THE PRACTICAL REPLACEMENT OF A BONUS-MALUS SYSTEM

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ABSTRACT

In this paper we will show how to set up a practical bonus-malus system with a finite number of classes. We will use the actual claim amount and claims frequency distributions in order to predict the future observed claims frequency when the new bonus-malus system will be in use. The future observed claims frequency is used to set up an optimal bonus-malus system as well as the transient and stationary distributions of the drivers in the new bonus-malus system. When the number of classes as well as the transition rules of the new bonus-malus system have been adopted, the premium levels are obtained by minimizing a certain distance between the levels of the practical bonus-malus system and the corresponding optimal bonus-malus system. Some iterations are necessary in order to reach stabilization of the future observed claims frequency and the levels of the practical bonus-malus system.

KEYWORDS

Optimal bonus-malus system, practical bonus-malus system, hunger for bonus, observed claims frequency distribution, actual claims frequency distribution, iterative algorithm, Hofmann Distribution, non-parametric mixed Poisson fit.

1. INTRODUCTION

A practical bonus-malus system consists of a table with s classes. A premium level is associated with each class. The entry in the bonus-malus system is generally at level 100%. The levels under 100% are bonuses. The levels above 100% are maluses. A rule is chosen in order to let the drivers move within the bonus-malus system according to the number of claims they report each year to the Company. This is the transition rule. A classical reference for bonus-malus systems is Lemaire (1995).

In this paper we are interested in setting up a methodology so that we may change a bonus-malus system. This is a typical problem that Belgian

Companies have to face. Indeed, for a long time, they have used a unique bonus-malus system which is now in contravention of the European directives. They may thus be forced to adapt their bonus-malus system.

Throughout the paper, we will use a numerical example in order to explain our methodology. The following characteristics will be used.

Let us assume that the current bonus-malus system is the following:

- The number of classes s is 9, numbered 0, 1, ..., 8. 0 is the minimum class. 8 is the maximum class.
- Entry of the system is in class 4.
- In the case of a claims free year, the policyholder comes down one class.
- In the case of claim(s) being reported, the policyholder climbs up 3 classes per claim.
- The bonuses and maluses (in percentage) are given in the following table:

TABLE 1

PREMIUM LEVELS OF THE CURRENT BONUS-MALUS SYSTEM

i	0	1	2	3	4	5	6	7	8
C_i	75	80	90	95	100	150	170	185	250

Let us assume that a new bonus-malus system will have the following characteristics:

- $s = 9$: 9 classes numbered 0, 1, ..., 8. 0 is the minimum class. 8 is the maximum class.
- Entry of the system is in class 4.
- In the case of a claims free year, the policyholder comes down one class.
- In the case of claim(s) being reported, the policyholder goes to level 8 whatever the number of claims.

Our problem is to find the bonuses and maluses (in percentage) for the new bonus-malus system.

In Walhin and Paris (2001) it has been shown how to find the actual claim amount and claims frequency distributions of a motor portfolio if a bonus-malus system is used. The reason why the actual claims frequency distribution is not observed is that the bonus-malus system induces the hunger for bonus (see Lemaire (1977)) for some drivers. Lemaire (1977) derived an algorithm giving the optimal retention of a driver in function of his bonus-malus level. See also Walhin and Paris (2001) for a short description of Lemaire's algorithm.

We will use the following portfolio which is observed within the current bonus-malus system:

TABLE 2

OBSERVED CLAIMS FREQUENCY DISTRIBUTION

Number of accidents	Number of policyholders
0	103704
1	14075
2	1766
3	255
4	45
5	6
6	2

We also assume that the following data set of claim amounts has been observed on our portfolio:

TABLE 3

OBSERVED CLAIM AMOUNT DISTRIBUTION

6	6	10	11	17	18	20	26	27	34
42	44	47	54	59	60	61	61	61	61
64	64	65	66	67	68	71	71	73	75
76	81	85	87	93	94	101	103	105	109
110	110	113	116	116	129	134	134	141	141
151	154	156	159	167	171	172	173	174	179
181	183	185	187	195	195	203	226	235	240
251	255	273	340						

This is the data set we used in Walhin and Paris (2001) in order to obtain the actual claim amount and frequency distributions of the drivers within the bonus-malus system.

We also obtained the proportion of drivers using their optimal retention given by Lemaire's algorithm. Indeed, we assume that, for psychological reasons, some drivers never apply Lemaire's algorithm. Instead they always use their insurance, even if it is not optimal.

These results are described and refined in section 3 after having recalled the conception of an optimal bonus-malus system in section 2. Section 4 shows how to set up a practical bonus-malus system from an optimal one. Section 5 uses Lemaire's algorithm in order to find iteratively the practical bonus-malus system that will be in accordance with the future observed claims frequency distribution within the new bonus-malus system. Section 6 is devoted to the conclusion.

We will use the following notations:

- N is the random variable representing the number of reported claims within the current bonus-malus system.
- N' is the random variable representing the number of actual claims.
- N'' is the random variable representing the number of claims under the optimal policy for the drivers using Lemaire's algorithm within the current bonus-malus system.
- N''' is the random variable representing the number of future observed claims within the new bonus-malus system.
- N'''' is the random variable representing the number of claims under the optimal policy for the drivers using Lemaire's algorithm within the new bonus-malus system.
- X is the random variable representing the observed claim amounts within the current bonus-malus system.
- Z is the random variable representing the actual claim amounts.
- X' is the random variable representing the observed claim amounts within the new bonus-malus system.

2. An optimal bonus-malus system

This section is mainly based on Walhin and Paris (1999b) where further details can be found.

It is generally considered that the number of claims, $N(t)$, in the interval $(0, t]$ is a mixed Poisson process:

$$\mathbb{P}[N(t)=k\,|\,\Lambda] = \Pi(k,t\,|\,\Lambda) = e^{-\Lambda t}\frac{(\Lambda t)^k}{k!},$$

$$\mathbb{P}[N(t)=k] = \Pi(k,t) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,dU(\lambda),$$

where Λ is a random variable with cumulative density function (cdf) $U(\lambda)$.

For the t^{th} year, we take into account the information which consists of the number of accidents during the first t years. The a posteriori premium is:

$$\begin{aligned}\mathbb{E}[N(t+1)-N(t)\,|\,N(t)=k] &= \mathbb{E}(\Lambda\,|\,N(t)=k)\\ &= \frac{k+1}{t}\,\frac{\Pi(k+1,t)}{\Pi(k,t)}.\end{aligned}$$

This is an application of the expected value premium principle.

By giving the premium levels such a norm that the a priori premium is 100 it is possible to set up a table with two entries: t the number of observed years and k the number of observed claims. The table gives the premium levels in percentage in function of t and k. We will denote these levels by

$$P_{(k,t)} = \frac{100}{\mathbb{E}\Lambda} \frac{k+1}{t} \frac{\Pi(k+1,t)}{\Pi(k,t)}. \tag{1}$$

TABLE 4

OPTIMAL BONUS-MALUS TABLE: $P_{(K,T)}$

t/k	0	1	2	3	4	…
0	100	—	—	—	—	…
1	$P_{(0,1)}$	$P_{(1,1)}$	$P_{(2,1)}$	$P_{(3,1)}$	$P_{(4,1)}$	…
2	$P_{(0,2)}$	$P_{(1,2)}$	$P_{(2,2)}$	$P_{(3,2)}$	$P_{(4,2)}$	…
⋮	⋮	…				

This formula is general and may be applied e.g. to

– a non-parametric mixed Poisson process

$$\Pi(k,t) = \sum_{j=1}^{r} p_j e^{-\lambda_j t} \frac{(\lambda_j t)^k}{k!}, \quad k \geq 0,$$

where Λ takes discrete values λ_j with probabilities p_j, $j = 1, \ldots, r$.

– the Hofmann process

$$\Pi(0,t) = e^{-\theta(t)},$$

$$(k+1)\Pi(k+1,t) = \frac{pt}{(1+ct)^a} \sum_{i=0}^{k} \frac{\Gamma(a+i)}{\Gamma(a) i!} \left(\frac{ct}{1+ct}\right)^i \Pi(k-i,t), \quad k \geq 0.$$

In this case Λ is continuous and such that $\Pi(0,t) = e^{-\theta(t)}$ with $\theta(t)' = \frac{p}{(1+ct)^a}$. It is possible to write the density function of Λ but it is tedious and unnecessary.

In Walhin and Paris (1999b) it is shown that the conception of an optimal bonus-malus system is not smooth enough if the non-parametric mixed Poisson fit is used. Therefore we will concentrate on the Hofmann process for the set up of an optimal bonus-malus system.

The problem is that the reported claims frequency distribution will be described using a non-parametric mixed Poisson distribution. We will choose the parameters p, c, and a by matching the first three moments of the non-parametric mixed Poisson distribution and the Hofmann distribution:

$$\sum_{j=1}^{r} p_j \lambda_j = p,$$

$$\sum_{j=1}^{r} p_j \lambda_j (\lambda_j + 1) = p + acp + p^2,$$

$$\sum_{j=1}^{r} p_j \lambda_j (\lambda_j^2 + 3\lambda_j + 1) = p(p + acp) + acp(2 + p) + p(1 + p)^2 + acp(1 + p) + (1 + a)ac^2 p.$$

Another solution might be to equate the mean and the probabilities of 0 observation and 1 observation:

$$\sum_{j=1}^{r} p_j \lambda_j = p,$$

$$\sum_{j=1}^{r} p_j e^{-\lambda_j} = e^{-\frac{p}{c(1-a)}((1+c)^{1-a}-1)},$$

$$\sum_{j=1}^{r} p_j \lambda_j e^{-\lambda_j} = \frac{p}{(1+c)^a} e^{-\frac{p}{c(1-a)}((1+c)^{1-a}-1)}.$$

Note that formula (1) is based on the expected value principle. Other premium principles may be used in order to set up optimal bonus-malus tables. In Walhin and Paris (1999b) the principle of zero utility with an exponential utility function is used but numerical examples show that this does not produce significantly different tables from the ones obtained with the expected value principle.

In Walhin and Paris (1999a) optimal bonus-malus tables are obtained with the use of exponential loss distributions. This gives a free parameter which the actuary can play with in order to introduce some solidarity in the bonus-malus table.

3. The hunger for bonus and the actual claims frequency distribution

This section is essentially based on Walhin and Paris (2001) where details can be found.

With the following hypotheses the algorithm of Lemaire (1977) gives the optimal policy (retention and frequency) of the driver as a function of his bonus-malus level.

Let – a bonus-malus system be with s classes: $i = 0, \ldots, s-1$,

– the claims frequency of a policyholder be Poisson distributed with mean λ,

- the claim amount distribution be Z, with cdf $F_Z(x)$,
- β be the discount rate forecast for the future,
- P be the commercial premium, i.e. the base premium at level 100%, including security loading, administration expenses, profit and taxes,
- $1-t$ with $0 \le t < 1$, be the time remaining until the next premium payment.

The observed claims frequency distribution (N) given in table 2 can be fitted by a mixed Poisson distribution. An interesting choice for the mixing distribution is the one leading to the non-parametric mixed Poisson fit, i.e. a mixture of various Poisson distributions. This makes the calculations with Lemaire's algorithm easy because it has to be run only for certain values of the parameter λ. Moreover, the stationary or transient distributions of the drivers within the bonus-malus system are easy to calculate for the same reason (see also Walhin and Paris (1999b)). For our data set we find

TABLE 5

NON-PARAMETRIC MAXIMUM LIKELIHOOD ESTIMATION FOR THE OBSERVED PORTFOLIO

$\lambda_1 = 0.05461$	$p_1 = 0.56189$
$\lambda_2 = 0.24599$	$p_2 = 0.41463$
$\lambda_3 = 0.95618$	$p_3 = 0.02348$

i.e. we find three types of drivers: 56% of the drivers show a claims frequency of 0.0546, 41% of the drivers show a claims frequency of 0.2459 and 2% of the drivers show a claims frequency of 0.9561. In general there are r types of drivers. Note that for classical motor portfolios r will be equal to 3 or 4 because the frequency is low, leading to a maximum number of claims per policyholder equal to about 7. The reason for being only 3 or 4 classes is given in Simar (1976) and discussed in Walhin and Paris (1999b).

We assume that the commercial premium is $P = 35$. This premium includes profit loading, administration expenses, brokerage, ...

We assume that the portfolio is at steady state. Using the observed claims frequency, it is easy to show that the stationary distribution is

$$\mathbf{e}_\infty = \{0.5729, 0.0562, 0.0661, 0.0784, 0.0421, 0.0441, 0.0457, 0.0429, 0.0516\}.$$

With this stationary distribution, the premium income at steady state is 100.16% of the commercial premium. So the bonus-malus system is financially equilibred. Let us make some precisions regarding our numerical hypotheses:

- the bonus-malus system is the one given in table 1.
- the claim amount distribution Z is exponentially distributed with mean μ:

$$\begin{aligned} F_Z(x) &= 0 \quad \text{if } x < 0, \\ &= 1 - e^{-\frac{x}{\mu}} \quad \text{else.} \end{aligned}$$

- the data set of table 3 has been used for the observed claim amounts.
- $\beta = 6\%$.
- $t = \frac{1}{2}$.
- the proportions $p_j, j = 1, \ldots, r$ remain the same with the observed and actual claims frequencies.

Taking into account the hunger for bonus and using this non-parametric fit, Walhin and Paris (2001) derived the actual claims frequency distribution of the drivers (N') using essentially the following relations:

(2) $$\lambda_j = \pi\lambda_j' + (1-\pi)\lambda_j'', \quad j=1,\ldots,r,$$

(3) $$f_X(x) = \pi f_Z(x) + (1-\pi)\frac{f_Z(x)}{1-F_Z(c)}\mathbb{I}_{\{x \geq c\}},$$

where

- c is the average of the optimal retentions of the drivers:
$$c = \sum_{j=1}^{r} p_j c_j,$$
where c_j, $j = 1, \ldots, r$ is the optimal retention for the drivers of type j.
- π is the proportion of drivers not using Lemaire's algorithm.
- λ_j is the Poisson parameter of the jth type of driver for the random variable N.
- λ_j' is the Poisson parameter of the jth type of driver for the random variable N'.
- λ_j'' is the Poisson parameter of the jth type of driver for the random variable N''.

In this paper we propose to modify the relation (3) as

(4) $$f_X(x) = \sum_{j=1}^{r}\left\{\frac{\pi p_j \lambda_j'}{\mathbb{E}N} f_Z(x) + \frac{(1-\pi)p_j\lambda_j''}{\mathbb{E}N}\frac{f_Z(x)}{1-F_Z(c_j)}\mathbb{I}_{\{x \geq c_j\}}\right\},$$

where

$$\begin{aligned}\mathbb{E}N &= \sum_{j=1}^{r} p_j \lambda_j \\ &= \sum_{j=1}^{r} p_j\left[\pi\lambda_j' + (1-\pi)\lambda_j''\right].\end{aligned}$$

Using this modified procedure, we obtain after some iterations the actual claims frequency distribution as well as the optimal claims frequency distribution and the optimal retentions:

TABLE 6

PARAMETERS FOR THE ACTUAL AND OPTIMAL CLAIMS FREQUENCY DISTRIBUTIONS

j	λ_j'	λ_j''	c_j	p_j
1	0.0650	0.0492	26.0270	0.56189
2	0.3840	0.1733	78.7517	0.41463
3	1.1293	0.8651	26.9551	0.02348

The actual claim amount distribution is seen to be exponential with parameter $\mu = 84.86$.

The proportion $(1 - \pi)$ of drivers using the algorithm of Lemaire is also obtained:

$$1 - \pi = 65.52\%.$$

Without a bonus-malus system the pure premium would be:

$$\mathbb{E}N' \times \mathbb{E}Z = 0.2223 \times 84.86 = 18.8612.$$

Within the bonus-malus system the pure premium writes:

$$\mathbb{E}N \times \mathbb{E}X = 0.1551 \times 114.116 = 17.7035,$$

where

$$\mathbb{E}X = \sum_{j=1}^{r} \frac{\pi p_j \lambda_j'}{\mathbb{E}N} \mathbb{E}Z + \frac{(1-\pi) p_j \lambda_j''}{\mathbb{E}N} \mathbb{E}\left[Z \middle| Z \geq c_j\right].$$

It is not illogical that the pure premium be lower within the bonus-malus system: indeed some claims are retained by the drivers and so the aggregate claims is lower for the insurer.

The rest of the paper will be based on the following hypotheses regarding the introduction of a new bonus-malus system:

- the proportions $p_j, j = 1, \ldots, r$ remain the same for predicted future observed and optimal claims frequencies.
- the proportion of drivers using Lemaire's algorithm remains the same. In fact this is an important information given by the current bonus-malus system. If we have to implement a bonus-malus system for the first time we have no idea of the value of p and a guess is needed. This in fact explains the title of this paper: The practical *replacement* of a bonus-malus system.

When the drivers use Lemaire's algorithm they will report claims with an optimal frequency distribution: $(\lambda_j'''', p_j, j=1, \ldots, r)$. The actual claims frequency distribution $(\lambda_j', p_j, j=1, \ldots, r)$ is given in table 6. The future observed claims frequency distribution $(\lambda_j''', p_j, j=1, \ldots, r)$ will be given by

$$\lambda_j''' = \pi\lambda_j' + (1-\pi)\lambda_j'''', \qquad j=1, \ldots, r, \tag{5}$$

where the notation $(\lambda_j^*, p_j), j=1, \ldots, r$ gives the non-parametric mixed Poisson fit of the random variable N^* defined in the introduction. The optimal retention of the drivers using Lemaire's algorithm are also given by Lemaire's algorithm: $(c_j', j=1, \ldots, r)$.

4. A PRACTICAL BONUS-MALUS SYSTEM

The optimal bonus-malus system described in section 2 seems difficult to apply because of the infinite number of classes in both directions k and t. Such a system will certainly be too complicated for the policyholders. Therefore most European countries use bonus-malus systems with a finite number of classes.

Basically, the bonus-malus systems with finite number of classes (s) have the Markov property (if not, a redefinition of the system may be needed to show the Markov property). They are presented as the example of the introduction:

TABLE 7

PREMIUM LEVELS

s	0	1	2	3	...	$s-1$
C_s	C_0	C_1	C_2	C_3	...	C_{s-1}

Transition rules are given indicating how the drivers move inside the bonus-malus system according to their annual number of reported claims.

If we assume that the number of classes has been chosen as well as the transition rules, the premium levels need to be calculated. This is explained in Coene and Doray (1996) where they minimize a certain distance between the C_i and the $P_{(k,t)}$ in order to determine the C_i.

We will use the same methodology as Coene and Doray (1996) with some amendments.

Firstly, it is necessary to set up a table of $C_{(k,t)}$ parallel to the table of $P_{(k,t)}$. $C_{(k,t)}$ gives the different possible values for C_i in function of k and t. For example with the new bonus-malus system described in the introduction we have

TABLE 8

$C_{(k,t)}$

t/k	0	1	2	...
0	C_4	—	—	...
1	C_3	C_8	C_8	...
2	C_2	C_8 / C_7	C_8 / C_7	...
3	C_1	C_8 / C_7 / C_6	C_8 / C_7 / C_6	...
4	C_0	C_8 / C_7 / C_6 / C_5	C_8 / C_7 / C_6 / C_5	...
5	C_0	C_8 / C_7 / C_6 / C_5 / C_4	C_8 / C_7 / C_6 / C_5 / C_4	...
⋮				

Coene and Doray (1996) suggest to choose the maximum class for $C_{(k,t)}$ because it is often the most probable class.

However it is not difficult to find the probabilities of the following events: $[N(t) = k, C_{(k,t)} = C_i]$. Let us define

$$w(k, t, i) = \mathbb{P}[N(t) = k, C_{(k,t)} = C_i].$$

The following general formula is valid for a mixed Poisson process:

$$\mathbb{P}[N(n) - N(n-1) = k_n, \ldots, N(1) - N(0) = k_1] = \frac{\Pi\left(\sum_{i=1}^{n} k_i, n\right)}{n^{\sum_{i=1}^{n} k_i}} \frac{\left(\sum_{i=1}^{n} k_i\right)!}{k_1! \ldots k_n!}.$$

Therefore it is always possible to write the probabilities $w(k, t, i)$ as $a\Pi(k, t)$ where a depends on the different ways to reach C_i at time t with k claims. For example

$$\begin{aligned}
\mathbb{P}\left[C_{(1,1)} = C_8 \middle| N(1) = 1\right] &= 1, \\
\mathbb{P}\left[C_{(1,1)} = C_i \middle| N(1) = 1\right] &= 0 \quad \forall i < 8, \\
\mathbb{P}\left[C_{(1,2)} = C_8 \middle| N(2) = 1\right] &= \frac{\mathbb{P}[N(1) - N(0) = 0, N(2) - N(1) = 1]}{\mathbb{P}[N(2) = 1]} = \frac{1}{2}, \\
\mathbb{P}\left[C_{(1,2)} = C_7 \middle| N(2) = 1\right] &= \frac{\mathbb{P}[N(1) - N(0) = 1, N(2) - N(1) = 0]}{\mathbb{P}[N(2) = 1]} = \frac{1}{2}, \\
\mathbb{P}\left[C_{(1,2)} = C_i \middle| N(2) = 1\right] &= 0 \quad \forall i < 7.
\end{aligned}$$

The most obvious way of calculating the C_i is to minimize the following quadratic error with natural weights:

$$\sum_{(k,t,i)} w(k, t, i) \left[P_{(k,t)} - C_i\right]^2.$$

This minimization procedure is quite similar to the one derived in Coene and Doray (1996) but it is more natural because we use the exact values of $C_{(k,t)}$ and we do not use the stationary distribution of the drivers in the bonus-malus system.

As mentioned in Coene and Doray (1996) we now have a problem of optimization with s variables. Of course some constraints have to be taken into account:

- The entry class in the bonus-malus system is generally at level 100. So in our example we have to constrain $C_4 = 100$.
- Corresponding to the natural properties

$$\frac{\partial}{\partial t} P_{(k,t)} \leq 0 \quad \forall k,$$
$$P_{(k+1,t)} \geq P_{(k,t)} \quad \forall t,$$

  we have the natural constraint: $C_i \leq C_i + 1 \quad \forall i$.
- As we are interested in having integer percentages C_i, it may be interesting to constrain the C_i to being integer, i.e. to make optimization with integers.
- As a consequence of the financial equilibrium of the optimal bonus-malus system

$$\sum_{k=0}^{\infty} \Pi(k,t) P_{(k,t)} = 100 \quad \forall t,$$

  the following constraints may be imposed

$$\sum_{i=0}^{s-1} e_\infty(i) C_i \geq 100,$$
$$\sum_{i=0}^{s-1} e_t(i, \mathbf{e}_0) C_i \geq 100 \quad \text{for some t with } \mathbf{e}_0 \text{ given,}$$

  where $e_\infty(i)$ denotes the i^{th} component of the stationary distribution of the drivers in the bonus-malus system ($\mathbf{e}_\infty$) and $e_t(i, \mathbf{e}_0)$ denotes the i^{th} component of the transient distribution at time t of the drivers in the bonus-malus system ($\mathbf{e}_t(\mathbf{e}_0)$) with initial distribution $\mathbf{e}_0$. It should be clear that in the present paper we work with closed Markov chains, i.e. there are no entries other than at level 100% and there are no exits from the portfolio. In case one has more information regarding these practical aspects, there is no problem to take them into account in the calculation of the transient and stationary distributions.

It should be clear that there will always be a solution to this problem unless one works with very illogical constraints. If the contraints are logical, there should be a feasible solution implying that the minimization procedure has a solution.

5. An iterative algorithm to find the new bonus-malus system

Let us assume that the number of classes of the new bonus-malus system has been chosen. The transition rules are also known.

On the one hand we have to determine the premium levels (C_i, $i = 0, \ldots, s-1$) associated with each class of the bonus-malus system.

On the other hand the conception of the bonus-malus system is based on the future observed claims frequency distribution $((\lambda_j''', p_j), j = 1, \ldots, r)$ within the bonus-malus system. The problem is that the future observed claims frequency distribution is not known and depends on the premium levels C_i.

The solution is thus to find the bonus-malus levels as well as the future observed frequency distribution iteratively.

We also have to take into account some loading for the bonus hunger. Indeed we observed in section 3 that the presence of a bonus-malus system induces the retention of some claims by the policyholders allowing for a lower pure premium. If the bonus-malus system is changed, it probably will happen that the pure premium will be changed. Then the commercial premium should be adapted.

We will assume that there is a proportional relation between the commercial premium and the pure premium. In our example, the loading of the pure premium is given by

$$a = \frac{35}{17.7035} = 1.977.$$

We will demand that this proportion remains the same wathever the new bonus-malus system. So the commercial premium will be adapted as

$$P^{new} = P^{old} \frac{\text{Pure premium}^{new}}{\text{Pure premium}^{old}}.$$

The following algorithm is proposed.

Algorithm

Initializing step

Choose the number of classes of the practical bonus-malus system as well as the transition rules.

Use the actual claims frequency distribution as an initial guess for the future observed claims frequency distribution.

Iterations

Do

Use the future observed claims frequency distribution obtained as a non-parametric mixture of Poisson distributions in order to find the observed claims frequency distribution under the form of a Hofmann distribution by matching the first three moments.

With the Hofmann distribution obtain an optimal bonus-malus table.

From the optimal bonus-malus table obtain the premium levels of the practical bonus-malus system.

Adapt the commercial premium in function of the predicted pure premium.
Make an initial choice for the future observed claims frequency distribution.
Run Lemaire's algorithm to obtain the optimal claims frequency distribution of the drivers and thus the future observed frequency distribution.

Do

Use the observed claims frequency distribution obtained by Lemaire's algorithm in order to rerun Lemaire's algorithm.

Until convergence

Until convergence.

We will now set up the practical bonus-malus system based on the hypotheses made in the introduction.

The optimization program used in order to match as best as possible the $P_{(k,t)}$ and the C_i is the following:

$$Min \sum_{k=0}^{4} \sum_{t=0}^{10} \sum_{i=0}^{8} w(k,t,i)\left[P_{(k,t)} - C_i\right]^2. \tag{6}$$

under the following constraints

$$\begin{aligned} C_i &\quad \text{are integers,} \\ C_4 &= 100 \\ C_{i+1} - C_i &\geq 0, \\ \sum_{i=0}^{8} e_\infty(i) C_i &\geq 100, \\ \sum_{i=0}^{8} e_0(i) C_i &\geq 100, \\ \sum_{i=0}^{8} e_2(i, \mathbf{e}_0) C_i &\geq 100, \\ \sum_{i=0}^{8} e_4(i, \mathbf{e}_0) C_i &\geq 100. \end{aligned}$$

where $\mathbf{e}_0 = \{0.5729, 0.0562, 0.0661, 0.0784, 0.0421, 0.0441, 0.0457, 0.0429, 0.0516\}$ i.e. the stationary distribution of the drivers in the current bonus-malus system.
We will now fully describe the first iteration of our algorithm.
The hypotheses for the algorithm of Lemaire are the following

- $\beta = 6\%$.
- $P = 35$.
- $t = \frac{1}{2}$.
- X is exponentially distributed with mean 84.86.

As an initializing step we use the actual claims frequency distribution:

TABLE 9

PARAMETERS OF THE ACTUAL CLAIMS FREQUENCY DISTRIBUTION (NON-PARAMETRIC FIT)

$\lambda_1' = 0.0650$	$p_1 = 0.56189$
$\lambda_2' = 0.3840$	$p_2 = 0.41463$
$\lambda_3' = 1.1293$	$p_3 = 0.02348$

The corresponding Hofmann distribution is given by matching the first three moments:

TABLE 10

PARAMETERS OF THE ACTUAL CLAIMS FREQUENCY DISTRIBUTION (HOFMANN PARAMETRIC FORM)

$p = 0.2223$
$c = 0.1897$
$a = 1.0452$

An optimal bonus-malus table is immediately obtained with the expected value premium principle:

TABLE 11

OPTIMAL BONUS-MALUS TABLE

k/t	0	1	2	3	4
1	83	158	232	305	377
2	71	136	199	262	325
3	62	119	175	230	285
4	55	106	156	205	254
5	50	96	140	185	229
6	45	87	128	168	209
7	41	80	117	154	192
8	38	74	108	143	177
9	35	68	101	133	164
10	33	64	94	124	154

The practical bonus-malus table is derived by solving the minimization procedure (6):

TABLE 12

PRACTICAL BONUS-MALUS LEVELS ASSOCIATED WITH TABLE 10

s	0	1	2	3	4	5	6	7	8
C_s	87	87	94	100	100	121	130	148	182

By iterating Lemaire's algorithm, we find the future observed and optimal claims frequency distributions:

TABLE 13

PARAMETERS FOR THE ACTUAL AND OPTIMAL CLAIMS FREQUENCY DISTRIBUTIONS

j	λ_j'''	λ_j''''	c_j	p_j
1	0.0407	0.0280	73.4642	0.56189
2	0.3021	0.2590	36.0752	0.41463
3	1.0680	1.0358	7.6594	0.02348

The pure premium within this bonus-malus system would be

$$\mathbb{E}N''' \times \mathbb{E}X' = 0.1732 \times 104.59 = 18.119.$$

So we have to correct the commercial premium by a factor

$$\frac{18.119}{17.7035} = 1.0235.$$

This is the end of the first iteration of our algorithm.
The following iterations are summarized in the next table:

TABLE 14

ITERATIONS OF THE ALGORITHM

Iteration	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	λ_1'''	λ_2'''	λ_3'''
1	87	87	94	100	100	121	130	148	182	0.0407	0.3021	1.0680
2	82	87	93	94	100	126	163	163	202	0.0351	0.2739	1.0566
3	79	87	91	100	100	137	156	173	217	0.0335	0.2636	1.0441
4	79	81	91	100	100	139	153	176	222	0.0336	0.2618	1.0395
5	78	84	93	100	100	142	154	177	223	0.0331	0.2603	1.0399
6	78	83	93	100	100	141	155	177	224	0.0331	0.2600	1.0390
7	78	83	93	100	100	141	155	177	224	0.0331	0.2600	1.0390

The optimal frequency and retention for the last iteration write:

TABLE 15

OPTIMAL FREQUENCY AND RETENTION

j	λ_j'''	c_j	p_j
1	0.0163	122.78	0.56189
2	0.1946	66.5305	0.41463
3	0.9915	11.9066	0.02348

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The pure premium within this new bonus-malus system is 17.229. This pure premium is lower than the one in use within the old bonus-malus system (17.7035) and therefore it has been possible to reduce the commercial premium within the new bonus-malus system to

$$35 \times \frac{17.229}{17.7035} = 34.0619.$$

Due to the hard transition rules of the new bonus-malus system, we observe that there is a more important hunger for bonus which translates to a lower pure premium.

We observe that the premium levels are not well differentiated. Should one demand more differentiated premium levels, it suffices to add some constraints in the optimization algorithm e.g. like

$$C_{i+1} - C_i \geq 5.$$

Let us note that with this bonus-malus system the future premium income, in percentage of the pure premium income, is

TABLE 16

FUTURE PREMIUM INCOME

t	Income
0	100%
1	113.77%
2	116.34%
3	117.82%
4	118.19%
5	118.46%
6	118.58%
∞	118.60%

This means that in such a situation the insurer will be able to propose a reduction of the premium because the financial desequilibrium is in his favour. Note that if the insurer decides to do so, it may have an influence on the behaviour of the drivers using Lemaire's algorithm. Then a new observed frequency distribution should be calculated as well as the new transient and stationary distributions.

Let us have a look at the bonus-malus system obtained if we ease the constrain

$$\sum_{i=0}^{8} e_0(i) C_i \geq 100.$$

We find

TABLE 17

BONUS-MALUS PROPOSED

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
57	72	80	92	100	135	152	180	229

Observed and optimal distributions are:

TABLE 18

OBSERVED AND OPTIMAL DISTRIBUTIONS

j	λ_j'''	λ_j''''	c_j	p_j
1	0.0301	0.0118	154.052	0.56189
2	0.2423	0.1678	84.9733	0.41463
3	1.0296	0.9771	13.447	0.02348

The pure premium within this bonus-malus system is

$$0.1416 \times 118.372 = 16.761$$

whereas it was 17.7035 within the old bonus-malus system. This allows a reduction of the commercial premium within the new bonus-malus system:

$$35 \times \frac{16.761}{17.7035} = 33.1367.$$

In percentage of the pure premium, the future pure premium income is

TABLE 19

FUTURE PREMIUM INCOME

t	Income
0	85.85%
1	100.02%
2	102.48%
3	104.13%
4	104.78%
5	105.14%
6	105.32%
∞	105.36%

If the company adopts this bonus-malus system, it accepts to lose money during the first year. The profit from the subsequent years will rapidly absorb the losses of the first year.

It should be noted that we have no proof that the proposed algorithm will converge for any case. It did converge for all numerical examples we used and as Lemaire's algorithm is convergent, we do not see a reason why the present algorithm would not converge to a solution.

6. Conclusion

We have derived in this paper an iterative algorithm based on Lemaire's algorithm in order to calculate the premium levels associated with a practical bonus-malus system. The actual claims frequency distribution, obtained from the algorithm of Walhin and Paris (2001), is used to predict the future observed claims frequency distribution. The latter is a function of the premium levels because of the hunger for bonus. The premium levels are a function of the future observed claims frequency distribution because the optimal bonus-malus table and the transient and stationary distributions of the drivers in the bonus-malus system are obtained with the use of the future observed claims frequency distribution. Therefore the need for an iterative algorithm.

The technique used by Coene and Doray (1996) in order to set up a practical bonus-malus system from an optimal bonus-malus table has been reviewed and extended.

Parametric and non-parametric mixed Poisson distributions are used throughout the paper.

It is important to have an estimate of the proportion of the policyholders using Lemaire's algorithm. This estimate is available from the algorithm of Walhin and Paris (2001) in case a bonus-malus system was in use before. If not, a guess has to be done.

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