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On unavoidable families of meromorphic functions

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Abstract. We prove several results on unavoidable families of meromorphic functions. For instance, we give new examples of families of cardinality 3 that are unavoidable with respect to the set of meromorphic functions on $\mathbb{C}$. We further obtain families consisting of less than three functions that are unavoidable with respect to certain subsets of meromorphic functions. In the other direction, we show that for every meromorphic function f, there exists an entire function that avoids f on $\mathbb{C}$.

1 Introduction and notation

For a domain $D \subset \mathbb{C}$, we denote by $H(D)$ and $M(D)$ the spaces of holomorphic and meromorphic functions on D, respectively, and further set $M_\infty(D) := M(D) \cup \{f_\infty|_D\}$, where $f_\infty \equiv \infty$. Given two functions f and g defined on D, we say that g avoids f on D, if $g(z) \neq f(z)$ for every $z \in D$. A function f is called unavoidable with respect to $A \subset M_\infty(D)$, if there is no $g \in A$ that avoids f on D. Furthermore, we say that a family F of functions is unavoidable with respect to $A \subset M_\infty(D)$, if there is no $g \in A$ that avoids every function $f \in F$ on D, that is, if for every $g \in A$, there exists $f \in F$ such that the equation $g(z) = f(z)$ has at least one solution in D (including the possibility that both functions take the value ∞).

Unavoidable families seem to first have been investigated by Rubel and Yang [12], who proved that for two functions f_1 and $f_2 \in M(\mathbb{C})$, the family $\{f_1, f_2\}$ is never unavoidable with respect to $M(\mathbb{C})$. On the other hand, they also showed that any family consisting of three polynomials p_1, p_2, and p_3, such that $p_1 - p_2$ and $p_2 - p_3$ are not both constant, is unavoidable with respect to $M(\mathbb{C})$. Thus, the minimum cardinality of a family $F \subset M(\mathbb{C})$ that is unavoidable with respect to $M(\mathbb{C})$ is 3. Hayman and Rubel [5] considered similar questions for general domains $D \subset \mathbb{C}$ and proved that a family $F \subset M(D)$ consisting of two functions cannot be unavoidable with respect to $M(D)$. In the other direction, it is shown in [5] that there exists a function $f \in H(D)$, such that for every function $g \in M(D)$, at least one of the three equations $g(z) = f(z), g(z) = -f(z)$, and $g(z) = \infty$ has infinitely many solutions in D. In particular, it follows that the family $\{f, -f, f_\infty|_D\}$ is unavoidable with respect to $M_\infty(D)$. Note that a result from [7] shows that if $F \subset M_\infty(D)$ is a family of three functions that is unavoidable with respect to $M_\infty(D)$, the three functions cannot avoid each other on D. The aforementioned result from [5] also implies that the

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family $\{f, -f\}$ is unavoidable with respect to the set of meromorphic functions in D that have at most finitely many poles. In particular, $\{f, -f\}$ is unavoidable with respect to $H(D)$, and because for a function $g \in H(D)$, the function $g + 1 \in H(D)$ avoids g on D, it follows that the minimum cardinality of a family $F \subset H(D)$ that is unavoidable with respect to $H(D)$ is 2. Again, according to [7], the two functions cannot avoid each other on D in this case. We further mention some results from [9], where unavoidable families of rational functions are investigated. For example, it is shown that the minimum cardinality of a family of rational functions, that no rational function can avoid on $\mathbb{C}$, is 2. On the other hand, given two rational functions r_1 and r_2, and a bounded domain $D \subset \mathbb{C}$, there is a rational function that avoids both r_1 and r_2 on D. Finally, for any domain $D \subset \mathbb{C}$, there exist two rational functions r_1 and r_2, such that $\{r_1|_D, r_2|_D\}$ is unavoidable with respect to $H(D)$; hence, $\{r_1|_D, r_2|_D, f_\infty|_D\}$ is unavoidable with respect to $M_\infty(D)$.

The abovementioned results show, in particular, that no single function $f \in M(D)$ is unavoidable with respect to $M_\infty(D)$. However, a result of Lappan [8] shows that for every simply connected domain $D \subset \mathbb{C}$, there exists a continuous function on D that is unavoidable with respect to $M_\infty(D)$. In a previous paper [7], the same author constructed a function continuous on the unit disk $\mathbb{D}$, which is unavoidable with respect to $H(\mathbb{D})$. In a similar vein, there may exist single functions $f \in M(D)$ that are unavoidable with respect to certain subsets of $M_\infty(D)$. For example, in [6], it was proved that there exists a function $f \in M(\mathbb{D})$ that is unavoidable with respect to the set of all normal functions in $M(\mathbb{D})$, and a corresponding result for $H(\mathbb{D})$ is also given.

In this note, we prove further results on unavoidable families of meromorphic functions. We give new examples of families of cardinality 3 that are unavoidable with respect to $M_\infty(\mathbb{C})$ and further construct families containing less than three functions that are unavoidable with respect to certain subsets of $M_\infty(\mathbb{C})$.

2 Unavoidable functions and zero–one sets

Let (a_n) and (b_n) be two (finite or infinite) disjoint sequences of complex numbers having no finite limit point. We say that $((a_n), (b_n))$ is a zero–one set, if there exists an entire function, whose zeros are exactly given by (a_n) and whose ones are exactly given by (b_n), where multiple occurrences of elements a_n and b_n correspond to zeros and ones of the corresponding multiplicity. This notation was introduced in [12], where it was shown that given sequences (a_n) and (b_n), the set $((a_n), (b_n))$ is not, in general, a zero–one set. More precisely, [12, Theorem 1] states that given any infinite sequence (a_n) in $\mathbb{C}$ without a finite limit point, there exists an infinite disjoint discrete sequence (b_n), such that $((a_n), (b_n))$ is not a zero–one set. For further results related to zero–one sets, we refer the reader, for example, to [10, 11, 14].

In the following, we show how the existence of sequences (a_n) and (b_n), such that $((a_n), (b_n))$ is not a zero–one set, can be used to obtain unavoidable functions. Note that we always assume that (a_n) and (b_n) are disjoint and have no finite limit point.

Proposition 1 *Let be given sequences (a_n) and (b_n) in $\mathbb{C}$ such that $((a_n), (b_n))$ is not a zero–one set. Consider a function $f \in H(\mathbb{C})$ whose zeros are exactly given by (a_n).*

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Then, f is unavoidable with respect to the set of entire functions whose zeros are exactly given by (b_n).

Proof Assuming that the statement is not correct, there exists a function $g \in H(\mathbb{C})$ whose zeros are exactly given by (b_n) such that $f(z) \neq g(z)$ for every $z \in \mathbb{C}$. Then, $f(z) - g(z) \neq 0$, for all $z \in \mathbb{C}$, and the function

$$F(z) := \frac{f(z)}{f(z) - g(z)}$$

is an entire function, whose zeros are exactly given by (a_n) and whose ones are exactly given by (b_n). This contradicts the assumption that $((a_n), (b_n))$ is not a zero–one set. ■

To give an example, we recall that a classic result states that for two distinct rays L_0 and L_1 emanating from the origin, there is no transcendental entire function for which all zeros lie on L_0 and all ones lie on L_1, while any (nonconstant) polynomial having this property is of degree 1 (e.g., [1, 2]). Hence, given two sequences $(a_n) \subset L_0$ and $(b_n) \subset L_1$ having no finite limit point and such that (a_n) has at least two elements, the set $((a_n), (b_n))$ is not a zero–one set. Thus, if f is an entire function having at least two zeros and all whose zeros lie on a ray L_0, we infer from Proposition 1 that f is unavoidable with respect to any entire function, all whose zeros are located on a ray that is different from L_0.

Using a similar idea as in Proposition 1, we obtain the following result that gives examples of families $F \subset M(\mathbb{C})$ of cardinality 3 that are unavoidable with respect to $M_\infty(\mathbb{C})$.

Theorem 1 *Let be given sequences (a_n) and (b_n) in $\mathbb{C}$ such that $((a_n), (b_n))$ is not a zero–one set. Consider functions f_1 and $f_2 \in H(\mathbb{C})$, whose zeros are exactly given by (a_n) and (b_n), respectively. Then, the family $\{f_1, f_2, \frac{2f_1f_2}{f_1+f_2}\}$ is unavoidable with respect to $M_\infty(\mathbb{C})$.*

Proof Let be given functions f_1 and $f_2 \in H(\mathbb{C})$, such that the zeros of f_1 are exactly given by (a_n) and the zeros of f_2 are exactly given by (b_n). Let further $g \in M(\mathbb{C})$, and suppose that g avoids both f_1 and f_2 in $\mathbb{C}$. Then, $g(z) - f_1(z) \neq 0$ and $g(z) - f_2(z) \neq 0$, for every $z \in \mathbb{C}$; in particular, g and f_1, as well as g and f_2, have no common zeros. Consider now the function

$$F(z) := \frac{f_1(z)(g(z) - f_2(z))}{f_1(z)(g(z) - f_2(z)) + f_2(z)(g(z) - f_1(z))}. \tag{1}$$

It follows from the assumptions that the zeros of F are exactly given by (a_n), whereas its ones are exactly given by (b_n). Because $((a_n), (b_n))$ is not a zero–one set, we must have $F \in M(\mathbb{C}) \setminus H(\mathbb{C})$. Thus, F must have at least one pole, implying that there exists $z_0 \in \mathbb{C}$ such that $f_1(z_0)(g(z_0) - f_2(z_0)) + f_2(z_0)(g(z_0) - f_1(z_0)) = 0$. It follows

$$g(z_0)(f_1(z_0) + f_2(z_0)) = 2f_1(z_0)f_2(z_0),$$

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and because f_1 and f_2 have no common zeros, we obtain

$$g(z_0) = \frac{2f_1(z_0)f_2(z_0)}{f_1(z_0)+f_2(z_0)}.$$

Hence, if the function $g \in M(\mathbb{C})$ avoids f_1 and f_2, it cannot avoid $\frac{2f_1f_2}{f_1+f_2}$, and the family $\{f_1, f_2, \frac{2f_1f_2}{f_1+f_2}\}$ is unavoidable with respect to $M(\mathbb{C})$. By Proposition 1, the functions f_1 and $-f_2$ cannot avoid each other, and because they further have no common zeros, there exists $z_1 \in \mathbb{C}$ with $f_1(z_1) + f_2(z_1) = 0$ and $f_1(z_1) \neq 0 \neq f_2(z_1)$. Thus, z_1 is a pole of $\frac{2f_1f_2}{f_1+f_2}$, so that the family $\{f_1, f_2, \frac{2f_1f_2}{f_1+f_2}\}$ is unavoidable with respect to $M_\infty(\mathbb{C})$. ■

Remark If, in (1), we consider the function

$$F(z) := \frac{f_1(z)(g(z)-f_1(z))}{f_1(z)(g(z)-f_1(z))+f_2(z)(g(z)-f_2(z))},$$

we obtain that the family $\{f_1, f_2, \frac{f_1^2+f_2^2}{f_1+f_2}\}$ is unavoidable with respect to $M_\infty(\mathbb{C})$. Indeed, in this case, there exists $z_0 \in \mathbb{C}$ such that $f_1(z_0)(g(z_0)-f_1(z_0)) + f_2(z_0)(g(z_0)-f_2(z_0)) = 0$, and hence

$$g(z_0)(f_1(z_0)+f_2(z_0)) = f_1^2(z_0) + f_2^2(z_0).$$

Because f_1 and f_2 have no common zeros, it follows

$$g(z_0) = \frac{f_1^2(z_0)+f_2^2(z_0)}{f_1(z_0)+f_2(z_0)}.$$

3 Unavoidable functions with multiple zeros and poles

In the following, we will use some standard terminology from Nevanlinna Theory (e.g., [3, 4, 15]), that is, for a nonconstant function $f \in M(\mathbb{C})$, a value $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ and $r \geq 0$, we denote by $n(r,a,f)$ and $\overline{n}(r,a,f)$ the number of a-points (counting multiplicity) and of distinct a-points of f, respectively, in $\{z : |z| \leq r\}$. The corresponding integrated counting functions will be denoted by $N(r,a,f)$ and $\overline{N}(r,a,f)$, respectively. We write $T(r,f)$ for the characteristic function of f, and we recall that the deficiency $\delta(a,f)$ and the branching index $\Theta(a,f)$ of a are defined by

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)} \qquad \text{and} \qquad \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a,f)}{T(r,f)}.$$

Note that $\delta(a,f) \leq \Theta(a,f)$, and it is a consequence of the Second Fundamental Theorem that $\sum_{a\in\mathbb{C}_\infty} \Theta(a,f) \leq 2$.

As mentioned in the introduction, a result from [5] states the existence of an entire function f, such that for every $g \in M(\mathbb{C})$ having at most finitely many poles, at least one of the two equations $g(z) = f(z)$ and $g(z) = -f(z)$ has infinitely many solutions in $\mathbb{C}$. Using similar ideas as in [5], we show the following result.

Theorem 2 *Let $f \in M(\mathbb{C})$ have infinitely many zeros, of which at most finitely many are simple. Suppose further that $g \in M(\mathbb{C})$ has at most finitely many simple poles. Then,*

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at least one of the two equations $g(z) = f(z)$ and $g(z) = -f(z)$ has infinitely many solutions in $\mathbb{C}$. In particular, the family $\{f, -f\}$ is unavoidable with respect to the set of meromorphic functions having at most finitely many simple poles.

Proof Suppose that $f \in M(\mathbb{C})$ has infinitely many zeros, of which at most finitely many are simple. Assuming that the statement is not correct, there exists a function $g \in M(\mathbb{C})$ that has at most finitely many simple poles, such that both equations $g(z) = f(z)$ and $g(z) = -f(z)$ have at most finitely many solutions in $\mathbb{C}$. In particular, f and g then have at most finitely many common zeros, and it follows from the assumptions that the function

$$F(z) = \frac{f(z)}{g(z)}$$

has infinitely many zeros, of which at most finitely many are simple. Using the First Fundamental Theorem, this implies

$$\overline{N}(r, 0, F) \leq \frac{1}{2} N(r, 0, F) + \mathcal{O}(1) \leq \frac{1}{2} T(r, F) + \mathcal{O}(1),$$

and hence $\Theta(0, F) \geq \frac{1}{2}$. Furthermore, it follows that the equations $F(z) = 1$ and $F(z) = -1$ have at most finitely many roots, so that we obtain $\Theta(1, F) = 1$ and $\Theta(-1, F) = 1$, because F is transcendental. Finally, $\Theta(0, F) + \Theta(1, F) + \Theta(-1, F) > 2$, in contradiction to the Second Fundamental Theorem. ■

Remark As previously mentioned, it was shown in [12] that a family consisting of two meromorphic functions is never unavoidable with respect to $M(\mathbb{C})$. In particular, given $f \in M(\mathbb{C})$, there exists $g \in M(\mathbb{C})$ such that $g(z) \neq f(z)$ and $g(z) \neq -f(z)$, for every $z \in \mathbb{C}$. It follows from the above theorem that if f has infinitely many zeros, of which at most finitely many are simple, then such a function g necessarily has infinitely many simple poles.

The following is an immediate consequence of Theorem 2.

Corollary 1 *Let $f \in M(\mathbb{C})$ have infinitely many zeros, of which at most finitely many are simple. Let further $g \in M(\mathbb{C})$ and $n, m \in \mathbb{N}$ with $m \geq 2$ be given. Then, the following hold:*

(i) *At least one of the two equations $g^{(n)}(z) = f(z)$ and $g^{(n)}(z) = -f(z)$ has infinitely many solutions in $\mathbb{C}$.*
(ii) *At least one of the two equations $g^m(z) = f(z)$ and $g^m(z) = -f(z)$ has infinitely many solutions in $\mathbb{C}$.*

In particular, for every $n, m \in \mathbb{N}$ with $m \geq 2$, the family $\{f, -f\}$ is unavoidable with respect to the sets $\{g^{(n)} : g \in M(\mathbb{C})\}$ and $\{g^m : g \in M(\mathbb{C})\}$.

If we make stronger assumptions on the zeros and poles, we can also obtain single functions $f \in M(\mathbb{C})$ that are unavoidable with respect to certain subsets of $M(\mathbb{C})$.

Proposition 2 *Let $f \in M(\mathbb{C})$ be a meromorphic function having infinitely many zeros and poles, at most finitely many of which have multiplicity less than 3. If $g \in M(\mathbb{C})$ has at most finitely many zeros and poles with multiplicity less than 3, the equation $g(z) = f(z)$ has infinitely many solutions in $\mathbb{C}$.*

Proof Let be given a function $f \in M(\mathbb{C})$ having infinitely many zeros and poles, at most finitely many of which have multiplicity less than 3. Assuming that the statement does not hold, there exists a function $g \in M(\mathbb{C})$ that has at most finitely many zeros and poles with multiplicity less than 3, such that the equation $g(z) = f(z)$ has at most finitely many solutions in $\mathbb{C}$. In particular, f and g then have at most finitely many common zeros and poles, so that the function

$$F(z) = \frac{f(z)}{g(z)}$$

has infinitely many zeros and poles, at most finitely many of which have multiplicity less than 3. As before, this implies

$$\overline{N}(r, 0, F) \le \frac{1}{3} N(r, 0, F) + \mathcal{O}(1) \le \frac{1}{3} T(r, F) + \mathcal{O}(1),$$

and hence $\Theta(0, F) \ge \frac{2}{3}$. A similar argumentation gives $\Theta(\infty, F) \ge \frac{2}{3}$, and because F is a transcendental function taking at most finitely many times the value 1, we further have $\Theta(1, F) = 1$. Hence, $\Theta(0, F) + \Theta(1, F) + \Theta(\infty, F) > 2$, in contradiction to the Second Fundamental Theorem. ∎

Corollary 2 *Let $f \in M(\mathbb{C})$ have infinitely many zeros and poles, and consider $g \in M(\mathbb{C})$. Then, for every $n, m \in \mathbb{N}$ with $n, m \ge 3$, the equation $g^n(z) = f^m(z)$ has infinitely many solutions in $\mathbb{C}$.*

Note that it follows from Theorem 2 that if $f \in M(\mathbb{C})$ has infinitely many zeros, then for $g \in M(\mathbb{C})$ and every $n, m \in \mathbb{N}$ with $n, m \ge 2$, at least one of the two equations $g^n(z) = f^m(z)$ and $g^n(z) = -f^m(z)$ has infinitely many solutions in $\mathbb{C}$.

It is easily seen that similar results can be obtained for entire functions.

Proposition 3 *Let $f \in H(\mathbb{C})$ be an entire function having infinitely many zeros, at most finitely many of which have multiplicity less than 3. If $g \in H(\mathbb{C})$ has at most finitely many simple zeros, the equation $g(z) = f(z)$ has infinitely many solutions in $\mathbb{C}$.*

Indeed, assuming that the statement does not hold, there exists a function $g \in H(\mathbb{C})$ that has at most finitely many simple zeros, such that the equation $g(z) = f(z)$ has at most finitely many solutions in $\mathbb{C}$. The function $F(z) = \frac{f(z)}{g(z)}$ is then transcendental, and it follows as before that $\Theta(F, 0) \ge \frac{2}{3}$, $\Theta(F, \infty) \ge \frac{1}{2}$ and $\Theta(F, 1) = 1$, which leads to a contradiction.

Corollary 3 *Let $f \in H(\mathbb{C})$ have infinitely many zeros, and consider $g \in H(\mathbb{C})$. Then, for every $n, m \in \mathbb{N}$ with $n \ge 2$ and $m \ge 3$, the equation $g^n(z) = f^m(z)$ has infinitely many solutions in $\mathbb{C}$.*

In the following, we will show the existence of meromorphic functions that are unavoidable with respect to functions in $M(\mathbb{C})$ whose number of zeros and poles are bounded in some sense. We therefore introduce the following notation. Given a continuous increasing function $\varphi : [0,\infty) \to [1,\infty)$ with $\varphi(r) \to \infty$ for $r \to \infty$, we denote by $M_\varphi(\mathbb{C})$ the set of all functions $f \in M(\mathbb{C})$ such that $n(r,0,f) = \mathcal{O}(\varphi(r))$ and $n(r,\infty,f) = \mathcal{O}(\varphi(r))$. Our result will be an immediate consequence of the following more general statement.

Theorem 3 *Let be given a continuous increasing function $\varphi : [0,\infty) \to [1,\infty)$ with $\varphi(r) \to \infty$ for $r \to \infty$. There exists a function $f \in M(\mathbb{C})$, such that for every function $g \in M_\varphi(\mathbb{C})$, we have $\Theta(a, \frac{f}{g}) = 0$ for every $a \in \mathbb{C} \setminus \{0\}$.*

Proof Because $M_{\varphi_1}(\mathbb{C}) \subset M_{\varphi_2}(\mathbb{C})$ if $\varphi_1(r) \le \varphi_2(r)$ for $r \in [0,\infty)$, we may assume that $r \le \varphi(r)$ for $r \in [0,\infty)$. We shall show that any function $f \in M(\mathbb{C})$ with "sufficiently few" distinct zeros and poles, but "sufficiently many" zeros and poles (counting multiplicity) has the claimed property. We therefore consider a function $f_1 \in H(\mathbb{C})$ such that for every $n \in \mathbb{N}$, the function f_1 has a zero at $z = n$ with multiplicity $[\varphi(n+2)]!$ and $f_1(z) \ne 0$ for $z \in \mathbb{C} \setminus \mathbb{N}$, where, here and in the following, we denote by $[r]$ the integer part of the real number r. We then define $f_2(z) := f_1(-z)$ and set $f(z) = \frac{f_1(z)}{f_2(z)}$. We claim that f satisfies the statement of the theorem.

Let therefore $g \in M_\varphi(\mathbb{C})$ be given. There exist g_1 and $g_2 \in H(\mathbb{C})$, such that g_1 and g_2 have no common zeros and such that $g = \frac{g_1}{g_2}$. Because $g \in M_\varphi(\mathbb{C})$, we have $n(r,0,g_1) = \mathcal{O}(\varphi(r))$ and $n(r,0,g_2) = \mathcal{O}(\varphi(r))$, and considering the function

$$F(z) := \frac{f(z)}{g(z)} = \frac{f_1(z)}{f_2(z)} \frac{g_2(z)}{g_1(z)},$$

we obtain for $r > 0$ sufficiently large

$$\overline{n}(r,0,F) \le \overline{n}(r,0,f_1) + \overline{n}(r,0,g_2) \le [r] + c_1\varphi(r) \le c_2\varphi(r), \tag{2}$$

where $c_1 > 0$ and $c_2 > 0$ are suitable constants. Moreover, for $r > 0$ sufficiently large, we have

(3)

$$n(r,0,F) \ge n(r,0,f_1) - n(r,0,g_1) \ge \sum_{i=3}^{[r]+2} [\varphi(i)]! - c_3\varphi(r) \ge \sum_{i=3}^{[r]+1} [\varphi(i)]! > [\varphi([r]+1)]!$$

for a suitable constant $c_3 > 0$.

Let now $\varepsilon > 0$ be given. Then, there exists $k \in \mathbb{N}$ such that $\varepsilon > \frac{1}{k} > 0$, and from (2) and (3), we obtain that there exists $R_k > 0$, such that for $r > R_k$, we have

$$\overline{n}(r,0,F) \le \frac{1}{k} n(r,0,F),$$

which, using the First Fundamental Theorem, implies

$$\overline{N}(r,0,F) \le \frac{1}{k} N(r,0,F) + \mathcal{O}(1) \le \frac{1}{k} T(r,F) + \mathcal{O}(1).$$

This finally yields $\Theta(0,F) \ge \frac{k-1}{k} > 1 - \varepsilon$, and thus $\Theta(0,F) = 1$.

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By a similar argumentation, we obtain that $\Theta(\infty, F) = 1$; hence, $\Theta(\infty, F) + \Theta(0, F) = 2$, so that $\Theta(a, F) = 0$ for every $a \in \mathbb{C} \setminus \{0\}$ by the Second Fundamental Theorem. ■

Corollary 4 *Let be given a continuous increasing function $\varphi : [0, \infty) \to [1, \infty)$ with $\varphi(r) \to \infty$ for $r \to \infty$. There exists a function $f \in M(\mathbb{C})$, such that for every function $g \in M_\varphi(\mathbb{C}) \cup \{0, f_\infty\}$, the equation $g(z) = f(z)$ has infinitely many solutions in $\mathbb{C}$. In particular, f is unavoidable with respect to $M_\varphi(\mathbb{C}) \cup \{0, f_\infty\}$.*

It is easily seen that the function f from the proof of Theorem 3 satisfies the requirement, for, assuming this is not the case, there exists a function $g \in M_\varphi(\mathbb{C}) \cup \{0, f_\infty\}$ such that the equation $g(z) = f(z)$ has at most finitely many solutions in $\mathbb{C}$. Because f has infinitely many zeros and poles, we have that $g \in M_\varphi(\mathbb{C})$, and the function $\frac{f}{g}$ takes at most finitely many times the value 1. Because $\frac{f}{g}$ is transcendental, this implies $\Theta(1, \frac{f}{g}) = 1$, which is in contradiction to Theorem 3.

Remark

(i) Note that if the function φ is such that for every $n \in \mathbb{N}$, we have $\frac{\varphi(r)}{r^n} \to \infty$ for $r \to \infty$, the set $M_\varphi(\mathbb{C})$ contains every meromorphic function g of the form $g = g_1 e^{g_2}$, where $g_1 \in M(\mathbb{C})$ is of finite order of growth and $g_2 \in H(\mathbb{C})$.

(ii) We further mention that it is a consequence of a generalization of the Second Fundamental Theorem (e.g., [4, Theorem 2.5]) that given a function $f \in M(\mathbb{C})$ and three functions g_1, g_2, and $g_3 \in M(\mathbb{C})$ that are "small" with respect to f, that is, functions satisfying $T(r, g_i) = o(T(r, f))$ for $r \to \infty$, at least one of the three equations $g_1(z) = f(z)$, $g_2(z) = f(z)$, and $g_3(z) = f(z)$ has infinitely many solutions in $\mathbb{C}$. Hence, a meromorphic function can avoid at most two small functions; in particular, a meromorphic function of infinite order can avoid at most two functions of finite order. Moreover, it follows from the deficiency relation for small functions (e.g., [15, p. 41]) that if $f \in M(\mathbb{C})$ is a function that takes every value $a \in \mathbb{C}_\infty$ infinitely many times and satisfies $\sum_{a \in \mathbb{C}_\infty} \delta(a, f) = 2$, the equation $g(z) = f(z)$ has infinitely many solutions in $\mathbb{C}$ for every function $g \in M(\mathbb{C})$ that is small with respect to f. In particular, such a function f is unavoidable with respect to the set of small functions.

4 No $f \in M(\mathbb{C})$ is unavoidable with respect to $H(\mathbb{C})$

In the other direction, we have the following result.

Theorem 4 *Given $f \in M(\mathbb{C})$, there exists a function $g \in H(\mathbb{C})$ that avoids f on $\mathbb{C}$. In particular, no $f \in M(\mathbb{C})$ is unavoidable with respect to $H(\mathbb{C})$.*

Proof Let $f \in M(\mathbb{C})$ be given. Then, there exist entire functions f_1 and f_2 that have no common zeros such that $f = \frac{f_1}{f_2}$. Denote by A the set of zeros of f_2, and denote the order of a zero $a \in A$ by p_a. We can assume that $A \neq \emptyset$, because otherwise f is entire and the function $g = f + 1$ avoids f on $\mathbb{C}$. Because f_1 and f_2 have no common zeros, for every $a \in A$, there exists $\varepsilon_a > 0$ such that $f_1(z) \neq 0$ in $D_a := \{z : |z - a| < \varepsilon_a\}$. Hence,

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for every $a \in A$, there exists a function $g_a \in H(D_a)$ such that $e^{g_a(z)} = f_1(z)$ holds for every $z \in D_a$. For $a \in A$ and $z \in D_a$, we have the expansion

$$g_a(z) = \sum_{n=0}^{\infty} c_n^{(a)}(z-a)^n = \sum_{n=0}^{p_a-1} c_n^{(a)}(z-a)^n + \sum_{n=p_a}^{\infty} c_n^{(a)}(z-a)^n.$$

According to a classic interpolation result (e.g., [13, p. 304]), there exists a function $\varphi \in H(\mathbb{C})$ that has, at every $a \in A$, the same power series development up to $(z-a)^{p_a-1}$ as g_a. Hence, for every $a \in A$ and $z \in D_a$, we obtain

$$\begin{aligned} \varphi(z) &= \sum_{n=0}^{p_a-1} c_n^{(a)}(z-a)^n + \sum_{n=p_a}^{\infty} d_n^{(a)}(z-a)^n \\ &= g_a(z) + \sum_{n=p_a}^{\infty} (d_n^{(a)} - c_n^{(a)})(z-a)^n \\ &= g_a(z) + Q_a(z), \end{aligned}$$

where Q_a has a zero of order p_a at the point a. Thus, $e^{\varphi(z)} = f_1(z)\, e^{Q_a(z)}$ for $a \in A$ and $z \in D_a$, so that the function $f_1 - e^{\varphi}$ has a zero of order p_a at the point a. It follows that the function

$$g(z) := \frac{f_1(z) - e^{\varphi(z)}}{f_2(z)}$$

is entire, and because

$$f(z) - g(z) = \frac{f_1(z)}{f_2(z)} - \frac{f_1(z) - e^{\varphi(z)}}{f_2(z)} = \frac{e^{\varphi(z)}}{f_2(z)}$$

is zero-free, the function g avoids f on $\mathbb{C}$. ∎

Remark

(i) Because the interpolation result we use in the proof also holds for arbitrary domains $D \subset \mathbb{C}$, it follows that the result also holds in this case; hence, no $f \in M(D)$ is unavoidable with respect to $H(D)$. In [7], it is shown that there exists a continuous function f on $\mathbb{D}$ that is unavoidable with respect to $H(\mathbb{D})$. Theorem 4 shows that there exists no $f \in M(\mathbb{D})$ with such a property.

(ii) As stated in the introduction, results from [5, 12] show that given f_1 and $f_2 \in M(\mathbb{C})$, there exists $g \in M(\mathbb{C})$ that avoids both f_1 and f_2 on $\mathbb{C}$. The proof given in [5, 12] does, however, not directly apply in case that $f_1 = f_\infty$ or $f_2 = f_\infty$, from which Theorem 4 would immediately follow. Nevertheless, we use similar ideas in our proof of Theorem 4.

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