## 108.09 A visual proof that $b^e < e^b$ when $b > e$

In a recent visual proof ([1]), the author provided a visual proof of the inequality $\pi^e < e^\pi$. However, their visual proof can be used to show the more general inequality $b^e < e^b$, where $e < b$.

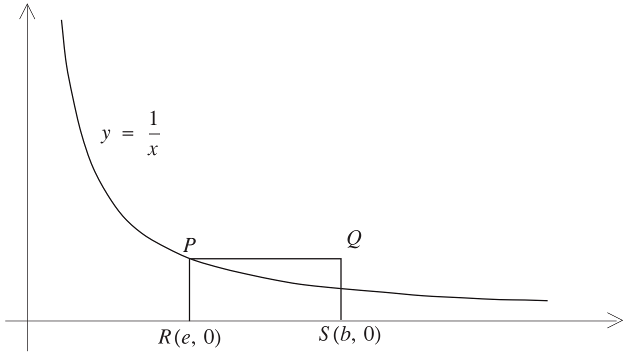


FIGURE 1

$$\ln b - 1 = \int_e^b \frac{dx}{x} < \frac{1}{e}(b - e) = \frac{b}{e} - 1$$

and so $$b^e < e^b.$$

*Reference*

1. Bikash Chakraborty, A visual proof that $\pi^e < e^\pi$, *Mathematical Intelligencer* **41** (2019) p. 60.

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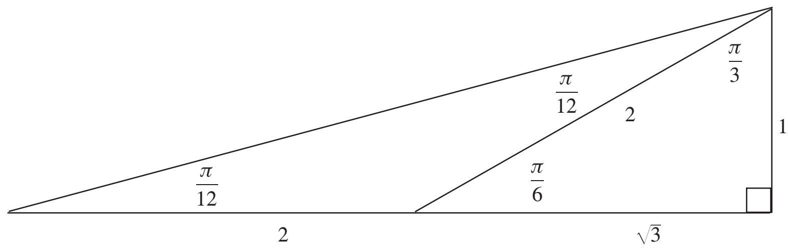
## 108.10 Proof without words: $\tan\frac{\pi}{12} = 2 - \sqrt{3}$, $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$

The standard proof of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ is to use the less well-known formula

$$\tan\alpha = \frac{-1 + \sqrt{1 + \tan^2 2\alpha}}{\tan 2\alpha}$$

for $\alpha = \frac{\pi}{12}$ and the well-known value $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$. Using only the last fact,




from the diagram the readers can readily see that

$$\tan\frac{5\pi}{12} = 2 + \sqrt{3} \qquad \text{and} \qquad \tan\frac{\pi}{12} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

There is another PWW from Garcia Capitan Francisco Javier [1]. Paul Stephenson [2] and Nick Lord [3] have offered other demonstrations of the identity of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$, for which Nick Lord gives four Proofs without words, with quite different ideas. For many useful principles and comments about Proofs without words, see [4].

*References*


1. F. J. Garcia Capitan, Proof without Words: tangents 15 and 17 degrees, *Coll. Math. J.* **48**:**1** (2017) p. 35.
2. P. Stephenson, Feedback: On what makes a good Proof without Words, *Math. Gaz.* **107** (March 2023) p. 165.
3. Nick Lord, Feedback, *Math. Gaz.* **107** (July 2023) p. 356.
4. G. Leversha, What makes a good Proof without Words, *Math. Gaz.* **105** (July 2021) pp. 271-281.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## 108.11 Euler's limit—revisited

Let $e_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. It is well known that the sequence $(e_n)$ is monotone increasing and bounded, hence it is convergent. The limit of this sequence is the famous Euler number $e$. Here we establish a generalisation of this limit.

*Theorem*: Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers such that $a_n \to +\infty$ and $b_n$ satisfies the asymptotic formula $b_n \sim k \cdot a_n$, where $k > 0$. Then

$$\lim_{n\to\infty}\left(1 + \frac{1}{a_n}\right)^{b_n} = e^k.$$