STOCHASTIC PENSION FUNDING: PROPORTIONAL CONTROL AND BILINEAR PROCESSES

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Université Laval, August 1999

Abstract

In this paper, we find explicit expressions for the moments of the fund level and the value of the total contribution when arithmetic or geometric rates of return are modeled by a moving average process of order q and when a proportional control is applied to the contributions. Our approach is based on the bilinear Markovian representation.

Keywords

Bilinear Markovian representation, geometric bilinear processes, moving average rate of return, pension funding.

1. Introduction

Defined benefit pension plans are considered in this paper. For these types of plans, actuarial valuations determine the annual amounts of contributions and reserves. The safe aspect of these plans, for employees, makes them very popular in many countries such as Canada, the Netherlands, the UK and the USA. Our valuation method is an individual one, where normal costs and actuarial liabilities are calculated separately for each member and are then summed to give total amounts for the whole population of the plan. Periodic valuation methods (here annual) rely on some demographic and financial hypothesis. Essentially, our hypothesis will be that, we have a pension plan with random rates of return, no inflation, a stationary population and a fixed valuation rate. In reality, actuarial hypotheses do not exactly come through. This has for effect of causing deficits (or surplus). In order to attenuate those deficits, a control is usually applied to contributions. Here, we use the proportional control which is common in Great Britain. The main purpose of this paper is to obtain, for our plan, expressions for the first two moments of the fund level and of the total contribution when arithmetic or geometric rates of return form a moving average process. This is achieved through fund levels which correspond to bilinear processes and geometric bilinear processes.

We begin this paper by introducing, in Section 2, the concepts of bilinear processes, geometric bilinear processes and bilinear Markovian representations. In Appendix, we explain how to formulate a bilinear Markovian representation for bilinear processes and geometric bilinear processes. In Subsection 2.3, we discuss the stationarity and the moments of processes having a bilinear Markovian representation.

In Section 3, we apply the theory of the previous section to our plan in order to study the stationarity and the moments of the fund level and of the contributions. In Subsection 3.1, we see how the fund level at time t, F_t, can correspond to a bilinear process or to a geometric bilinear process. Then, in the next subsection, we give the Markovian representation of $\{F_t\}$ when arithmetic or geometric rates of return are modeled by a moving average process of order 0, 1 or 2. In Subsection 3.3, using the bilinear Markovian representation of $\{F_t\}$, we study the stationarity and the moments of the fund level and of the contributions. Finally, in Subsection 3.4, we observe the variability of those processes for different scenarios.

In practice, it is well known that random rates of return happen to be a major cause of deficit or surplus. This is why pension funding with random rates of return has been a popular research subject in actuarial sciences for the two past decades. In discrete time, essentially, three models have been considered for rates of return:

- i.i.d. rates of return: Dufresne (1986, 1988, 1989, 1990, 1994), Haberman (1993b), Zimbidis and Haberman (1993), Cairns and Parker (1997).
- Autoregressive rates of return: Dufresne (1993), Haberman (1993a, 1994), Gerrard and Haberman (1996), Cairns and Parker (1997).
- Moving average rates of return: Bédard (1997), Bédard and Dufresne (1999), Dufresne (1990), Haberman and Wong (1997).

Haberman and Wong (1997) obtained explicit expressions for the first two moments of the fund level and of the contributions under the proportional control when geometric rates of return formed a moving average process of order 1 or 2. Their approach was based on the fact that the white noise included in their moving average process is supposed to be normal. In this paper, using the bilinear processes theory, we solve similar problems for arithmetic and geometric moving average rates of return of order $q \geq 0$ having a white noise not necessarily normal. Cairns and Parker (1997) established expressions for the first two moments of the fund level and of the contributions under the proportional control when arithmetic rates of return formed a stationary i.i.d. (independent, identically distributed) process with a mean rate of return different from the valuation rate. Here, the bilinear theory allows us to find expressions for those moments without the stationarity condition. However, unlike Cairns and Parker (1997), our approach does not allow us to solve explicitly optimization problems.

2. Bilinear processes, geometric bilinear processes and the bilinear Markovian representation

We first define bilinear processes and geometric bilinear processes. Then, we investigate the probabilistic structure of those processes through their bilinear Markovian representation. Bilinear processes were introduced in 1978 by Granger and Andersen and the theory on geometric bilinear processes has been developed in Bédard (1997). As you will notice in Section 3, the Markovian representation of those two non-linear time series happen to be very helpful in pension funding with arithmetic and geometric moving average rates of return.

2.1. Bilinear Processes and Geometric Bilinear Processes

Let $t \in \mathbb{Z}$. The sum $\sum_{l=n}^{m}$ and the product $\prod_{l=n}^{m}$ are defined as zero for $m < n$.

Definition 2.1: The process $\{X_t\}$ is a *bilinear process of order p, q, P, Q*, denoted: $\{X_t\} \sim \mathrm{BLg}(p, q, P, Q)$, if it satisfies

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + \sum_{h=0}^{q} b_h e_{t-h} + \sum_{j=0}^{Q} \sum_{k=1}^{P} \beta_{j,k} X_{t-k} e_{t-j} + \alpha \tag{2.1}$$

where $\{e_t\}$ is a sequence of i.i.d. random variables, usually but not always with zero mean, and $\{a_k\}$, $\{b_h\}$, $\{\beta_{j,k}\}$ and α are real constants.

Definition 2.2: The process $\{X_t\}$ is a *geometric bilinear process of order p, q, $\tilde{P}$* denoted: $\{X_t\} \sim \mathrm{BLg}(p, q, \tilde{P})$, if it satisfies:

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + r(e_t) + \left(\prod_{j=0}^{q} g_j(e_{t-j}) \right) \left(\sum_{k=1}^{\tilde{P}} b_k X_{t-k} + c \right) + \alpha \tag{2.2}$$

where $r : \mathbb{R} \to \mathbb{R}$ and $g_h : \mathbb{R} \to \mathbb{R}$, $h = 0, 1, ..., q$, are measurable functions in e_t, $\{e_t\}$ is a sequence of i.i.d. random variables, usually but not always with zero mean, and $\{a_k\}$, $\{b_k\}$, c and α are real constants.

Because Expressions (2.1) and (2.2) are not very tractable when we want to examine the probabilistic structure of bilinear processes and geometric bilinear processes, we usually work with their bilinear Markovian representation that is defined below.

2.2. Bilinear Markovian Representation

Definition 2.3: Let $\{X_t\}$ be a bilinear process or a geometric bilinear process. The *bilinear Markovian representation* of $\{X_t\}$, if it exists, is in the form of:

$$\begin{cases} Z_t = A(e_t)Z_{t-1} + H(e_t) \\ X_t = B(e_t)Z_{t-1} + C(e_t) \end{cases} \tag{2.3}$$

where X_t represents the output of the system at time t and where:
(a) $\{e_t\}$ is a sequence of i.i.d. random variables (not necessarily with zero mean);
(b) Z_t is a state vector of dimension $n \times 1$. This vector is not always uniquely defined;
(c) The matrices $A(e_t)$, $H(e_t)$ and $B(e_t)$ are respectively matrices of measurable functions in e_t of dimensions $n \times n$, $n \times 1$ and $1 \times n$; and $C(e_t)$ is a measurable function in e_t;
(d) $\{e_t\}$ is independent of $\{Z_{t-k}\}$, $k = 1, 2, \ldots$

The Representation (2.3) is said to be "bilinear" since it contains matrices which have for components measurable functions in e_t. Moreover, the present state of the System (2.3) (i.e. Z_t), together with the future inputs $\{e_{t+k},\ k = 1, 2, \ldots\}$ is sufficient to obtain $\{X_{t+k},\ k = 0, 1, \ldots\}$. Since the inputs are independent random variables, the Representation (2.3) is based on the Markov property, which implies that given the present state, the future of the system is independent of its past. Hence, the name bilinear "Markovian" representation for the two equations in (2.3). The Markovian representation is considered as a very general approach in the modeling of time series. Its Markovian property allows to have a better knowledge of the probabilistic structure of some processes such as bilinear processes and geometric bilinear processes. This representation plays an important role in time series, especially in Kalman filtering.

Theorem 2.4: Pham (1986): If $\{X_t\}$ is a bilinear process (Equation 2.1) then $\{X_t\}$ has a bilinear Markovian Representation (2.3).

Theorem 2.5: Bédard (1997): If $\{X_t\}$ is a geometric bilinear process (Equation 2.2) then $\{X_t\}$ has a bilinear Markovian Representation (2.3).

The proof of Theorems 2.4 and 2.5 are given in Appendix. They indicate the procedure for obtaining a bilinear Markovian Representation from a bilinear process and from a geometric bilinear process. We will refer to those proofs in Subsection 3.2.

2.3. Moments of Processes Satisfying a Bilinear Markovian Representation

Guégan (1987) studied processes $\{X_t\}$ having the bilinear Markovian Representation (2.3) where $E[C(e_t)] = 0$ and obtained results about the stationarity and the moments of those processes. In Bédard (1997), it was found that her theorems remain valid when $E[C(e_t)] \neq 0$. We will state three of her theorems. The first gives recursive expressions for the first two moments of processes $\{X_t\}$ given by the Representation (2.3). The two other theorems give explicit expressions for the first two moments of processes having a Representation (2.3) and which are first-order or second-order stationary.

We begin by presenting concepts that are necessary to the understanding of those theorems.

Definition 2.6: Nicholls and Quinn (1982): Let S and T be two matrices of order $m \times n$ and $p \times q$ respectively. The *Kronecker product* of *S with T*, $S \otimes T$, is the $mp \times nq$ matrix whose (i, j)th block is the $p \times q$ matrix $S_{ij}T$, if S_{ij} is the (i, j)th element of S.

Let M and N be two matrices.

vec M is the vector obtained from M by stacking its columns one on top of the other, in order, from left to right

$\rho(M)$ supposed that M is a squared matrice, $\rho(M)$ represents the maximum modulus of the eigenvalues of the matrix M

$\overline{M \otimes N} = E[M(e_t) \otimes N(e_t)]$

$\overline{MN} = E[M(e_t)N(e_t)]$

$\overline{M} = E[M(e_t)], \quad \forall t.$

$\overline{M \otimes N}$, $\overline{MN}$ and $\overline{M}$ are in fact functions of $\mu_e = E[e_t]$ and $\sigma_e^2 = \text{var}[e_t]$ where we omit μ_e and σ_e^2 in order to simplify the notations.

Definition 2.7: A process $\{X_t, t \in \mathbb{Z}\}$ is *first-order stationary* if

(a) $E[X_t]$ is constant,

and $\{X_t, t \in \mathbb{Z}\}$ is *second-order stationary* if condition (a) holds and if:

(b) $E[X_t X_s]$ depends only on the value of $(t - s)$.

Theorem 2.8: Guégan (1987): Suppose $\{X_t\}$ satisfies (2.3). Then

$$E[X_t] = \overline{B}E[Z_{t-1}] + \overline{C}, \tag{2.4}$$

https://doi.org/10.2143/AST.29.2.504615 Published online by Cambridge University Press

where

$$\begin{aligned}\mathrm{E}[Z_t] &= \overline{A}\mathrm{E}[Z_{t-1}] + \overline{H} \\ &= \left(I + \overline{A} + \ \dots \ + \overline{A}^{t-1}\right)\overline{H} + \overline{A}^t\mathrm{E}[Z_0] \\ &= \left(\overline{A} - I\right)^{-1}\left(\overline{A} - I\right)^t\overline{H} + \overline{A}^t\mathrm{E}[Z_0],\end{aligned} \tag{2.5}$$

and

$$\mathrm{E}\left[X_t^2\right] = \overline{B \otimes B}\,\mathrm{vec}\tilde{\Lambda}_{t-1} + 2\overline{B \otimes C}\mathrm{E}[Z_{t-1}] + \overline{C \otimes C} \tag{2.6}$$

where

$$\mathrm{vec}\tilde{\Lambda}_t = \mathrm{vecE}\left[Z_t Z_t'\right] = \overline{A \otimes A}\,\mathrm{vec}\tilde{\Lambda}_{t-1} + \left(\overline{H \otimes A} + \overline{A \otimes H}\right)\mathrm{E}[Z_{t-1}] + \overline{H \otimes H}. \tag{2.7}$$

Theorem 2.9: Guégan (1987): If $\rho(\overline{A}) < 1$, then the process $\{Z_t\}$, and consequently $\{X_t\}$ of Representation (2.3), are first-order stationary and

$$\begin{aligned}x &= \lim_{t\to\infty} \mathrm{E}[X_t] = \overline{B}z + \overline{C} \qquad \text{with} \\ z &= \lim_{t\to\infty} \mathrm{E}[Z_t] = \left(I - \overline{A}\right)^{-1}\overline{H}.\end{aligned}$$

Theorem 2.10: If $\rho(\overline{A}) < 1$ and $\rho\left(\overline{A \otimes A}\right) < 1$, then the process $\{Z_t\}$, and consequently $\{X_t\}$ in (2.3), are both second-order stationary and

(a) Guégan (1987): if $s = 0$:

$$\begin{aligned}\lim_{t\to\infty} \mathrm{cov}(X_t, X_{t+s}) &= \lim_{t\to\infty} \mathrm{var}[X_t] \\ &= \overline{B \otimes B}\lim_{t\to\infty} \mathrm{vec}\tilde{\Lambda}_t + 2\overline{B \otimes C}z + \overline{C \otimes C} - x^2\end{aligned}$$

where

$$\begin{aligned}\lim_{t\to\infty} \mathrm{vec}\tilde{\Lambda}_t &= \lim_{t\to\infty} \mathrm{vecE}\left[ZZ_t'\right] \\ &= \left(I - \overline{A \otimes A}\right)^{-1}\left[\left(\overline{A \otimes H} + \overline{H \otimes A}\right)z + \overline{H \otimes H}\right].\end{aligned}$$

(b) Bédard and Dufresne (1999): if $s \neq 0$:

$$\lim_{l\to\infty} \mathrm{cov}(X_t, X_{t+s}) = \overline{B}\left[\overline{A}^{|s|-1}\xi + \sum_{k=0}^{|s|-2}\overline{A}^k x\overline{H}\right] + x\overline{C} - x^2$$

https://doi.org/10.2143/AST.29.2.504615 Published online by Cambridge University Press

where

$$\xi = \overline{B \otimes A} \lim_{t \to \infty} \mathrm{vec}\tilde{\Lambda}_t + \left(\overline{C \otimes A} + \overline{B \otimes H}\right) z + \overline{H \otimes C}.$$

Remark: Based on Guégan's (1987) results, it is also possible to find expressions for the moments of even degree $k = 2n$, ($n \in \mathbb{N}$) for the k-order stationary processes $\{X_t\}$ satisfying (2.3).

3. Pension Funding: Proportional Control and Moving Average Rates of Return

3.1. Applications of Bilinear Processes and Geometric Bilinear Processes to Pension Funding

In the following, we apply the results of Section 2 to study the moments of the fund level and of the contributions for the pension plan described in the introduction. We consider this a defined benefit pension plan with an individual valuation method under Assumptions A1 to A4 given below. From now on we let $t \in \mathbb{N}$.

Let C, D and F be processes representing respectively the total contribution, the unfunded liability and the fund level. The symbols AL, B and NC refer to the actuarial liability, the benefits and the normal cost, respectively.

The unfunded liability at time t, D_t, is the excess of actuarial liabilities over assets (which may be positive or negative): $D_t = AL_t - F_t$.

The total contribution made at time t is given by

$$C_t = NC_t + ADJ_t \tag{3.1}$$

where NC_t and ADJ_t represent respectively the normal cost and the adjustment made to the contributions at time t.

Let R_t be the rates of return for the period $(t-1, t)$, $\delta_t = \ln(1 + R_t)$, $r = \mathrm{E}[R_t]$, $\delta = \mathrm{E}[\delta_t]$ and i be the valuation rate of interest. We assume that contributions and benefits are paid in full at the beginning of each year, and therefore

$$F_t = (1 + R_t)(F_{t-1} + C_{t-1} - B_{t-1}) \tag{3.2}$$

We make the following assumptions:

A1. $\{R_t - r\} \sim \mathrm{MA}(q)$ i.e. $R_t - r = \sum_{j=0}^{q} d_j e_{t-j}$,
where $d_0 = 1$, $\{d_j,\ j = 1, 2, \ldots, q\}$ are real constants, and $\{e_t, t > 0\}$ is a zero mean white noise process, or

A1'. $\{\delta_t - \delta\} \sim \text{MA}(q)$ i.e. $\delta_t - \delta = \sum_{j=0}^{q} d_j e_{t-j}$.

A2. There is no inflation on benefits and on salaries.
A3. The population is static.
A4. The valuation rate, i, is fixed.

Assumptions A2 to A4, which also have been made by Dufresne (1986, 1988, 1989, 1990, 1994), Haberman (1993a, b, 1994), Haberman and Wong (1997), Zimbidis and Haberman (1993) and Cairns and Parker (1997), are stronger than the ones usually met in practice. However, they allow us to detect more easily the effect of a random rate of return on the value of the fund and on the contributions. Moreover, Assumptions A2 to A4 imply that AL_t, NC_t and B_t do not depend on time.

Equivalently, as mentioned in Cairns and Parker (1997), one may suppose that salaries are inflated by imagining that the valuation rate and the random rate of return are linked to the rate of increase of salaries. For example, if s is the rate of increase of salaries, we could replace respectively R_t and i in all formulas by $H_t = \frac{1+R_t}{1+s} - 1$ and by $j = \frac{1+i}{1+s} - 1$. Approximately, H_t can be seen as the rate of return over the increase of salaries (i.e. $H_t = R_t - s$), and j as the valuation rate over the increase of salaries (i.e. $j = i - s$). In our numerical example of Subsection 3.4, the rates R_t and i can both be thought of as "net" rates.

When a proportional control is applied to the contributions, the adjustment made to the contributions at time t corresponds to a proportion of the deficit at time t:

$$ADJ_t = kD_t \tag{3.3}$$

where $k \in (0, 1]$.

Replacing the above equation in Expressions (3.1) and (3.2), we find under Assumptions A2 to A6 that the processes $\{C_t\}$ and $\{F_t\}$ are such that:

$$C_t = NC + kD_t, \qquad \text{and} \tag{3.4}$$

$$\begin{aligned} F_t &= (1 + R_t)(F_{t-1} + NC + k(AL - F_{t-1}) - B) \\ &= (1 + R_t)((1 - k)F_{t-1} + h), \end{aligned} \tag{3.5}$$

where $h = NC + kAL - B$.

Equation (3.5) allows us to establish the two following theorems:

Theorem 3.1: Dufresne (1990): If $\{R_t - r\} \sim \text{MA}(q)$, then $\{F_t\} \sim \text{BL}(1, q, 1, q)$.

Proof: We first replace $\{R_t - r\}$ by a moving average process of order q in (3.5):

$$\begin{aligned} F_t &= (1 + R_t)((1-k)F_{t-1} + h) \\ &= \left(1 + r + \sum_{j=0}^{q} d_j e_{t-j}\right)((1-k)F_{t-1} + h) \\ &= (1+r)(1-k)F_{t-1} + (1-k)\sum_{j=0}^{q} d_j e_{t-j}F_{t-1} + h\sum_{j=0}^{q} d_j e_{t-j} + (1+r)h \end{aligned}$$

where $d_0 = 1$.
Setting $a_1 = (1+r)(1-k)$, $b_j = hd_j$, $\beta_{j,1} = (1-k)d_j$ and $\alpha = (1+r)h$ in the preceding equation, we obtain

$$F_t = a_1 F_{t-1} + \sum_{j=0}^{q} b_j e_{t-j} + \sum_{j=0}^{q} \beta_{j,1} F_{t-1} e_{t-j} + \alpha \tag{3.6}$$

Thus $\{F_t\} \sim \mathrm{BL}(1, q, 1, q)$. □

Theorem 3.2: If $\{\delta_t - \delta\} \sim \mathrm{MA}(q)$, then $\{F_t\} \sim \mathrm{BL}g(0, q, 1)$ (Equation 2.2) where $g_j(e_{t-j}) = e^{d_j e_{t-j}}$ with $d_0 = 1$.

Proof: We replace $\{\delta_t - \delta\}$ by a moving average process of order q in (3.5):

$$F_t = (1 + R_t)((1-k)F_{t-1} + h)$$

$$F_t = e^{\delta + \sum_{j=0}^{q} d_j e_{t-j}}((1-k)F_{t-1} + h), \qquad (\delta_t = \ln(1 + R_t) \text{ and } d_0 = 1).$$

Setting $b_1 = e^{\delta}(1-k)$ and $c = e^{\delta}h$ in the above equation implies that:

$$F_t = \prod_{j=0}^{q} e^{d_j e_{t-j}}(b_1 F_{t-1} + c). \tag{3.7}$$

Thus $\{F_t\} \sim \mathrm{BL}g(0, q, 1)$ where $g_j(e_{t-j}) = e^{d_j e_{t-j}}$ and $r(e_t) = \alpha = 0$ (see Equation 2.2). □

3.2. Markovian Representation of Fund Levels

In this section, we apply the procedures given in the proofs of Theorems 2.4 and 2.5 in order to show how to obtain a bilinear Markovian Representation (2.3) for the process $\{F_t\}$ with $\{R_t - r\} \sim \mathrm{MA}(q)$ ($q = 0$ or 1) (Equation 3.6) and for the process $\{F_t\}$ with $\{\delta_t - \delta\} \sim \mathrm{MA}(2)$ (Equation 3.7).

3.2.1. Case: $\{R_t - r\} \sim \mathrm{MA}(0)$ *and* $\{R_t - r\} \sim \mathrm{MA}(1)$

For the case $\{R_t - r\} \sim \mathrm{MA}(1)$, Equation (3.6) results in:

$$F_t = a_1 F_{t-1} + b_0 e_t + b_1 e_{t-1} + \beta_{0,1} F_{t-1} e_t + \beta_{1,1} F_{t-1} e_{t-1} + \alpha.$$

And, the bilinear Markovian representation for $\{F_t\}$ is given by:

$$\begin{aligned} Z_t &= A(e_t) Z_{t-1} + H(e_t) \\ F_t &= B(e_t) Z_{t-1} + C(e_t), \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} Z_t &= \begin{pmatrix} F_t \\ a_1 F_t + b_1 e_t + \beta'_{1,0} F_t e_t \end{pmatrix} \\ &= \begin{pmatrix} \beta_{0,1} e_t & 1 \\ (a_1 + \beta'_{1,0} e_t)\beta_{0,1} e_t & a_1 + \beta'_{1,0} e_t \end{pmatrix} Z_{t-1} + \begin{pmatrix} b_0 e_t + \alpha \\ b_1 e_t + (a_1 + \beta'_{1,0} e_t)(b_0 e_t + \alpha) \end{pmatrix} \\ F_t &= (\beta_{0,1} e_t, 1) Z_{t-1} + (b_0 e_t + \alpha) \end{aligned}$$

where $\beta'_{1,0} = \beta_{1,1}$.

Remark: Naturally, when $\{R_t - r\} \sim \mathrm{MA}(0)$, the Markovian Representation is obtained by setting $b_1 = 0$ and $\beta_{0,1} = 0$ in the representation above.

3.2.2. Case: $\{\delta_t - \delta\} \sim \mathrm{MA}(2)$

When $\{\delta_t - \delta\} \sim \mathrm{MA}(2)$, Equation (3.7) becomes

$$F_t = e^{d_0 e_t} e^{d_1 e_{t-1}} e^{d_2 e_{t-2}} \, (b_1 F_{t-1} + c).$$

With the later equation, we obtain the following bilinear Markovian representation for $\{F_t\}$:

$$\begin{aligned} Z_t &= A(e_t) Z_{t-1} + H(e_t) \\ F_t &= B(e_t) Z_{t-1} + C(e_t), \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} Z_t &= \begin{pmatrix} Z_t^{(0)} \\ Z_t^{(0)} e^{d_2 e_t} \\ e^{d_2 e_t} \end{pmatrix} \\ &= \begin{pmatrix} A^{(0)}(e_t) & C^{(0)}(e_t) & D^{(0)}(e_t) \\ A^{(0)}(e_t) e^{d_2 e_t} & C^{(0)}(e_t) e^{d_2 e_t} & D^{(0)}(e_t) e^{d_2 e_t} \\ 0_{1\times 2} & 0_{1\times 2} & 0 \end{pmatrix} Z_{t-1} + \begin{pmatrix} B^{(0)}(e_t) \\ B^{(0)}(e_t) e^{d_2 e_t} \\ e^{d_2 e_t} \end{pmatrix} \\ F_t &= \left(0,\ e^{d_0 e_t},\ 0,\ 0,\ 0\right) Z_{t-1} + 0 \end{aligned}$$

https://doi.org/10.2143/AST.29.2.504615 Published online by Cambridge University Press

where $0_{1\times 2} = (0,\ 0)$ and

$$Z_t^{(0)} = \begin{pmatrix} F_t \\ e^{d_1 e_t} e^{d_2 e_{t-1}}(b_1 F_t + c) \end{pmatrix}$$
$$= A^{(0)}(e_t) Z_{t-1}^{(0)} + B^{(0)}(e_t) + \left(C^{(0)}(e_t) Z_{t-1}^{(0)} + D^{(0)}(e_t)\right) e^{d_2 e_{t-1}}$$
$$= \begin{pmatrix} 0 & e^{d_0 e_t} \\ 0 & 0 \end{pmatrix} Z_{t-1}^{(0)} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \left(\begin{pmatrix} 0 & 0 \\ 0 & b_1 e^{(d_0+d_1)e_t} \end{pmatrix} Z_{t-1}^{(0)} + \begin{pmatrix} 0 \\ c e^{d_1 e_t} \end{pmatrix} \right) e^{d_2 e_{t-1}}.$$

3.3. Moments of $\{F_t\}$ and $\{C_t\}$

Proposition 3.3: Under Assumptions A1 to A4, if a proportional control is applied to the contributions and that $\{F_t\}$ is a second-order stationary process, it follows from Theorems 2.9 and 2.10, the following explicit expressions for the first two moments of $\{F_t\}$ and $\{C_t\}$:

$$f = \lim_{t\to\infty} \mathrm{E}[F_t] = \overline{B}(I - \overline{A})^{-1}\overline{H} + \overline{C}, \tag{3.8}$$

$$\lim_{t\to\infty} \operatorname{cov}[F_t, F_{t+s}]$$
$$= \begin{cases} \overline{B \otimes B} \lim_{t\to\infty} \operatorname{vec}\tilde{\Lambda}_t + 2\overline{B \otimes C}(I - \overline{A})^{-1}\overline{H} + \overline{C \otimes C} - f^2 & \text{if} \quad s = 0, \\ \overline{B}\left[\overline{A}^{|s|-1}\xi + \sum_{k=0}^{|s|-2} \overline{A}^k f \overline{H}\right] + f\overline{C} - f^2 & \text{if} \quad s > 0, \end{cases} \tag{3.9}$$

where $\lim_{t\to\infty} \operatorname{vec}\tilde{\Lambda}_t = \left(I - \overline{A \otimes A}\right)^{-1}\left[\left(\overline{A \otimes H} + \overline{H \otimes A}\right)\left(I - \overline{A}\right)^{-1}\overline{H} + \overline{H \otimes H}\right]$ and $\xi = \overline{B \otimes A} \lim_{t\to\infty} \operatorname{vec}\tilde{\Lambda}_t + \left(\overline{C \otimes A} + \overline{B \otimes H}\right)\left(I - \overline{A}\right)^{-1}\overline{H} + \overline{H \otimes C}$.
And, from Equation (3.4):

$$\lim_{t\to\infty} \mathrm{E}[C_t] = NC + k(AL - f) \quad \text{and} \tag{3.10}$$

$$\lim_{t\to\infty} \operatorname{cov}[C_t, C_{t+s}] = k^2 \lim_{t\to\infty} \operatorname{cov}[F_t, F_{t+s}], \qquad s \geq 0. \tag{3.11}$$

Proof: In Subsection 3.2 we have shown, under Assumptions A1 to A4, that when the adjustment made to the contributions is a proportional control, $\{F_t\}$ has a bilinear Markovian Representation (2.3). Using this representation, we apply Theorems 2.9 and 2.10 to obtain the above formulas for the first two moments of $\{F_t\}$. Then, with Expression (3.4), we find the moments of $\{C_t\}$. □

Remark: Obviously, since $\{F_t\}$ has a bilinear Markovian representation, it is always possible to calculate the moments of $\{F_t\}$ and $\{C_t\}$ recursively. For example, for the first two moments, we use formulas (2.12), (2.14) and (3.4).

Remark: The first-order moments of F_t and C_t exist only if the process $\{e_t\}$ is such that the expectations: $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{H}$ exist. And, the second-order moments of F_t and C_t exist only if the process $\{e_t\}$ is such that the expectations: $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{H}$, $\overline{A \otimes A}$, $\overline{B \otimes B}$, $\overline{C \otimes C}$, $\overline{H \otimes H}$, $\overline{B \otimes C}$, $\overline{A \otimes H}$, $\overline{H \otimes A}$, $\overline{B \otimes A}$, $\overline{C \otimes A}$, $\overline{B \otimes H}$ and $\overline{H \otimes C}$ exist.

3.4. Numerical Results

In this section, we use the theory of the previous subsection to study the variability of the fund level and of the contributions. This is done in the context of arithmetic moving average rates of return of order 0, 1 or 2.

Our proportional control is the "spread control" which is commonly used in the UK and which has been investigated by Cairns and Parker (1997), Dufresne (1986, 1988, 1990, 1994), Gerrard and Haberman (1996), Haberman (1993a, 1994) and Haberman and Wong (1997). With this method, the deficit is spread over a certain number of years M (in practice: around 20-25 years), i.e.

$$ADJ_t = kD_t = \frac{1}{\ddot{a}_{\overline{M}|}} D_t = \frac{\frac{i}{1+i}}{1-(1+i)^{-M}} D_t,$$

where $M \in \mathbb{N}$ and i is the valuation rate. Hence, the formulas for F_t and C_t are obtained by setting

$$k = \frac{1}{\ddot{a}_{\overline{M}|}}$$

in Expressions (3.5) and (3.4), respectively.

According to Equations (3.9) and (3.11), var$[F_t]$ and var$[C_t]$ are both functions of the fraction k and implicitly, of the spread period M. In studies such as Cairns and Parker (1997), Dufresne (1986, 1988, 1994), Haberman (1994) and Haberman and Wong (1997), it was shown that for i.i.d. rates of return, autoregressive rates of return and moving average rates of return, there is usually a value M^* at which var$[C_t]$ reaches its minimum. And, usually for all M between 1 and that M^*, var$[F_t]$ increases and var$[C_t]$ decreases as the period M increases and, when $M > M^*$, var$[F_t]$ and var$[C_t]$ both increases as M increases. This is why the period $[1, M^*]$ is the optimal spread period for an actuary.

In the following, we study the variability of the fund level and of the contributions with respect to the spread period M and the moving average parameters d_1 and d_2.

In Examples 1 to 4, we consider arithmetic rates of return when $\sqrt{\text{var}[R_t]} = 0.05$ or 0.1. As you will see, for all those examples, results indicate that we have optimal spread periods $[1, M^*]$.

Since $\sqrt{\text{var}[F_t]}$ and $\sqrt{\text{var}[C_t]}$ are proportional to AL and that we only want to minimize $\sqrt{\text{var}[F_t]}$ and $\sqrt{\text{var}[C_t]}$, we therefore proceed as in Dufresne (1986, 1988, 1994), i.e. we calculate

$$\frac{\sqrt{\text{var}[F_t]}}{AL} \quad \text{and} \quad \frac{\sqrt{\text{var}[C_t]}}{NC}.$$

In Figures 3-1 to 3-4, those values are given in percent of AL and NC, respectively, where $AL = 451\%$ and $NC = 14.5\%$ of the payroll.

For the MA(q) processes considered in our calculations, we set the values of the parameters d_1, d_2, ..., d_q and $\text{var}[R_t]$. We suppose that $\{e_t\} \sim$ Beta (2,2) over (−2,2), this is, a density equal to

$$\frac{3}{4b^3}(b^2 - x^2)I_{(-b,b)}(x);$$

where the value of b is determined by the following relation:

$$\text{var}[e_t] = \frac{b^2}{5} = \frac{\text{var}[R_t]}{\left(1 + d_1^2 + d_2^2 + \ \dots \ + d_q^2\right)}.$$

As it is often the case in actuarial papers dealing with random rates of return, we suppose in our calculations, except for Example 2, that the mean rate of return is equal to the valuation rate, i.e. $r = i$.

Example 1

We study the process $\{F_t\}$ given by (3.5) when rates of return are i.i.d. and have a mean rate, r, equal to the valuation rate of return i. Our calculations support Dufresne's (1994) conclusions. We have optimal spread periods $[1, M^*]$; this is, for all M between 1 and M^*, $\text{var}[F_t]$ increases and $\text{var}[C_t]$ decreases as the period M increases.

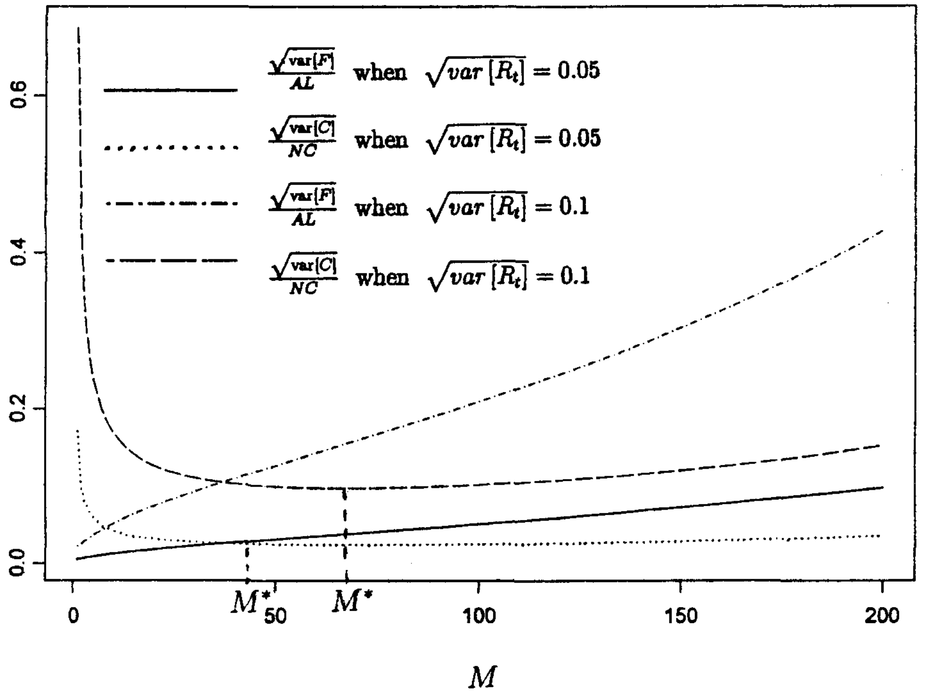

FIGURE 3-1: Case: $\{R_t - r\} \sim \text{MA}(0)$, $r = i = 0.01$

Example 2

Here, we also consider i.i.d. rates of return. But, unlike Example 1, the mean rate of return, r, is greater than the valuation rate, i. Increasing r to 0.02 increases var$[F]$ and var$[C]$ and reduces significantly M^*.

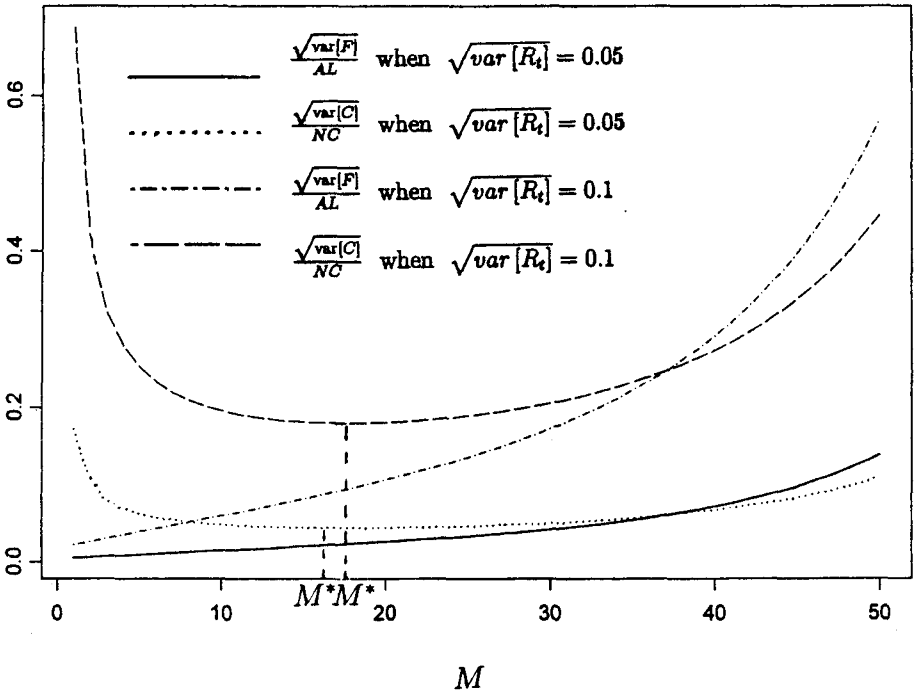

FIGURE 3-2: Case: $\{R_t - r\} \sim$ MA(0), $r = 0.02$, $i = 0.01$

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Example 3
We set $R_t = r + e_t + e_{t-1}$ in order to have $\mathrm{Corr}[R_t, R_{t-1}] = \frac{1}{2} > 0$. As expected, according to Figure 3-1, this positive correlation between successive rates of return increases the variances of the fund levels and of the contributions.

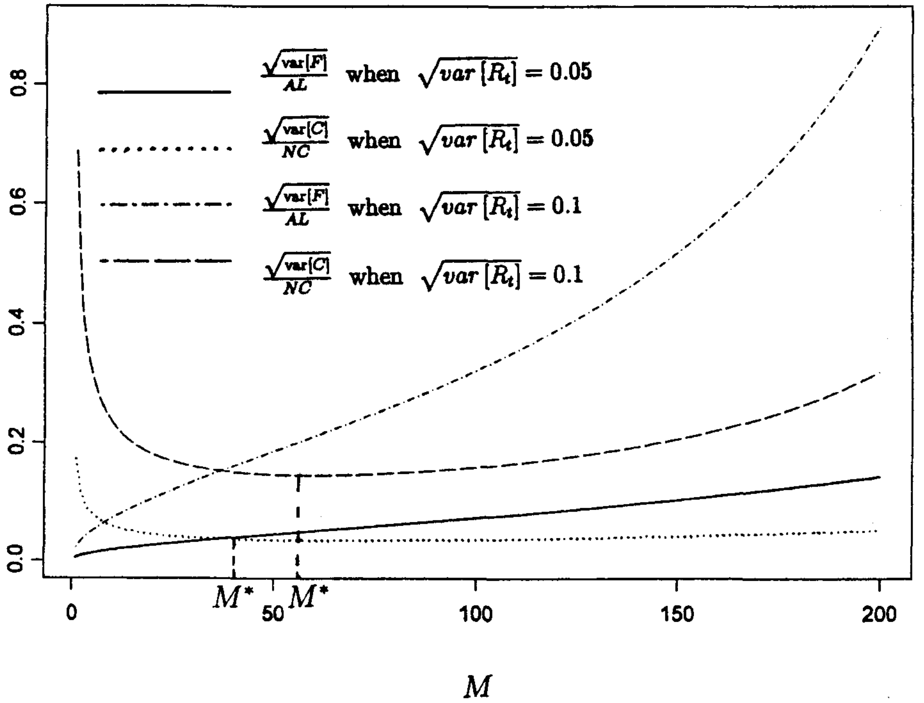

FIGURE 3-3: Case: $\{R_t - r\} \sim \mathrm{MA}(1)$, $r = i = 0.01$, $d_1 = 1$

Example 4
We set $R_t = r + e_t + e_{t-1} + 0.3e_{t-2}$ and naturally, increasing d_2 to 0.3 increases the variances of both the fund level and of the contributions.

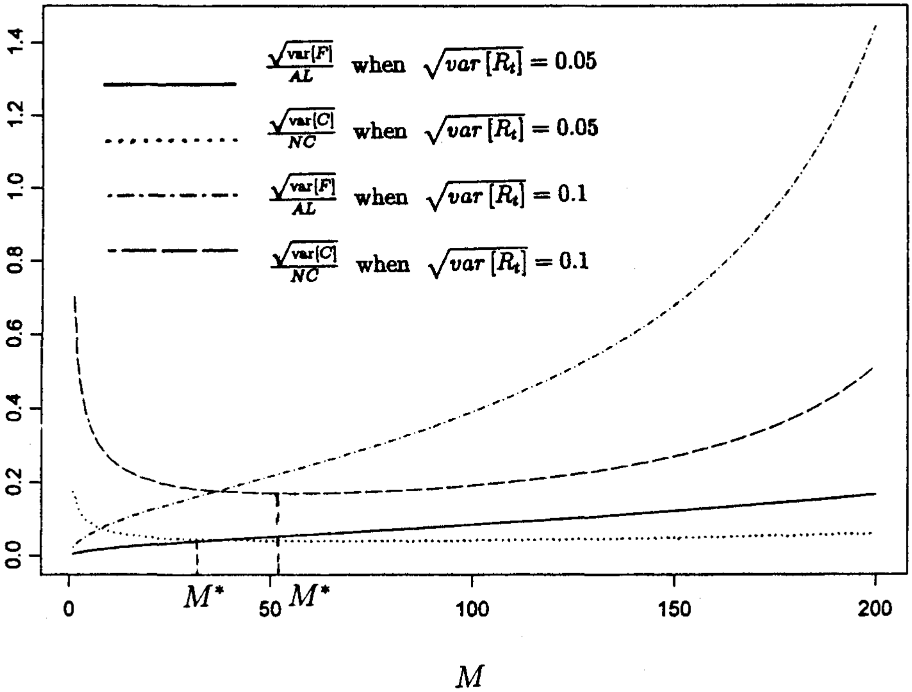

FIGURE 3-4: Case: $\{R_t - r\} \sim \text{MA}(2)$, $r = i = 0.01$, $d_1 = 1$, $d_2 = 0.3$

3.4.1. Analysis of results

Let $A(e_t)$ be the matrix of Representation (2.3). We observe that for $|d_1 \leq 1|$, $|d_2 \leq 1|$, $M \leq 200$, $0 < i,\ r,\ \delta < 0.5$ and $\{e_t\} \sim \text{N}(0, \text{var}[e_t] < \infty)$, we usually have that $\rho(\overline{A}) < 1$ and $\rho(\overline{A \otimes A}) < 1$, i.e. the processes $\{F_t\}$ and $\{C_t\}$ are second-order stationary. This implies that for this range of parameters, we usually have explicit expressions for the first two moments of $\{F_t\}$ and $\{C_t\}$ which are given by (3.8), (3.9), (3.10) and (3.11).

For all scenarios of Figures 3-1 to 3-4, it results that $\rho(\overline{A}) < 1$ and $\rho(\overline{A \otimes A}) < 1$. This means that for those scenarios $\{F_t\}$ and $\{C_t\}$ are second-order stationary processes and that we can use the explicit formulas (3.8) to (3.11) to calculate, $\text{E}[F_t]$, $\text{E}[C_t]$, $\text{var}[F_t]$ and $\text{var}[C_t]$.

From Figures 3-1 to 3-4, we make the following conclusions:

(a) As mentioned earlier, we have an optimal spread period for all examples, i.e. we have a range of M where $\text{var}[F_t]$ increases and $\text{var}[C_t]$ decreases as the spread period M increases and the other parameters stay constant. This means that for each example, there is an increase of the spread period which implies an increase of the variability of the fund level and a decrease of the variability of the contributions. What is interesting is that this period is always longer than 15 years. From Figures 3-1 to 3-4, we observe that increasing $\sqrt{\text{var}[R_1]}$ from 0.05 to 0.1 increases significantly var$[F_t]$, var$[C_t]$ and also the optimal period $[1, M^*]$.

(b) As we could naturally expect, the variability of the fund level and of the contributions seems to increase with the moving average parameters d_1 and d_2.

(c) We made analogue calculations for geometric rates of return and we observed similar results than those obtained for arithmetic rates of return.

4. Conclusion

Haberman and Wong (1997) obtained, under the proportional control, explicit expressions for the first two moments of the fund level and of the contributions when geometric rates of return formed a moving average process of order 1 or 2. Their method was based on the fact that the white noise included in their moving average processes is supposed to be normal. Here, we use the bilinear Markovian representation to find similar results. We found, under the proportional control, explicit and recursive expressions for the moments of the fund level and of the contributions when arithmetic or geometric rates of return were modeled by a moving average process of order $q \geq 0$ with a white noise not necessarily normal. It was Dufresne (1990) who suggested the Markovian approach. Here, the Markovian approach happened to be remarkable since it allowed us to easily resolve problems which were considered difficult. Unfortunately, this approach has some limitations. It cannot really be used for autoregressive rates of return since it would involve infinite order matrices and it does not allow to solve explicitly optimization problems.

Acknowledgments

The author would like to express her thanks to the referees and to Prof. Daniel Dufresne for their helpful comments. The author is also grateful to the Chaire en Assurance l'Industrielle-Alliance and to the National Sciences and Engineering Research Council of Canada for their financial support.

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APPENDIX

Proof of Theorem 2.4:

The aim is to find a procedure which allows to obtain a bilinear Markovian Representation (2.3) for the bilinear process (Equation 2.1):

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + \sum_{h=0}^{q} b_h e_{t-h} + \sum_{j=0}^{Q} \sum_{k=1}^{P} \beta_{j,k} X_{t-k} e_{t-j} + \alpha.$$

Let $(*) = \sum_{j=0}^{Q} \sum_{k=1}^{P} \beta_{j,k} X_{t-k} e_{t-j}$. It is easier to obtain a Representation (2.3) when we first split the double sum $(*) = \sum_{j=0}^{Q} \sum_{k=1}^{P} \beta_{j,k} X_{t-k} e_{t-j}$ into three parts.

The first part is the term $e_t \sum_{k=1}^{P} \beta_{0,k} X_{t-k}$. The second part consists in the terms in (*) where the subscript of X is less than or equal to the subscripts of e (cases where $1 \leq j \leq k$). The remaining terms form the third part of (*). Define P', Q', $\beta'_{k,j}$ and $\beta''_{k,j}$ as follows (for $k \geq 1$):

$$P' = Q - 1$$

$$Q' = \min\left[P, \max\left\{k | \beta_{j+k,k} \neq 0 \text{ for at least one value of } j,\ j = 1,\ 2,\ \ldots,\ P'\right\}\right]$$

$$\beta'_{k,j} = \begin{cases} \beta_{k,j+k} & \text{if } j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta''_{k,j} = \begin{cases} \beta_{j+k,k} & \text{if } j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then Equation (2.1) becomes

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + b_0 e_t + \sum_{h=1}^{q} b_h e_{t-h} + e_t \sum_{k=1}^{P} \beta_{0,k} X_{t-k}$$
$$+ \sum_{j=0}^{P} \sum_{k=1}^{Q} \beta'_{k,j} X_{t-k-j} e_{t-k} + \sum_{j=1}^{P'} \sum_{k=1}^{Q'} \beta''_{k,j} X_{t-k} e_{t-j-k} + \alpha. \qquad \text{(A.1)}$$

Let $n = \max(p, P+q, P+Q, P+Q')$ and $m = n - \max(q, Q, Q')$, and define a vector $Z_t^{(0)}$ as:

$$Z_{t,[i]}^{(0)} = X_{t-m+i}, \qquad 1 \leq i \leq m,$$

if $n - m = 0$ and $\max(p, P) = 1$: $Z_{t,[m+1]}^{(0)} = \sum_{k=1}^{p} a_k X_{t-k+1}$,

$$\text{if } n - m \geq 1 \ : \ Z_{t,[m+i]}^{(0)} = \sum_{k=1}^{p} a_k X_{t-k+i} + \sum_{k=i}^{n-m} \left[b_k + \sum_{j=0}^{P} \beta'_{k,j} X_{t+i-j-k} \right] e_{t+i-k}$$

$$+ \sum_{k=i}^{n-m} \left[\sum_{j=1}^{P'} \beta''_{k,j} e_{t+i-j-k} \right] X_{t+i-k}, \qquad 1 \leq i \leq n-m.$$

We obtain

$$X_t = \left(\sum_{j=m-P+1}^{m} \beta_{0,m-j+1} Z_{(t-1),[j]}^{(0)} \right) e_t + Z_{(t-1),[m+1]}^{(0)} + b_0 e_t + \alpha \qquad \text{(A.2)}$$

and

$$Z_t^{(0)} = A^{(0)}(e_t)Z_{t-1}^{(0)} + B^{(0)}(e_t) + \sum_{j=1}^{P'} \left[C_j^{(0)}(e_t)Z_{t-1}^{(0)} + D_j^{(0)}(e_t)\right] e_{t-j}, \qquad \text{(A.3)}$$

where $A^{(0)}(e_t)$, $B^{(0)}(e_t)$, $C_j^{(0)}(e_t)$ and $D_j^{(0)}(e_t)$ are matrices or vectors in e_t. Let $F^{(k)}(t) = (e_t, \ldots\ldots, e_{t+k-P'})'$, $\quad k = 1, \ldots, P'$, and define

$$Z_t^{(k)} = \left(Z_t^{(k-1)}, Z_t^{(k-1)} \otimes F^{(k)}(t), F^{(k)}(t)\right)', \qquad k = 1, \ldots, P'. \qquad \text{(A.4)}$$

Then, we claim that

$$Z_t^{(k-1)} = M^{(k)}(e_t)Z_{t-1}^{(k)} + N^{(k)}(e_t), \qquad k = 1, \ldots, P', \qquad \text{(A.5)}$$

where $M^{(k)}(e_t)$ and $N^{(k)}(e_t)$ are matrices of finite degree polynomials in e_t. Using the definition of $Z_t^{(k)}$, we can show by induction that Equation (A.5) is valid. Indeed, according to Expression (A.3), the equation is verified for $k = 1$. And if Equation (A.5) is true for k then it will be valid for $k + 1$ since $F_{[1]}^{(k)}(t) = e_t$, and

$$F_{[j+1]}^{(k)}(t) = F_{[j]}^{(k-1)}(t-1), \qquad 1 \leq j \leq P' - k.$$

Expression (A.5) is therefore verified for $k = 1, \ldots, P'$.
Finally, setting

$$Z_t = \begin{cases} Z_t^{(0)} & \text{if} \quad P' = -1, \\ Z_t^{(P')} = Z_t^{(0)} & \text{if} \quad P' = 0, \\ Z_t^{(P')} = \left(Z_t^{(P'-1)}, Z_t^{(P'-1)} e_t, e_t\right)' & \text{if} \quad P' > 0, \end{cases}$$

and taking X_t given by Equation (A.2), we obtain a bilinear Markovian Representation (2.3). □

Remark: In Pham (1986), the expression for $Z_t^{(k)}$, which is analogous to (A.4), is slightly incorrect. The right expression is given by (A.4).

Proof of Theorem 2.5:
Consider $\{X_t\} \sim \mathrm{BLg}(p, q, \tilde{P})$, i.e.:

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + \left(\prod_{j=0}^{q} g_j(e_{t-j})\right)\left(\sum_{k=1}^{\tilde{P}} b_k X_{t-k} + c\right) + r(e_t) + \alpha.$$

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Setting $m = \max\{p, \tilde{P}\}$, we define the vector $Z_t^{(0)}$ with the following components:

$$Z_{t,[i]}^{(0)} = X_{t-m+i}, \qquad 1 \le i \le m,$$

$$Z_{t,[m+1]}^{(0)} = \begin{cases} \sum_{k=1}^{\tilde{P}} b_k X_{t-k+1} + c & \text{if} \quad q = 0 \\ \left(\prod_{j=0}^{q-1} g_{j+1}(e_{t-j}) \right) \left(\sum_{k=1}^{\tilde{P}} b_k X_{t-k+1} + c \right) & \text{if} \quad q \ge 1. \end{cases}$$

The components of the vector $Z_t^{(0)}$ allow us to write:

$$X_t = \left(\sum_{j=m-p+1}^{m} a_{m-j+1} Z_{(t-1),[j]}^{(0)} \right) + g_0(e_t) Z_{(t-1),[m+1]}^{(0)} + r(e_t) + \alpha \tag{A.6}$$

and

$$Z_t^{(0)} = A^{(0)}(e_t) Z_{t-1}^{(0)} + B^{(0)}(e_t) + \left(C^{(0)}(e_t) Z_{t-1}^{(0)} + D^{(0)}(e_t) \right) \left(\prod_{j=1}^{q-1} g_{j+1}(e_{t-j}) \right) \tag{A.7}$$

where $A^{(0)}(e_t)$, $B^{(0)}(e_t)$, $C^{(0)}(e_t)$ and $D^{(0)}(e_t)$ are matrices of measurable functions in e_t of dimensions $(m+1) \times (m+1)$, $(m+1) \times 1$, $(m+1) \times (m+1)$ and $(m+1) \times 1$, respectively. We define for $q \ge 2$:

$$Z_t^{(k)} = \left(Z_t^{(k-1)},\ Z_t^{(k-1)} \left(\prod_{j=k}^{q-1} g_{j+1}(e_{t-j+k}) \right),\ \left(\prod_{j=k}^{q-1} g_{j+1}(e_{t-j+k}) \right) \right)',$$

$$k = 1, \ldots, q-1. \tag{A.8}$$

Then, we claim that:

$$Z_t^{(k-1)} = M^{(k)}(e_t) Z_{t-1}^{(k)} + N^{(k)}(e_t), \qquad k = 1, \ldots, q-1 \tag{A.9}$$

where $M^{(k)}(e_t)$ and $N^{(k)}(e_t)$ are matrices of measurable functions in e_t. Using the definition of $Z_t^{(k)}$, we can show by induction that Equation (A.9) is valid. Indeed, according to Expression (A.7), the equation is verified for $k = 1$. And if Equation (A.9) is true for k then it will be valid for $k + 1$. Expression (A.9) is therefore verified for $k = 1, \ldots, q-1$.

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Finally, taking Z_t defined by:

$$Z_t = \begin{cases} Z_t^{(0)} & q = 0, \\ Z_t^{(q-1)} = Z_t^{(0)} & q = 1, \\ Z_t^{(q-1)} = \left(Z_t^{(q-2)},\ Z_t^{(q-2)} g_q(e_t),\ g_q(e_t)\right)' & q > 1, \end{cases}$$

and X_t given by Equation (A.6), we obtain a bilinear Markovian Representation (2.3). □

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