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ON DUAL BAER MODULES

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Abstract. In this paper we introduce $\mathcal{T}$-non-cosingular modules, dual Baer modules and $\mathcal{K}$-modules. We prove that a module M is lifting and $\mathcal{T}$-non-cosingular if and only if it is a dual Baer and $\mathcal{K}$-module. Rings for which all modules are dual Baer are precisely determined. We also give a necessary condition for a finite direct sum of dual Baer modules to be dual Baer.

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1. Introduction. Throughout this paper S will denote the endomorphism ring of any module M. In [**11**] and [**12**], the authors investigate Baer modules and $\mathcal{K}$-non-singular modules. Motivated by these works, we introduce dual notions, these of dual Baer modules and $\mathcal{T}$-non-cosingular modules. A module M is called a *dual Baer module* if for every $N \leq M$, there exists an idempotent e in S such that $\mathrm{D}(N) = \{\varphi \in S \mid \mathrm{Im}\varphi \subseteq N\} = eS$. The module $\mathbb{Z}_\mathbb{Z}$ is not dual Baer, because for every integer $n \geq 2$, $\mathrm{D}(n\mathbb{Z})$ is a non-zero and proper right ideal of $End(\mathbb{Z})$. On the other hand, the modules $\mathbb{Q}_\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ are dual Baer for every prime p (see Corollary 2.4). A module M is called a $\mathcal{T}$-non-cosingular module if, for every non-zero endomorphism φ of M, $\mathrm{Im}\varphi$ is not small in M. Following [**14**], the module M is called *non-cosingular* if for every non-zero module N and every non-zero homomorphism $f : M \to N$, $\mathrm{Im}f$ is not a small submodule of N. It is clear that every non-cosingular module is $\mathcal{T}$-non-cosingular.

A module M is called a *lifting* module if for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$, or equivalently, for every submodule N of M there is a direct summand K of M such that $N/K \ll M/K$.

A ring R is called a right *Harada* ring if every injective right R-module is lifting (see [**2**, 28.1 and 28.10]).

The aim of this paper is to study dual Baer modules. $\mathcal{T}$-non-cosingular modules will be studied in a subsequent paper.

Section 2 is devoted to the study of dual Baer modules. We will begin by providing an equivalent formulation of dual Baer modules (Theorem 2.1). Then we show that R_R is dual Baer if and only if the ring R is semi-simple. We also prove Theorem 2.14 which exhibits the connections between dual Baer modules and lifting modules. Moreover, we characterize right hereditary right Harada rings in terms of lifting dual Baer modules (Proposition 2.15).

In Section 3 we will be concerned with the direct sums of dual Baer modules. The structure of dual Baer modules over Dedekind rings is described explicitly.

2. Dual Baer modules.

Rizvi and Roman introduced the concept of Baer modules in [**11**]. Let M be a module. According to [**11**], M is called a *Baer module* if for all $N \leq M$, the left annihilator of N in S, $l_S(N) = Se$, with $e^2 = e \in S$. Let $N \leq M$. In this paper we introduce the right ideal $\mathrm{D}(N) = \{\varphi \in S \mid \mathrm{Im}\varphi \subseteq N\}$ of S as the dual notion of left annihilator $l_S(N)$ of N in S. Clearly, $\mathrm{D}(e(M)) = eS$ for any idempotent e in S. A module M is called *dual Baer* if for every $N \leq M$, there exists an idempotent e in S such that $\mathrm{D}(N) = eS$. It is obvious that any module with semi-simple endomorphism ring is dual Baer. The module M is said to have the *(strong) summand sum property*, denoted briefly by (SSSP) SSP, if the sum of (any family of) two direct summands of M is a direct summand of M. Next, we provide a characterization of dual Baer modules in terms of SSSP.

THEOREM 2.1. *The following are equivalent for a module M:*
(i) *M is dual Baer.*
(ii) *For every subset A of S, $\sum_{f \in A} Imf = e(M)$ where $e = e^2 \in S$.*
(iii) *For every right ideal I of S, $\sum_{f \in I} Imf = e(M)$ where $e = e^2 \in S$.*
(iv) *M has the SSSP and for every $\varphi : M \to M$, $Im\varphi$ is a direct summand of M.*

Proof. (i) $\Rightarrow$ (ii) Let $A \subseteq S$. Let $N = \sum_{f \in A} \mathrm{Im}f \leq M$. Since M is dual Baer, there exists an idempotent $e \in S$ such that $\mathrm{D}(N) = eS$. Thus $e(M) \subseteq N$. On the other hand, for every $f \in A$, we have $f \in \mathrm{D}(N) = eS$. Therefore for every $f \in A$, there exists $s \in S$ such that $f = es$. It follows that for every $f \in A$, $\mathrm{Im}f \subseteq e(M)$. This gives that $N \subseteq e(M)$. Consequently, $N = e(M)$.

(ii) $\Rightarrow$ (iv) It is a consequence of the fact that every direct summand of M is an epimorphic image of M.

(iv) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let N be a submodule of M. Consider the right ideal $I = \mathrm{D}(N) = \{f \in S \mid \mathrm{Im}f \subseteq N\}$ of S. By hypothesis, $\sum_{f \in I} \mathrm{Im}f = e(M)$ for some $e = e^2 \in S$. Then $e \in \mathrm{D}(N)$ and hence $eS \subseteq \mathrm{D}(N)$. Now if $f \in \mathrm{D}(N) = I$, then $\mathrm{Im}f \subseteq e(M)$. Moreover, since $S = eS \oplus (1-e)S$, we get $f = es_1 + (1-e)s_2$ for some $s_1, s_2 \in S$. Therefore $f = es_1$ because $\mathrm{Im}f \subseteq e(M)$. Hence $f \in eS$. So $eS = \mathrm{D}(N)$. This completes the proof. □

From Theorem 2.1 it follows easily that all semi-simple modules are dual Baer.

Note that if R is a commutative ring, then for any dual Baer module M and any $r \in R$, Mr is a direct summand of M.

COROLLARY 2.2. *A module M is an indecomposable dual Baer module if and only if for every non-zero $\varphi \in S$, φ is an epimorphism.*

Proof. By Theorem 2.1. □

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In view of the above corollary, every indecomposable dual Baer module M is cohopfian (i.e. every monomorphism from M to M is an isomorphism). Let R be a commutative Noetherian local complete domain with maximal ideal m. Then $E(R/m)$ is dual Baer by [**13**, page 143, Corollary 2] and Corollary 2.2.

COROLLARY 2.3. *Every dual Baer module M is $\mathcal{T}$-non-cosingular.*

Proof. Let M be a dual Baer module and let $\varphi \in End(M)$ with $\mathrm{Im}\varphi \ll M$. By Theorem 2.1, $\mathrm{Im}\varphi$ is a direct summand of M. Therefore $\mathrm{Im}\varphi = 0$, and so $\varphi = 0$. □

COROLLARY 2.4. *Let M be an injective R-module over a right hereditary ring R. Then M is dual Baer if and only if M has the SSSP. In particular, every indecomposable injective R-module is dual Baer.*

Proof. Let f be an endomorphism of M. Since $\mathrm{Im}f$ is a factor module of M and R is right hereditary, $\mathrm{Im}f$ is injective. Hence $\mathrm{Im}f$ is a direct summand of M. The result follows from Theorem 2.1. □

COROLLARY 2.5. *Let M be a dual Baer module. Then every direct summand of M is also dual Baer.*

Proof. Let $M = N \oplus N'$. Since M has the SSSP, it is easy to see that N has the SSSP. Now let $f : N \to N$ be any endomorphism of N. Consider the homomorphism $f \oplus 0_{N'} : N \oplus N' \to N \oplus N'$ defined by $f \oplus 0_{N'}(n + n') = f(n)$. Now $f \oplus 0_{N'}(N \oplus N') = f(N)$ is a direct summand of M and hence it is a direct summand of N. Therefore N is dual Baer by Theorem 2.1. □

COROLLARY 2.6. (i) *Every dual Baer module is a direct sum of indecomposable modules.*

(ii) *Every dual Baer lifting module is a direct sum of hollow modules.*

Proof. (i) By Theorem 2.1 and [**10**, Theorem 2.17].

(ii) By (i) and [**2**, 22.2 and 22.6]. □

A module M is called a *regular* module if every cyclic submodule of M is a direct summand of M (see [**8**, Page 272, Exercise 16]).

COROLLARY 2.7. *If M is a regular dual Baer module, then M is semi-simple.*

Proof. Let $N \leq M$. Note that $N = \sum_{x \in N} xR$. By Theorem 2.1, N is a direct summand of M. □

PROPOSITION 2.8. *If R_R is dual Baer, then the ring R is von Neumann regular.*

Proof. Take any principal right ideal $I = aR$ of R. Consider the R-homomorphism $f : R_R \longrightarrow R_R$ defined by $f(r) = ar$, where $r \in R$. Then $\mathrm{Im}f = I$, which is a direct summand of R_R by hypothesis. So R is von Neumann regular. □

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COROLLARY 2.9. *Let R be a ring. Then the following are equivalent:*
(i) *R_R is dual Baer.*
(ii) *R_R is semi-simple.*
(iii) *${}_RR$ is dual Baer.*
(iv) ${}_RR$ is semi-simple.

Proof. (i) ⇔ (ii) Assume R_R is dual Baer. By Proposition 2.8, R is von Neumann regular. Therefore R_R is semi-simple by Corollary 2.7. The converse is clear.
(iii) ⇔ (iv) The proof runs as before.
(ii) ⇔ (iv) By [**8**, Theorem 8.2.1]. □

COROLLARY 2.10. *The following are equivalent for any ring R:*
(i) *Every right R-module is dual Baer.*
(ii) *Every left R-module is dual Baer.*
(ii) *R is semi-simple.*

LEMMA 2.11. *Let N be a submodule of a $\mathcal{T}$-non-cosingular module M and let e be an idempotent in S. If $e(M) \le N$ and $N/e(M) \ll M/e(M)$, then $\mathrm{D}(N) = eS$.*

Proof. Since $e(M) \le N$, we have $eS \subseteq \mathrm{D}(N)$. Now let $\varphi \in \mathrm{D}(N)$ and let us prove that $\varphi \in eS$. Note that $M = e(M) \oplus (1-e)(M)$ and $N/e(M) \ll M/e(M)$. Therefore $N \cap (1-e)(M) \ll M$. Since $S = eS \oplus (1-e)S$, there exist s_1 and s_2 in S such that $\varphi = es_1 + (1-e)s_2$. Thus $\mathrm{Im}(1-e)s_2 \le N \cap (1-e)(M) \ll M$. By hypothesis, we get $(1-e)s_2 = 0$ and hence $\varphi = es_1 \in eS$. □

A module M is called a $\mathcal{K}$*-module* if, for every non-small submodule N of M, there exists a non-zero endomorphism φ of M such that $\varphi^{-1}(N) = M$. This is obviously equivalent to the condition that, $\mathrm{D}(N) \ne 0$ for every non-small submodule N of M. Note that $\mathbb{Q}_{\mathbb{Z}}$ is not a $\mathcal{K}$-module since it contains a proper non-small submodule and every non-zero endomorphism of $\mathbb{Q}_{\mathbb{Z}}$ is an isomorphism. On the other hand, $\mathbb{Z}$ is a $\mathcal{K}$-module since every non-zero submodule of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

LEMMA 2.12. *Every lifting module M is a $\mathcal{K}$-module.*

Proof. Let $N \le M$ such that $\mathrm{D}(N) = 0$. By the lifting property, there exists a direct summand K of M such that $N/K \ll M/K$. Now there exists an idempotent $e \in S$ such that $e(M) = K$. This implies that $e \in \mathrm{D}(N)$. By hypothesis, $e = 0$. Therefore $K = 0$ and hence $N \ll M$. □

PROPOSITION 2.13. *Let M be a dual Baer $\mathcal{K}$-module. Then M is lifting.*

Proof. Let N be any non-small submodule of M. Then $\mathrm{D}(N) = eS$ for some non-zero idempotent $e \in S$. Thus $\mathrm{Im}e$ is a non-zero direct summand of M which is contained in N. Moreover, if f is another idempotent in S such that $\mathrm{Im}e \le \mathrm{Im}f \le N$, then $f \in \mathrm{D}(N)$. Therefore there exists $s \in S$ such that $f = es$. So $\mathrm{Im}f \le \mathrm{Im}e$ and hence $\mathrm{Im}e = \mathrm{Im}f$. This proves that $\mathrm{Im}e$ is a maximal direct summand of M with $\mathrm{Im}e \le N$. By [**15**, 41.12], M is lifting. □

THEOREM 2.14. *The following statements are equivalent for a module M:*
(i) *M is a lifting $\mathcal{T}$-non-cosingular module.*
(ii) *M is a dual Baer and $\mathcal{K}$-module.*

Proof. By Lemmas 2.11, 2.12, Corollary 2.3 and Proposition 2.13. □

Recall that a module M is *uniserial* if its submodules are linearly ordered by inclusion and it is *serial* if it is a direct sum of uniserial submodules. The ring R is right (left) *serial* if the right (left) R-module R_R ($_RR$) is serial and it is *serial* if it is both right and left serial.

Theorem 2.14 is a useful source of examples of dual Baer modules. In fact, since every non-cosingular module is $\mathcal{T}$-non-cosingular, every non-cosingular lifting module is dual Baer. By using this fact, we will construct the following examples.

(1) If R is a right hereditary ring, then every injective module is non-cosingular by [**14**, Proposition 2.7]. Thus every injective lifting module is dual Baer.

(2) If R is a right Harada ring, then every injective module is lifting. Therefore every injective non-cosingular module is dual Baer.

(3) If the ring R is artinian serial with $(Rad(R))^2 = 0$, then every module is lifting by [**2**, 29.10]. So every non-cosingular module is dual Baer.

PROPOSITION 2.15. *The following statements are equivalent for a ring R:*
(i) *R is a right hereditary right Harada ring.*
(ii) *Every injective module is lifting dual Baer.*

Proof. (i) ⇒ (ii) Let M be an injective module. It is clear that M is lifting. By [**5**, Proposition 1.6], M has the SSP. Thus M has the SSSP by [**4**, Proposition 4.9]. Therefore M is dual Baer by Corollary 2.4.

(ii) ⇒ (i) It follows from [**5**, Proposition 1.6]. □

What is lacking is an explicit example. Let k be a field and let R be the ring of $n \times n$ upper triangular matrices over k. By [**9**, Example 2.36], R is an artinian right hereditary ring and $Rad(R)$ consists of all matrices in R with a zero diagonal. Thus $(Rad(R))^2 = 0$. On the other hand, R is a serial ring by [**3**, Example 1.21]. It follows that every injective R-module is dual Baer.

Theorem 2.14 and Proposition 2.15 show the importance of dual Baer modules in the theory of lifting modules, hereditary rings, and Harada rings.

In [**14**], the authors defined $\overline{Z}(M) = \cap\{\mathrm{Ker}(g) : g \in Hom(M, N), N \ll E(N)\}$ and $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$, where $E(N)$ is the injective hull of N.

PROPOSITION 2.16. *Let M be a lifting module. Then $\overline{Z}^2(M)$ is a direct summand of M which is dual Baer.*

Proof. By [**14**, Corollary 3.4 and Theorem 4.1] and Theorem 2.14. □

PROPOSITION 2.17. *Let M be an indecomposable dual Baer module with finite uniform dimension. Then S is semi-local.*

Proof. By Corollary 2.2 and [**1**, Theorem 5]. □

Note that the ring of endomorphisms of a dual Baer module may not be von Neumann regular. Also if the ring of endomorphisms of any module M is von Neumann regular, then M need not be dual Baer.

EXAMPLE 2.18. *(i) Let M be the Prüfer p-group $\mathbb{Z}(p^\infty)$. It is dual Baer by Corollary 2.4. But $S = End(M)$ is not von Neumann regular.*

*(ii) Let K be a field and let $R = \prod_{i=1}^{\infty} K_i$ with $K_i = K$ for $i = 1, 2, \ldots$. In [**8**, Page 264], it is proven that the ring R is von Neumann regular which is not semi-simple. Thus R_R is not dual Baer by Corollary 2.9. But $End(R)$ is von Neumann regular.*

In this vein we can give the following result.

PROPOSITION 2.19. (i) *Let M be a dual Baer module such that for every endomorphism f of M, $Kerf$ is a direct summand of M. Then S is von Neumann regular.*

(ii) *Let M be a module. If S is von Neumann regular and M has the SSSP, then M is dual Baer.*

Proof. By Theorem 2.1 and [**8**, Page 272, Exercise 17]. □

3. Direct sums of dual Baer modules. If R is a Dedekind domain, then R is said to be *proper* if R is not a field. If R is a proper Dedekind domain, then for each non-zero prime ideal P of R, $R(P^\infty)$ will denote the P-primary component of the torsion R-module K/R, where K is the quotient field of R. To prove Theorem 3.4 we need the following three results.

EXAMPLE 3.1. *Let R be a proper Dedekind domain. Let P be any non-zero prime ideal of R. Consider the module $M = R(P^\infty) \oplus R/P$ and the endomorphism $f : M \longrightarrow M$ defined by $f(x + \overline{y}) = cy$ with $x \in R(P^\infty)$, $y \in R$ and c is a non-zero element of $R(P^\infty)$ such that $cP = 0$. It is clear that $Imf = cR$ which is non-zero and small in M. So M is not a $\mathcal{T}$-non-cosingular module. In particular, for any prime integer p, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}/p\mathbb{Z}$ is not a $\mathcal{T}$-non-cosingular $\mathbb{Z}$-module.*

LEMMA 3.2. *Let $L = xR$ be a cyclic module over a commutative ring R. Then L is dual Baer if and only if L is semi-simple.*

Proof. Let $y \in L$. Then there exists $r \in R$ such that $y = xr$. Consider the endomorphism f of L defined by $f(x\alpha) = y\alpha$. The map f is well defined since R is commutative. As L is dual Baer, yR is a direct summand of L by Theorem 2.1. Applying Theorem 2.1 again, L has SSSP, and hence every submodule of L is a direct summand. Therefore L is semi-simple. The converse is clear. □

LEMMA 3.3. *Suppose that R is a commutative ring which is not semi-simple. Let M be an indecomposable module containing an element x such that $x \notin Rad(M)$ and $Ann_R(x) = 0$. Then M is not dual Baer.*

Proof. Suppose that M is dual Baer. Since $x \notin Rad(M)$, xR is not small in M. Let L be a proper submodule of M such that $xR + L = M$. If $xR \cap L = 0$, then $M = xR$ which is isomorphic to R since $Ann_R(x) = 0$. Thus R is dual Baer and hence R is semi-simple by Corollary 2.9. This contradicts our assumption. So $xR \cap L \neq 0$. Let $0 \neq r \in R$ such that $xr \in L$. Now consider the endomorphism of M defined by $f(y) = yr$ for every $y \in M$. Since $f \neq 0$, Im$f = M$ by Corollary 2.2. But Im$f = Mr = (xr)R + Lr \leq L$. Thus $L = M$, a contradiction. Consequently, M is not dual Baer. □

THEOREM 3.4. *Let R be a proper Dedekind domain with quotient field K. The following are equivalent for an R-module M:*

(i) *M is dual Baer.*

(ii) *M is a direct sum of copies of K, $(R(P_i^\infty))_{i\in I}$ and $(R/Q_j)_{j\in J}$ where $(P_i)_{i\in I}$ and $(Q_j)_{j\in J}$ are non-zero prime ideals of R with $P_i \neq Q_j$ for every couple $(i, j) \in I \times J$.*

Proof. (i) ⇒ (ii) Since M is dual Baer, $M = \oplus_{k\in\Lambda} M_k$ is a direct sum of indecomposable submodules $M_k (k \in \Lambda)$ by Corollary 2.6. By [**7**, Theorem 10], each M_k is either isomorphic to $R(P^\infty)$ or R/P^n for some prime ideal P or M_k is torsion-free. Note that if M_k is isomorphic to R/P^n, then $P^n = P$ by Corollary 2.5 and Lemma 3.2. Now if M_k is a non-divisible torsion-free module, then M_k is isomorphic to K (see [**7**, Theorem 7] and Lemma 3.3). The proof of the necessity is completed by Example 3.1 and Corollaries 2.3 and 2.5.

(ii) ⇒ (i) It is well known that over R, a module N is radical if and only if it is divisible if and only if it is injective. Let M be a module having the structure described in the statement. By [**7**, Theorem 8], M possesses a unique largest injective submodule $I(M)$. Note that $I(M)$ is the sum of all injective submodules of M. Moreover, $M = I(M) \oplus S(M)$ where $S(M) = [\oplus_{j\in J} M_{Q_j}]$ is semi-simple and M_{Q_j} is the Q_j-primary component of $T(M)$, the torsion submodule of M. Let N and L be submodules of M such that $M = N \oplus L$. Then $N = I(N) \oplus N_1$ and $L = I(L) \oplus L_1$. Hence $M = I(N) \oplus I(L) \oplus N_1 \oplus L_1$. By [**6**, Lemma 2.1], $I(M) = I(N) \oplus I(K)$. So $N_1 \oplus K_1 \cong S(M)$ and hence N_1 is a direct summand of $S(M)$. Note that injective R-modules and semi-simple R-modules all have the SSSP. Thus $I(M)$ and $S(M)$ have the SSSP. Therefore M has the SSSP. On the other hand, if f is an endomorphism of M, then $f(I(M))$ is injective since R is an hereditary ring. So $f(I(M))$ is a direct summand of $I(M)$. This gives that $f(M)$ is a direct summand of M since $f(S(M))$ is a direct summand of $S(M)$. Consequently, M is dual Baer by Theorem 2.1. □

Note that the last theorem gives many examples (see also Example 3.1) showing that a direct sum of dual Baer modules is not, in general, dual Baer.

COROLLARY 3.5. *A $\mathbb{Z}$-module M is dual Baer if and only if M is isomorphic to a direct sum of arbitrarily many copies of $\mathbb{Q}$ and $(\mathbb{Z}(p_i^\infty))_{i\in I}$ and $(\mathbb{Z}/q_j\mathbb{Z})_{j\in J}$, where $p_i (i \in I)$ and $q_j (j \in J)$ are primes with $p_i \neq q_j$ for every couple $(i, j) \in I \times J$.*

Proof. By Theorem 3.4. □

THEOREM 3.6. *Let R be a non-local Dedekind domain. The following are equivalent for a module M:*

(i) *M is dual Baer lifting.*

(ii) *M is torsion and every P-primary component of M is isomorphic either to $[R(P^\infty)]^{n_P}$ or $[R/P]^{(I_P)}$ for some natural number n_P and index set I_P.*

Proof. By Theorem 3.4, [**10**, Propositions A.7 and A.8]. □

COROLLARY 3.7. *A $\mathbb{Z}$-module M is dual Baer lifting if and only if M is torsion and each p-primary component M_p is isomorphic either to $[\mathbb{Z}(p^\infty)]^{n_p}$ or $[\mathbb{Z}/p\mathbb{Z}]^{(I_p)}$ for some natural number n_p and index set I_p.*

Proof. By Theorem 3.6. □

Let A and B be modules. If for every homomorphism $\varphi : A \longrightarrow B$, $\mathrm{Im}\varphi$ is a direct summand of B, then we say that A is *relative d to B*. We call the modules A and B *relatively d-modules* if, A is relative d to B and B is relative d to A.

LEMMA 3.8. *Let M_1 and M_2 be dual Baer relatively d-modules. Assume that M_2 is M_1-projective (or M_1 is M_2-projective). Then $M = M_1 \oplus M_2$ is dual Baer.*

Proof. Let $I = \{\varphi_j \mid j \in J\}$ be any subset of S. Let $K = \sum_{j\in J} \mathrm{Im}\varphi_j$. We want to prove that K is a direct summand of M. Let $i_1 : M_1 \to M$, $i_2 : M_2 \to M$ be the canonical inclusions and let $\pi_2 : M \to M_2$ be the canonical projection. Let $j \in J$. Since M_2 is dual Baer, $\mathrm{Im}(\pi_2\varphi_j i_2) = \pi_2(\varphi_j(M_2))$ is a direct summand of M_2. Since M_2 is relative d to M_1, we have $\mathrm{Im}(\pi_2\varphi_j i_1) = \pi_2(\varphi_j(M_1))$ is a direct summand of M_2. As M_2 has SSSP, $\pi_2(\mathrm{Im}\varphi_j) = \pi_2(\varphi_j(M_2)) + \pi_2(\varphi_j(M_1))$ is a direct summand of M_2. Hence $\pi_2(K)$ is a direct summand of M_2 since M_2 has SSSP. It follows that $\pi_2(K) + M_1$ is a direct summand of M. But it is clear that $\pi_2(K) + M_1 = K + M_1$. Thus $K + M_1$ is a direct summand of M. Let L and E be two submodules of M such that $M = (K + M_1) \oplus L$ and $K + M_1 = E \oplus M_1$. Then $M = E \oplus L \oplus M_1$. Thus $M_2 \cong E \oplus L$. Since M_2 is M_1-projective, E is M_1-projective. So there exists a submodule $K' \leq K$ such that $K' \oplus M_1 = K + M_1$ (see [**2**, 4.12]). Thus $M = K' \oplus M_1 \oplus L$. Hence $K = K' \oplus [(M_1 \oplus L) \cap K] = K' \oplus (M_1 \cap K)$. Now we consider the homomorphism $\pi\varphi_j : M \to M_1$ where $\pi : M = K' \oplus M_1 \oplus L \to M_1$ is the canonical projection. Since M_1 is dual Baer and M_2 is relative d to M_1, $\mathrm{Im}(\pi\varphi_j) = \pi(\varphi_j(M_1)) + \pi(\varphi_j(M_2))$ is a direct summand of M_1 (Theorem 2.1). Hence $\sum_{j\in J} \mathrm{Im}(\pi\varphi_j)$ is a direct summand of M_1 because M_1 has SSSP. But $\sum_{j\in J} \mathrm{Im}(\pi\varphi_j) = \pi(\sum_{j\in J} \mathrm{Im}\varphi_j) = \pi(K) = M_1 \cap K$. Then $M_1 \cap K$ is a direct summand of M_1. Therefore $K = K' \oplus (M_1 \cap K)$ is a direct summand of M. □

LEMMA 3.9. *Let M_1, M_2 and M_3 be dual Baer relatively d-modules. Assume that M_2 is M_1-projective (or M_1 is M_2-projective). Then $M_1 \oplus M_2$ and M_3 are relatively d-modules.*

Proof. Let $\varphi : M_1 \oplus M_2 \longrightarrow M_3$ be any homomorphism. Then $\mathrm{Im}(\varphi i_1) = \varphi(M_1)$ and $\mathrm{Im}(\varphi i_2) = \varphi(M_2)$ are direct summands of M_3, where $i_1 : M_1 \to M_1 \oplus M_2$ and $i_2 : M_2 \to M_1 \oplus M_2$ are the canonical inclusions (M_1 and M_2 are relative d to M_3). Since M_3 has SSSP, $\mathrm{Im}\varphi$ is a direct summand of M_3. Now let $\psi : M_3 \longrightarrow M_1 \oplus M_2$ be any homomorphism. Since M_3 is relative d to M_2, $\mathrm{Im}(\pi_2\psi)$ is a direct summand of M_2, where $\pi_2 : M_1 \oplus M_2 \to M_2$ is the canonical projection. Therefore $\pi_2(\mathrm{Im}\psi)$ is a direct summand of M_2. Thus $\pi_2(\mathrm{Im}\psi) + M_1$ is a direct summand of $M_1 \oplus M_2$.

But $\pi_2(\mathrm{Im}\psi) + M_1 = \mathrm{Im}\psi + M_1$. Then $\mathrm{Im}\psi + M_1$ is a direct summand of $M_1 \oplus M_2$. Let L and E be submodules of $M_1 \oplus M_2$ such that $(\mathrm{Im}\psi + M_1) \oplus L = M_1 \oplus M_2$ and $\mathrm{Im}\psi + M_1 = E \oplus M_1$. Thus $E \oplus L \cong M_2$. Since M_2 is M_1-projective, E is M_1-projective. So there exists $F \leq \mathrm{Im}\psi$ such that $F \oplus M_1 = \mathrm{Im}\psi + M_1$ (see [**2**, 4.12]). Hence $F \oplus M_1 \oplus L = M_1 \oplus M_2$. Therefore $\mathrm{Im}\psi = F \oplus [\mathrm{Im}\psi \cap (M_1 \oplus L)] = F \oplus (\mathrm{Im}\psi \cap M_1)$. Let $\pi : F \oplus M_1 \oplus L \to M_1$ be the canonical projection. Consider the homomorphism $\pi\psi : M_3 \to M_1$. We have $\mathrm{Im}(\pi\psi) = \pi(\mathrm{Im}\psi) = \pi(F \oplus [\mathrm{Im}\psi \cap M_1]) = \mathrm{Im}\psi \cap M_1$. Since M_3 is relative d to M_1, $\mathrm{Im}\psi \cap M_1$ is a direct summand of M_1. Therefore $\mathrm{Im}\psi$ is a direct summand of $M_1 \oplus M_2$, and the proof is complete. □

THEOREM 3.10. *Let $M_1, \ldots, M_n$ be dual Baer modules, where $n \in \mathbb{N}$. Assume that, for any $i \neq j$, M_i and M_j are relatively d-modules and for any $i < j$, M_i is M_j-projective. Then $M = \oplus_{i=1}^{n} M_i$ is dual Baer.*

Proof. By Lemmas 3.8 and 3.9. □

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