theorem 3.1. We expand on this theme in Section 4.4.

4.4. The monoids I_n . We analyze here the monoids I_n , $n \in \mathbb{N}$, $n > 0$, defined in Section 2.3. As usual, we identify a natural number n with the set $\{0, \dots, n-1\}$. The monoid I_n is the monoid of all functions $f: n \rightarrow n$ such that $f(0) = 0$ and $f(i-1) \leq f(i) \leq f(i-1) + 1$, for all $0 < i < n$, taken with composition. We consider these monoids, on the one hand, to illustrate our notion of Ramsey monoids and, on the other hand, to answer a question of Lupini from [10]. We will prove the following theorem.

THEOREM 4.6. *The monoids I_n , for $n \geq 4$, are not Ramsey. The monoids I_1 , I_2 , and I_3 are Ramsey.*

We now state a theorem and question of Lupini [10] in our terminology. For $k \in \mathbb{N}$, let w_k be a finite word in the alphabet $n = \{0, 1, \dots, n-1\}$ that contains an occurrence $n-1$. Let $I_n(w_k)$ be equal to the set $\{f(w_k): f \in I_n\}$, where $f(w_k)$ is the word obtained from w_k by applying f letter by letter. We take $I_n(w_k)$ with the natural action of I_n and with w_k as the distinguished element. Note that if w_k is the word of length one whose unique letter is $n-1$, then $I_n(w_k) = n$ with the natural action of I_n on n .

THEOREM (Lupini [10]). Let $n > 0$, and let $w_k = (n-1)$. Then the sequence of pointed I_n -sets $(I_n(w_k))_k$ has the Ramsey property.

In [10], Lupini asked the following natural question: does $(I_n(w_k))_k$ have the Ramsey property for every choice of words w_k , $k \in \mathbb{N}$?

The following corollary to Theorem 4.6 answers this question in the negative.

COROLLARY 4.7. Let $n \geq 4$. For $k \in \mathbb{N}$, let $w_k = (01 \cdots (n - 1))$. Then the sequence of pointed I_n -sets $(I_n(w_k))_k$ does not have the Ramsey property.

Proof. By Theorem 4.6, I_n is not Ramsey for $n \geq 4$. It follows, by Corollary 4.5 and by R-triviality of I_n , that the sequence $X_k = I_n$, $k \in \mathbb{N}$, does not have the Ramsey property, where I_n is considered as a pointed I_n -set with the left multiplication action and with 1 as the distinguished element. Note that I_n is isomorphic as a pointed I_n -set with $I_n(01 \cdots (n - 1))$ as witnessed by the function

$$I_n \ni f \rightarrow f(01 \cdots (n - 1)) \in I_n(01 \cdots (n - 1)).$$

Thus, since $w_k = 01 \cdots (n - 1)$, the sequence $(I_n(w_k))_k$ does not have the Ramsey property. \square

We will give a recursive presentation of the monoid I_n that may be of some independent interest and usefulness for future applications. It will certainly make it easier for us to manipulate symbolically elements of I_n below. In the recursion, we will start with a trivial monoid and adjoin a tetris operation as in [4] at each step of the recursion.

First, we present a general extension operation that can be applied to certain monoids equipped with an endomorphism and a distinguished element. Let M be a monoid, let $f: M \rightarrow M$ be an endomorphism, and let $t \in M$ be such that for all $s \in M$ we have

$$st = tf(s). \tag{4.11}$$

Define

$$\mu(M, t, f)$$

to be the triple

$$(N, \tau, \phi),$$

where N is a monoid, τ is an element of N , and ϕ is an endomorphism of N that are obtained by the following procedure. Let N be the disjoint union of M and the set $\{\tau s: s \in M\}$, where τ is a new element and the expression τs stands for the ordered pair (τ, s) . For $s \in M$, we write $\tau^0 s$ for s and $\tau^1 s$ for τs . Define a function $\phi: N \rightarrow M \subseteq N$ by letting, for $s \in M$ and $e = 0, 1$,

$$\phi(\tau^e s) = t^e f(s),$$

where $t^e f(s)$ is a product computed in M . Define multiplication on N by letting, for $s_1, s_2 \in M$, and $e_1, e_2 = 0, 1$,

$$(\tau^{e_1} s_1) \cdot (\tau^{e_2} s_2) = \begin{cases} \tau^{e_1} (s_1 s_2), & \text{if } e_2 = 0, \\ \tau(\phi(\tau^{e_1} s_1) s_2), & \text{if } e_2 = 1, \end{cases}$$

where, on the right-hand side, $s_1 s_2$ and $\phi(\tau^{e_1} s_1) s_2$ are products computed in M . We write τ for $\tau 1_M$. Note that $s_1 \cdot s_2 = s_1 s_2$ for $s_1, s_2 \in M$, $\tau \cdot s = \tau s$ for $s \in M$ and $\tau \cdot \tau = \tau t$. We will omit writing \cdot for multiplication in N .

The following lemma is proved by a straightforward computation.

LEMMA 4.8. *N is a monoid, ϕ is an endomorphism of N , and, for all $\sigma \in N$, we have relation (4.11), that is,*

$$\sigma \tau = \tau \phi(\sigma).$$

Later, we will need the following technical lemma.

LEMMA 4.9. *For $\sigma \in N$ and $s \in M$, there exists $s' \in M$ such that $\tau s \sigma = \tau s s'$.*

Proof. If $\sigma \in M$, then we can let $s' = \sigma$. Otherwise, $\sigma = \tau s_0$ for some $s_0 \in M$. Note that

$$\tau s \sigma = \tau s \tau s_0 = \tau \tau f(s) s_0 = \tau t f(s) s_0 = \tau s t s_0,$$

and we can let $s' = t s_0$. □

By recursion, we define a sequence of monoids with distinguished elements and endomorphisms. Let M_1 be the unique one element monoid, let t_1 be its unique element, and let f_1 be its unique endomorphism. Assume we are given a monoid M_k for some $k \geq 1$ with an endomorphism f_k of M_k and an element t_k with (4.11). Define

$$(M_{k+1}, t_{k+1}, f_{k+1}) = \mu(M_k, t_k, f_k).$$

PROPOSITION 4.10. *For each $k \in \mathbb{N}$, $k > 0$, M_k is isomorphic to I_k .*

Proof. One views I_{k-1} as a submonoid of I_k , for $k > 1$, identifying I_{k-1} with its image under the isomorphic embedding $I_{k-1} \ni s \rightarrow s' \in I_k$, where

$$s'(i) = \begin{cases} 0, & \text{if } i = 0, \\ s(i-1) + 1, & \text{if } 0 < i < k. \end{cases}$$

One checks that $t_k \in I_k$ given by

$$t_k(i) = \begin{cases} 0, & \text{if } i = 0, \\ i - 1, & \text{if } 0 < i < k, \end{cases}$$

and $f_k: I_k \rightarrow I_k$ given by

$$f_k(s)(i) = \begin{cases} 0, & \text{if } i = 0, \\ s(i-1) + 1, & \text{if } 0 < i < k, \end{cases}$$

fulfill the recursive definition of (M_k, t_k, f_k) . □

Since, as proved in Section 2.3, I_n is R-trivial, the partial order $\mathbb{X}(I_n)$ can be identified with I_n . We will make this identification and write \leq_{I_n} for $\leq_{\mathbb{X}(I_n)}$. We have the following recursive formula for \leq_{I_n} . Obviously, \leq_{I_1} is the unique partial order on the one element monoid.

PROPOSITION 4.11. *Let $t_{n+1}^{e_1}s_1, t_{n+1}^{e_2}s_2 \in I_{n+1}$ with $s_1, s_2 \in I_n$ and $e_1, e_2 = 0, 1$. Then $t_{n+1}^{e_1}s_1 \leq_{I_{n+1}} t_{n+1}^{e_2}s_2$ if and only if*

$$e_2 \leq e_1 \quad \text{and} \quad s_1 \leq_{I_n} f_n^{e_1-e_2}(s_2).$$

Proof. By Proposition 4.10, we regard $(I_{n+1}, t_{n+1}, f_{n+1})$ as obtained from the triple (I_n, t_n, f_n) via operation μ . In particular, we regard I_n as a submonoid of I_{n+1} . We also have $f_{n+1}(s) = f_n(s)$, for $s \in I_n$.

(\Leftarrow) If $e_0 = e_1$, the implication is obvious. The remaining case is $e_2 = 0$ and $e_1 = 1$. In this case, we have $s_1 \leq_{I_n} f_n(s_2)$, that is, $s_1 = f_n(s_2)s'$ for some $s' \in I_n$. But then

$$t_{n+1}s_1 = t_{n+1}f_n(s_2)s' = s_2(t_{n+1}s'),$$

and $t_{n+1}s_1 \leq_{I_{n+1}} s_2$ as required.

(\Rightarrow) Note that it is impossible to have $s_1 \leq_{I_{n+1}} t_{n+1}s_2$ for $s_1, s_2 \in I_n$. Indeed, this inequality would give $s_1 = t_{n+1}s_2\sigma$ for some $\sigma \in I_{n+1}$, which would imply, by Lemma 4.9, that $s_1 = t_{n+1}s_2s'$ for some $s' \in I_n$. This is a contradiction since $s_2s' \in I_n$. Thus, $t_{n+1}^{e_1}s_1 \leq_{I_{n+1}} t_{n+1}^{e_2}s_2$ implies $e_2 \leq e_1$.

If $e_1 = e_2 = 0$, then we have $s_1 \leq_{I_{n+1}} s_2$, which means $s_1 = s_2\sigma$ for some $\sigma \in I_{n+1}$. If $\sigma \in I_n$, then $s_1 \leq_{I_n} s_2$, as required. Otherwise, $\sigma = \tau s'$ for some $s' \in I_n$, which gives

$$s_1 = s_2\tau s' = \tau(f_n(s_2)s'),$$

which is impossible since $f_n(s_2)s' \in I_n$.

If $e_1 = e_2 = 1$, then we have $t_{n+1}s_1 \leq_{I_{n+1}} t_{n+1}s_2$, which means $t_{n+1}s_1 = t_{n+1}s_2\sigma$ for some $\sigma \in I_{n+1}$. By Lemma 4.9, this equality implies $t_{n+1}s_1 = t_{n+1}(s_2s')$ for some $s' \in I_n$. Since $s_2s' \in I_n$, this equality gives $s_1 = s_2s'$, so $s_1 \leq_{I_n} s_2$.

The last case to consider is $e_1 = 1$ and $e_2 = 0$, that is, $t_{n+1}s_1 \leq_{I_{n+1}} s_2$. Then $t_{n+1}s_1 = s_2\sigma$ for some $\sigma \in I_{n+1}$. Note that $\sigma \notin I_n$, so $\sigma = t_{n+1}s'$ for some $s' \in I_n$. But then we have

$$t_{n+1}s_1 = s_2t_{n+1}s' = t_{n+1}f_n(s_2)s',$$

which implies $s_1 = f_n(s_2)s'$, that is, $s_1 \leq_{I_n} f_n(s_2)$. □

Proof of Theorem 4.6. It is easy to see from Proposition 4.11 that the orders $\leq_{I_1}, \leq_{I_2}, \leq_{I_3}$ are linear. So, by Corollary 4.5, I_1, I_2 , and I_3 are Ramsey.

By Corollary 4.5, it remains to check that the partial order (I_n, \leq_{I_n}) is not linear for $n \geq 4$. By Proposition 4.10, we regard $(I_{n+1}, t_{n+1}, f_{n+1})$ as obtained from

the triple (I_n, t_n, f_n) via operation μ . It follows from Proposition 4.11 that $\leq_{I_{n+1}}$ restricted to I_n is equal to \leq_{I_n} . Thus, it suffices to show that (I_4, \leq_{I_4}) is not linear. Note that the image of f_3 is equal to I_2 and I_2 has two elements. So there exists $s_0 \in I_3$ such that $f_3(s_0) \neq 1_{I_2}$. Thus, since $1_{I_2} = 1_{I_3}$, we get $f_3(s_0) <_{I_3} 1_{I_3}$. It then follows from Proposition 4.11 that t_4 and s_0 are not comparable with respect to \leq_{I_4} . Indeed,

$$t_4 = t_4^1 1_{I_3} \quad \text{and} \quad s_0 = t_4^0 s_0.$$

Since $0 < 1$, we have $s_0 \not\leq_{I_4} t_4$; since $1_{I_3} \not\leq_{I_3} f_3^{1-0}(s_0)$, we have $t_4 \not\leq_{I_4} s_0$. \square

The monoid I_4 is the first one among the monoids $I_n, n > 0$, that is not Ramsey. Using Propositions 4.10 and 4.11, one can compute \leq_{I_4} as follows. Since I_3 is linearly ordered, one can list the four elements of I_3 as $a_3 \leq_{I_3} a_2 \leq_{I_3} a_1 \leq_{I_3} 1$. Then I_4 is equal to the disjoint union $I_3 \cup t_4 I_3$ and the order \leq_4 is the transitive closure of the relations

$$\begin{aligned} a_3 &\leq_{I_4} a_2 \leq_{I_4} a_1 \leq_{I_4} 1; \\ t_4 a_3 &\leq_{I_4} t_4 a_2 \leq_{I_4} t_4 a_1 \leq_{I_4} t_4; \\ t_4 &\leq_{I_4} a_1; \\ t_4 a_1 &\leq_{I_4} a_3. \end{aligned}$$

One can check by inspection that $M_1 = I_4 \setminus \{t_4\}$ and $M_2 = I_4 \setminus \{a_2, a_3\}$ are submonoids of I_4 . They are R-trivial as submonoids of an R-trivial monoid [14]. One easily checks directly that \leq_{M_1} and \leq_{M_2} are linear, therefore, M_1 and M_2 are Ramsey by Corollary 4.5. Thus, I_4 is not itself Ramsey, but it is the union of two Ramsey monoids.

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