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LENS SPACES, ISOSPECTRAL ON FORMS BUT NOT ON FUNCTIONS

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Abstract

We study the p-form spectrum of the Laplace-Beltrami operator acting on lens spaces as considered by Ikeda [*Geometry of manifolds* (Academic Press, Boston, MA, 1989) 383–417]. Ikeda gave examples of such spaces that are non-isometric but isospectral for all $p \leqslant p_0$. In this paper we exhibit examples of such spaces that are not isometric, and are isospectral for various, but not for all, values of p. In particular, examples are given of non-isometric lens spaces that are isospectral for some values of p but not for the case $p = 0$.

1. *Introduction*

The purpose of this paper is to study the p-form spectrum of the Laplacian Δ_p acting on lens spaces. Examples of non-isometric lens spaces that are isospectral for all $p < p_0$ and not for $p \geqslant p_0$ (up to duality) were given by Ikeda in [16]. Here we consider the opposite situation: Can there exist non-isometric pairs of lens spaces that are isospectral for some $p_0 > 0$, and not for any $p < p_0$? We affirmatively answer this question by presenting examples of such pairs that have been found computationally. We discuss the approach to these computations, results of the computations and applications to representation theory, as well as the representation theory underlying the problem.

The Laplace spectrum of a closed Riemannian manifold is the set of eigenvalues of the Laplace–Beltrami operator Δ, counted with multiplicity. Two manifolds are isospectral on functions if they share the same Laplace spectrum. The operator Δ may be extended to act on smooth p-forms by $\Delta_p = d\delta + \delta d$, where δ is the (metric) adjoint of d. Two manifolds are p-isospectral if they share the same spectrum of Δ_p. Of interest is determining to what extent the p-spectrum of a manifold determines the geometry of the manifold.

Until the 1990s, almost all examples of isospectral manifolds were also strongly isospectral, that is, isospectral for any natural, elliptic differential operator; see [8]. This relation between the function and form spectrum on early examples was due to the purely representation-theoretic nature of early constructions of isospectral manifolds. As such, examples of manifolds isospectral on functions were also p-isospectral, for $p = 1, \ldots, \dim$. The notable exceptions were examples of Ikeda, who showed that for any $p_0 > 0$ there exist examples of pairs of lens spaces that are isospectral on p-forms for $p = 0, \ldots, p_0$ and not isospectral for $(p_0 + 1)$-forms. A lens space is a manifold of the type S^n/G, where G is a cyclic subgroup of isometries acting freely on S^n.

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Since the 1990s, there has been a growth of methods for producing isospectral manifolds, almost all of which involve bundles and submersions of some kind. See for example [**7**, **8**, **9**, **10**, **13**, **14**, **15**, **17**, **22**, **23**, **24**, **25**, **26**, **27**]. These more recent methods of Szabo, Gordon, and Schueth are for functions only; that is, the methods employed do not extend to the spectrum on forms. In fact, Schueth [**21**] shows the non-isometry of some of the examples by showing that the 1-form heat invariants cannot be equal, that is, by showing that they are not isospectral on forms. For other examples of manifolds with different p-spectra, see [**6**, **9**, **10**, **11**]. In particular, [**12**] exhibits examples of manifolds of even dimension n that are isospectral on $(n/2)$-forms but not on functions.

Most notably, in 2001, R. Miatello and J. P. Rossetti [**18**] methodically studied the function and p-form spectrum of compact flat manifolds. They produced beautiful examples of pairs of compact flat manifolds that are isospectral on forms, not isospectral on functions, and with different lengths in the length spectrum.

In this paper, we methodically study the function and p-form spectrum of lens spaces in a way that allows us to study just a single p-spectrum. We produce conditions under which we may have examples of lens spaces that are isospectral on p_0-forms for some p_0, and not p-isospectral for $p < p_0$. We show computationally that such examples exist. (For basics on the p-form spectrum, see, for example, [**4**, Appendix] or [**19**].)

This paper is organized as follows. In Section 2, we give background information for lens spaces and their generating functions. In Section 3 we present the computational results. In Section 4, we present applications of the existence of these examples to representation theory (see Corollary 2), and we present the representation theory behind the construction of p-isospectral lens spaces. In Appendix A we include the MATHEMATICA code used to find our examples. Notebooks containing this code are given in Appendix B; PDF files of the code are in Appendix C, and the original MAPLE code is in Appendix D.

2. *Lens spaces*

Ikeda [**16**] considered lens spaces, defined as follows. Let q be a positive integer, and let $p_1, \dots, p_\lambda$ be integers prime to q. Given

$$R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix},$$

define the matrix

$$g = \begin{pmatrix} R(\frac{p_1}{q}) & 0 & 0 & 0 & 0 \\ 0 & R(\frac{p_2}{q}) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & R(\frac{p_{\lambda-1}}{q}) & 0 \\ 0 & 0 & 0 & 0 & R(\frac{p_\lambda}{q}) \end{pmatrix}. \tag{1}$$

Let $G \subset O(2\lambda)$ be the cyclic group generated by g, and recall that $O(2\lambda)$ acts isometrically on the canonical sphere $\mathbb{S}^{2\lambda-1}$. We define a lens space

$$L = \mathbb{S}^{2\lambda-1}/G.$$

We generate known non-isometric pairs of lens spaces in a manner similar to Ikeda's.

Choose $k \in \mathbb{N}$, $k < \phi(q)/2 = q_0$, and define $n = q_0 - k$. Here ϕ denotes the Euler phi-function, so that $\phi(m)$ is the number of integers between 1 and m (inclusive) that are prime to m, which is always even when $m > 2$. Consider the set

$$RP = \left\{ \pm i \mid i \in \mathbb{N},\ i < \tfrac{q}{2},\ (i, q) = 1 \right\}.$$

Here $\mathbb{N} = \{1, 2, \ldots\}$. Note that the elements of RP form a representative set for the units in the ring $\mathbb{Z}_q$, so that the order of RP is $\phi(q) = 2q_0 = 2n + 2k$.

For a given q and k, we consider pairs of disjoint subsets R of order $2n$ and S of order $2k$ of RP, under the condition that: if $\alpha \in R$, then $-\alpha \in R$, and if $\alpha \in S$, then $-\alpha \in S$. Note that a choice of R determines a choice of S, and vice versa. We refer to the pairs (R, S) for convenience in what follows; it is enough just to consider the sets R or the sets S.

By viewing elements of RP as equivalence classes of $\mathbb{Z}_q$, we define an equivalence relation on subsets $\{t_1, \ldots, t_h\}$ of RP as follows: for all $j \in RP$,

$$\{t_1, \ldots, t_h\} \sim \{\pm j t_1, \ldots, \pm j t_h\} \qquad (\mathrm{mod}\, q), \tag{2}$$

where the choices of $\pm$ are independent; that is, some may be plus and others minus. For a given q and k, we then define an equivalence relation on the set of all pairs (R, S) by

$$(R, S) \sim (R', S') \text{ if and only if } R \sim R' \text{ and/or } S \sim S'$$

where $R \subset RP$ has order $2n$ and $S \subset RP$ has order $2k$.

Let $G(q, k)$ be a set of representative elements of equivalence classes of pairs (R, S), and denote the order of $G(q, k)$ by $|G(q, k)|$.

Now define a lens space $L_{R,S}$ using as the p_i of matrix (1) above, the elements of R. It is well known that L_{R_1,S_1} and L_{R_2,S_2} are isometric if and only if $R_1 \sim R_2$. (See references in [16].) Thus, distinct elements of $G(q, k)$ produce nonisometric associated lens spaces.

Our goal is to find distinct elements of $G(q, k)$ with associated lens spaces that are isospectral for isolated values of p.

We continue to follow the approach of Ikeda, who used a generating function in order to compare the p-form spectrum of pairs of lens spaces constructed as above. The primary difference between the work here and that of Ikeda is that Ikeda required that q be prime, which permitted a systematic analysis of the p-isospectrality behavior. We do not require that q be prime in what follows. The interested reader is encouraged to read Ikeda's paper, where the motivation and computation of the generating function are given in detail.

Given a pair (R, S) representative of an equivalence class, the generating function for the p-form spectrum of the Laplacian on a lens space $L_{R,S}$ is given by

$$\eta^p_{(R,S)}(z) = \sum_{a=0}^{2k} \sum_{t=0}^{p} (-1)^{t+a} \left(z^{a-t} - z^{a+t+2} \right) C^{a,p-t}_{(R,S)} \tag{3}$$

where

$$C^{\alpha,\beta}_{(R,S)} = q \left| \left\{ (A, B) : A \subset R,\ B \subset S,\ |A| = 2n - \beta, \right.\right.$$
$$\left.\left. |B| = 2k - a,\ \sum_{a \in A} a + \sum_{b \in B} b = 0\ (\mathrm{mod}\, q) \right\} \right|, \tag{4}$$

where by $|A|$ we mean the cardinality of A, and so on. Note that $|A| = 2n - \beta$ and $|B| = 2k - \alpha$, and that $C^{\alpha,\beta}_{(R,S)}$ is just

$$q * \text{Number of pairs } (A, B) \text{ with the sum of elements in } (A, B) = 0 \bmod(q),$$

so computing values for C requires only determining the number of such pairs of subsets that add correctly.

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THEOREM 1 (see [16], Proposition 2.1). *The lens spaces $L_{(R_1,S_1)}$ and $L_{(R_2,S_2)}$ are p-isospectral if and only if*

$$\eta^p_{(R_1,S_1)}(z) = \eta^p_{(R_2,S_2)}(z) \qquad \text{and} \qquad \eta^{p-1}_{(R_1,S_1)}(z) = \eta^{p-1}_{(R_2,S_2)}(z).$$

The reason that we need two generating functions has to do with closed and exact forms. In particular, for lens spaces the p-form spectrum splits as *Exact* $\oplus$ *Closed*, and for full isospectrality we need both to match. Again, see [16] for more details. Notice that if two lens spaces are 1-isospectral, they must also be 0-isospectral, that is, isospectral on functions.

We are left with the following problem.

PROBLEM 1. For what values of q and k is it possible to find non-equivalent pairs (R_1, S_1) and (R_2, S_2) in $G(q, k)$ with

(i) $\eta^p_{(R_1,S_1)}(z) = \eta^p_{(R_2,S_2)}(z)$, $\eta^{p-1}_{(R_1,S_1)}(z) = \eta^{p-1}_{(R_2,S_2)}$, for some $p > 0$ and
(ii) $\eta^m_{(R_1,S_1)}(z) \neq \eta^m_{(R_2,S_2)}(z)$, for all $m < p$?

3. *Computational results*

Our approach is to check the equivalence of all possible $\eta^p_{(R,S)}$ for various choices of q and k.

Checking the equivalence of the $\eta^p_{(R,S)}$ reduces to checking the coefficients of the powers of z, which are given by

$$\text{coefficient of } z^b = (-1)^b \sum_{t=0}^{p} \left(C^{b+t,p-t} - C^{b-t-2,p-t}\right). \tag{5}$$

These computations are related to the known problem of *subset sums*. Specifically, here we are computing *double subset sums* (see [2, 3] for information on multiple subset sums), and we could find no general results in the literature.

Our hope was that relatively low-dimensional examples would be found computationally, and then verified by hand. The simplest examples are for relatively large values of q, and thus are computable only using computers. Computations have been done using MAPLE, MUPAD and MATHEMATICA, with both integer and floating-point algorithms (see Appendix A for code in MATHEMATICA, which is most efficient and does not involve the use of floating-point numbers).

We have run computations for all $q \leqslant 100$ with $k = 2, 3$, and for all $q \leqslant 50$ with $k = 4$. What we know from these numerical computations is that there are many examples of p-isospectral lens spaces that are not isospectral for $m < p$ (see Table 1 for some examples when $k = 2$). No examples of spaces which are not isospectral on functions, but are isospectral for higher-degree forms, have been found when $k = 2$ or $k = 3$; however examples exist when $k = 4$ (see Table 3).

Also of interest are examples of 'half-isospectral' lens spaces; that is, examples where the lens spaces are isospectral on (say) closed 2-forms and nothing else. We know of no obvious application for this property, but mention it here for completeness; we discuss the representation theory behind these 'half-isospectral' examples in the next section. (See Remark 5).

Examples of this sort of behavior seem to exist in essentially any possible combination (as even a quick check using the MATHEMATICA code in Appendix A will demonstrate). In Table 2 we show the case with $k = 2$ and $q = 56$, where the rows are the equivalence

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classes of lens spaces and the columns are the values of p. The entries in the array track the η^p equivalence classes for each row, so if two rows have the same entry then the given lens spaces are isospectral for co-exact forms of that degree. In the example shown, the spaces represented by rows two and three are isospectral for forms of degree zero, one, and two, and also eight, nine, and ten, while the spaces for rows six and seven are also isospectral for forms of degree zero, one, and two, but then only 'half-isospectral' for degree six.

Table 1: Isospectralities for $k = 2$. For larger q, there are multiple isospectral pairs with varying p values.

q	Lens Space Dim	Isospectral for $p =$
43	37	0 – 5, 14
44	15	0 – 1, 6 – 8
47	41	0 – 5, 14
49	37	0 – 2, 13
53	47	0 – 7, 18
56	19	0 – 3, 8 – 10
58	23	0 – 3, 9 – 11
62	25	0 – 5, 10 – 13
64	27	0 – 5, 12 – 14

Table 2: Various isospectralities for $q = 56$ and $k = 2$, lens spaces of dimension 21. Each row represents an equivalence class of lens spaces, and the entries track the matching generating functions.

$$
\begin{array}{l}
p = \quad 0 \quad\;\; 1 \quad\;\; 2 \quad\;\; 3 \quad\;\; 4 \quad\;\; 5 \quad\;\; 6 \quad\;\; 7 \quad\;\; 8 \quad\;\; 9 \quad\;\; 10 \\
\begin{pmatrix}
(1) & (1) & (1) & (3) & (2) & (4) & (2) & (3) & (2) & (5) & (2) \\
(1) & (1) & (3) & (1) & (4) & (3) & (4) & (1) & (5) & (2) & (5) \\
(1) & (1) & (3) & (1) & (5) & (2) & (5) & (1) & (5) & (2) & (5) \\
(1) & (1) & (1) & (3) & (2) & (4) & (1) & (4) & (1) & (5) & (1) \\
(1) & (1) & (2) & (2) & (3) & (4) & (3) & (2) & (4) & (3) & (4) \\
(1) & (1) & (1) & (3) & (2) & (4) & (1) & (4) & (1) & (5) & (1) \\
(1) & (1) & (1) & (3) & (1) & (5) & (1) & (3) & (3) & (4) & (3) \\
(1) & (1) & (3) & (1) & (6) & (1) & (6) & (1) & (6) & (1) & (6)
\end{pmatrix}
\end{array}
$$

Table 3: Some examples of spaces isospectral for $p > 0$ but not for $p = 0$.

k	q	Isospectral for $p =$	Dimension	Defining sets (S)
4	39	2	17	$\{\pm1, \pm2, \pm4, \pm10\}$ $\{\pm1, \pm2, \pm14, \pm17\}$
4	44	2	15	$\{\pm1, \pm3, \pm7, \pm13\}$ $\{\pm1, \pm3, \pm17, \pm19\}$
4	49	6,13	37	$\{\pm1, \pm6, \pm8, \pm13\}$ $\{\pm1, \pm6, \pm8, \pm22\}$

We note that the first two examples in Table 3 are in some sense the 'best' possible, since it is not possible to have lens spaces isospectral on forms of degree one, and not on functions. Given the seemingly random behavior of the isospectrality exhibited, an obvious question that was formulated by Miatello and Rossetti in [17] is whether any possible combination of isospectrality is possible with the right choice of q and k?

To illustrate the techniques described above, and implemented in the code, we now present a typical calculation using $k = 2, 3$ and low values of q. The lens spaces involved do not yield any of the new examples, but as mentioned earlier, the new examples have such large values of q, that the computations must be done mechanically.

For $q = 28, k = 2$ the set $RP = \{\pm 1, \pm 3, \pm 5, \pm 9, \pm 11, \pm 13\}$. Since $k = 2$ and $q_0 = 6$, we have $n = 4$. We seek a list of inequivalent pairs (R, S) with $|R| = 8$ and $|S| = 4$ with the condition that if $\alpha \in R$ then $-\alpha \in R$, and likewise for S. Because the set RP comprises the units of $\mathbb{Z}_q$, using the equivalence relation (2) we may assume that one element of S is 1. After using (2) again, we see that a complete list of possible S sets is $S_3 = \{\pm 1, \pm 3\}$, $S_5 = \{\pm 1, \pm 5\}$ and $S_{13} = \{\pm 1, \pm 13\}$, and we compute the corresponding $R = RP - S$.

Note that because $k = 2$, the maximum p-value that we consider is $p = 3$, due to symmetries in the table of C values. For $S_3 = \{\pm 1, \pm 3\}$, we carefully compute $C^{\alpha,\beta} := C^{\alpha,\beta}_{(R,S)}$ for $0 \leqslant \alpha \leqslant 4$ and $0 \leqslant \beta \leqslant 3$. We merely state the C-values for the other two possible S-sets.

First note that if α is odd and β is even, then $C^{\alpha,\beta}_{(R,S)} = 0$, since in this case there is no way to take $(8 - \beta)$ odd elements of R and $(4 - \alpha)$ odd elements of S and add them to 0 mod 28: that is, an even number. Now $C^{0,0}/q = 1$ since there is only one way to take eight elements of R_3 and four elements of S_3, which must add to 0 mod 28. Likewise, $C^{0,2}/q = 4$ since there are only four ways to take 6 elements of R, and four elements of S_3, and still add to 0 mod q. In particular, $A = \{\pm 5, \pm 9, \pm 11\}$, $\{\pm 5, \pm 9, \pm 13\}$, $\{\pm 5, \pm 11, \pm 13\}$ or $\{\pm 9, \pm 11, \pm 13\}$ and $B = S_3$ when using (4).

The value $C^{1,1} = 0$, as there is no way to add seven elements of R_3 and three elements of S_3 to 0. The value $C^{1,3}/q = 12$. We list six (A, B) pairs that satisfy the necessary conditions for (4). The other six pairs are obtained by negating all elements of A and of B:

$A = \{5, 9, 11, \pm 13\}$ and $B = \{\pm 1, 3\}$;
$A = \{5, \pm 9, 11, 13\}$ and $B = \{-1, \pm 3\}$;
$A = \{5, 9, \pm 11, 13\}$ and $B = \{1, \pm 3\}$;
$A = \{5, 9, \pm 11, -13\}$ and $B = \{-1, \pm 3\}$;
$A = \{5, 9, -11, \pm 13\}$ and $B = \{\pm 1, -3\}$;
$A = \{5, \pm 9, 11, -13\}$ and $B = \{\pm 1, -3\}$.

The value $C^{2,0} = 2$, since there are only two ways to take eight elements of R_3, which must add to 0, and two elements of S_3 and still add to 0. In particular, $A = R_3$ and either $B = \{\pm 1\}$ or $B = \{\pm 3\}$. The value $C^{2,2}/q = 18$. We obtain eight of the (A, B) pairs by removing an arbitrary element of R_3 and its inverse, and an arbitrary element of S_3 and its inverse. We list five of the remaining (A, B) pairs, with the other five obtained by negating all A and B elements:

$A = \{\pm 5, -9, \pm 11, 13\}$ and $B = \{-1, -3\}$;
$A = \{-5, 9, \pm 11, \pm 13\}$ and $B = \{-1, -3\}$;
$A = \{\pm 5, -9, 11, \pm 13\}$ and $B = \{1, -3\}$;
$A = \{\pm 5, \pm 9, -11, 13\}$ and $B = \{1, -3\}$;
$A = \{\pm 5, \pm 9, 11, 13\}$ and $B = \{1, 3\}$.

The value $C^{3,1} = 0$, as there is no way to add seven elements of R_3 and one element of S_3 to have them add to 0. Using the definition of C, one easily checks that $C^{3,3} = C^{1,3} = 12q$. Likewise, one easily checks that $C^{4,0} = q$ and $C^{4,2} = C^{0,2} = 4q$.

We thus have the following values of $C^{\alpha,\beta}$ for $S_3 = \{\pm 1, \pm 3\}$ with $\alpha = 0, \ldots, 4$ and $\beta = 0, \ldots, 3$:

$$C^{\alpha,\beta} = \begin{pmatrix} q & 0 & 4q & 0 \\ 0 & 0 & 0 & 12q \\ 2q & 0 & 18q & 0 \\ 0 & 0 & 0 & 12q \\ q & 0 & 4q & 0 \end{pmatrix}.$$

A similar calculation shows that for $S_5 = \{\pm 1, \pm 5\}$, we have

$$C^{\alpha,\beta} = \begin{pmatrix} q & 0 & 4q & 0 \\ 0 & 0 & 0 & 16q \\ 2q & 0 & 16q & 0 \\ 0 & 0 & 0 & 16q \\ q & 0 & 4q & 0 \end{pmatrix},$$

and for $S_{13} = \{\pm 1, \pm 13\}$,

$$C^{\alpha,\beta} = \begin{pmatrix} q & 0 & 4q & 0 \\ 0 & 0 & 0 & 8q \\ 2q & 0 & 20q & 0 \\ 0 & 0 & 0 & 8q \\ q & 0 & 4q & 0 \end{pmatrix}.$$

Applying equation (3) to these C-values, we obtain

$$\eta^0_{(R_3,S_3)}(z) = \eta^0_{(R_5,S_5)}(z) = \eta^0_{(R_{13},S_{13})}(z)$$

and

$$\eta^1_{(R_3,S_3)}(z) = \eta^1_{(R_5,S_5)}(z) = \eta^1_{(R_{13},S_{13})}(z),$$

and hence the resulting three lens spaces are isospectral on functions and on 1-forms. However, there are no remaining η^p equalities for any pair of lens spaces; that is, no pair of lens spaces is isospectral (or half-isospectral) on 2-forms or 3-forms.

We now give two particular C-values for the case $q = 17$, and $k = 3$. In this case, the set $RP = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8\}$. Two inequivalent choices for S are $S = \{\pm 1, \pm 2, \pm 4\}$ and $S' = \{\pm 1, \pm 3, \pm 4\}$.

We compute the following values of $C^{\alpha,\beta}$ for $S = \{\pm 1, \pm 2, \pm 4\}$ with $\alpha = 0, \ldots, 6$ and $\beta = 0, \ldots, 5$:

$$C^{\alpha,\beta} = \begin{pmatrix} 17 & 0 & 85 & 68 & 272 & 170 \\ 0 & 0 & 340 & 578 & 1462 & 1360 \\ 51 & 136 & 833 & 1598 & 3366 & 3434 \\ 0 & 102 & 986 & 2176 & 4522 & 4828 \\ 51 & 136 & 833 & 1598 & 3366 & 3434 \\ 0 & 0 & 340 & 578 & 1462 & 1360 \\ 17 & 0 & 85 & 68 & 272 & 170 \end{pmatrix}.$$

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For $S' = \{\pm 1, \pm 3, \pm 4\}$, we have

$$C^{\alpha,\beta} = \begin{pmatrix} 17 & 0 & 85 & 102 & 238 & 238 \\ 0 & 0 & 306 & 646 & 1292 & 1462 \\ 51 & 102 & 799 & 1700 & 3230 & 3706 \\ 34 & 102 & 1020 & 2278 & 4318 & 4964 \\ 51 & 102 & 799 & 1700 & 3230 & 3706 \\ 0 & 0 & 306 & 646 & 1292 & 1462 \\ 17 & 0 & 85 & 102 & 238 & 238 \end{pmatrix}.$$

Using the computer to apply equation (3) to these C-values, we obtain

$$\eta^1_{(R,S)}(z) = \eta^1_{(R',S')}(z),$$

with no other η-polynomials equal. Hence the resulting lens spaces are isospectral on co-exact 1-forms (and hence also on exact 2-forms), and are not isospectral on functions. See the following section for a discussion of the representation theory behind these half-isospectral examples.

4. *Representation theory and the examples*

In this section we explain the behavior of the examples in Section 3 in terms of representation theory; that is, there is a representation-theoretic explanation as to why the examples in Section 3 behave the way they do. Most of the results below can be found in [20]; we include a summary and some proofs not found in [20], for expository purposes.

Let G be a unimodular Lie group, and let K be a compact subgroup of G. Recall that there are two basic procedures in representation theory that relate representations of G to representations of K.

Definition 1 (Restriction and induction). The *restriction* of a unitary representation (ρ, V) of G to K is is denoted $\mathrm{Res}^G_K(\rho)$. Let (τ, W) be a finite-dimensional unitary representation of K on the Hilbert space W with inner product $\langle\ ,\ \rangle$. Define

$$X_\tau := \left\{ f : G \to W : f \text{ is measurable} ; \int_{K\backslash G} \langle f, f\rangle \, dx < \infty; \right. \\ \left. f(kg) = \tau(k) f(g)\ \forall k \in K, \forall g \in G \right\}. \quad (6)$$

The space X_τ is a Hilbert space. Define an action ρ of G on X_τ by

$$(\rho(g) f)(x) = f(xg)$$

for all $x, g \in G$ and all $f \in X_\tau$. One may check that this is a unitary group representation, denoted $\mathrm{Ind}^G_K(\tau)$; it is called the representation of G *induced* from the representation τ of K. See [20] for details.

Definition 2 (Subrepresentation of ρ relative to τ, denoted ρ_τ). Let G be a unimodular Lie group, and let (ρ, V) be a unitary representation of G. Let K be a compact subgroup of G, and let (τ, W) be an irreducible unitary representation of K. We define V_τ to be the smallest closed G-invariant subspace of V that contains all K-invariant subspaces of $(\mathrm{Res}^G_K(\rho), V)$ that are isomorphic to τ. That is, if $(\mathrm{Res}^G_K(\rho), \hat{V})$ is a K-invariant subspace of $(\mathrm{Res}^G_K(\rho), V)$ and $(\mathrm{Res}^G_K(\rho), \hat{V})$ is unitarily equivalent to τ, then $\hat{V} \subset V_\tau$, and V_τ is the smallest closed G-invariant subspace with this property. If τ is not an irreducible representation of K, then we let V_τ be the closure of the sum of the V_μ where μ is an irreducible component of τ. Let ρ_τ denote the representation (ρ, V_τ) of G.

REMARK 1. Note that if $(\mathrm{Res}_K^G(\rho), V)$ has no irreducible components that are equivalent to τ, then $V_\tau = \{0\}$ and $\rho_\tau = 0$.

DEFINITION 3 (τ-EQUIVALENCE, K-EQUIVALENCE). Let ρ and π be unitary representations of G, and let τ be a unitary representation of K. The representations ρ and π are *τ-equivalent* if ρ_τ and π_τ are unitarily equivalent representations of G. The representations ρ and π are *K-equivalent* if they are $\mathbf{1}_K$-equivalent. That is, they are τ-equivalent in the special case where $\tau = \mathbf{1}_K$, the trivial representation of K.

DEFINITION 4 (STABILIZER). Let G act on the left on a manifold X. Let $x \in X$. The *stabilizer* of x is defined as

$$G_x := \{g \in G : g \cdot x = x\}.$$

If G acts properly and smoothly, then G_x is a compact subgroup of G. Note that G acts properly if and only if the mapping $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ for all $g \in G$ and all $x \in X$ is a proper map.

The following is a basic property; see [1].

PROPOSITION 1 (GENERIC STABILIZER). *Let G act properly and smoothly on the left on the manifold X. There exists a compact subgroup K of G such that the following properties hold.*

(i) *For all $x \in X$, K is conjugate to a subgroup of G_x.*

(ii) *There exists an open dense subset $U \subset X$ such that if $x \in U$, then K and G_x are conjugate.*

DEFINITION 5. Let $\pi_{\Gamma_i}^G$ denote the induced representation on G arising from the trivial representation on Γ_i; that is, for $i = 1, 2$,

$$\pi_{\Gamma_i}^G := \mathrm{Ind}_{\Gamma_i}^G(\mathbf{1}_{\Gamma_i}).$$

Theorems 2 and 3 are the main theorems of [20].

THEOREM 2 (ISOSPECTRALITY ON FUNCTIONS [20]). *Let (X, m) be a Riemannian manifold. Let G be a closed subgroup of isometries of (X, m), and K the generic stabilizer of the action of G on X. Let Γ_1 and Γ_2 be discrete subgroups of G such that $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are compact manifolds. If $\pi_{\Gamma_1}^G$ and $\pi_{\Gamma_1}^G$ are K-equivalent, then $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are isospectral on functions.*

REMARK 2. Pesce proved Theorem 2 using three different methods: one proof uses the method of transplantation developed by Berard; a second proof uses the Selberg trace formula; the last proof uses Frobenius reciprocity, which is valid only in the case where X is compact. Each proof has its advantages; for example, the transplantation proof extends to orbifolds. See [20] for more details.

THEOREM 3 (NECESSITY AND SUFFICIENCY ON FUNCTIONS [20]). *Let (X, m) be a compact Riemannian manifold, G a group of isometries of (X, m), K the generic stabilizer of the action of G on X, and Γ_1 and Γ_2 discrete subgroups such that $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are compact. If the real eigenspaces of (X, m) are irreducible under the action of G, then $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are isospectral on functions if and only if they are K-equivalent.*

REMARK 3. Actually, Pesce proves that Theorem 3 is necessary and sufficient in two additional cases:

(i) (X, m) is a rank one symmetric space of noncompact type, and

(ii) (X, m) is Euclidean space with the canonical metric.

THEOREM 4 (ISOSPECTRALITY ON FORMS [20]). *Let E be a natural fiber bundle over (X, m), and let D be a natural, self-adjoint, elliptic differential operator on E. Let G be a group of isometries of (X, m) and K the generic stabilizer of the action of G on X. Let τ be the representation of K on the fiber E_x over $x \in X$, where G_x, the stabilizer of x, is isomorphic to K. Let Γ_1 and Γ_2 be discrete subgroups of G such that $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are compact manifolds. If $\pi^G_{\Gamma_1}$ and $\pi^G_{\Gamma_2}$ are equivalent relative to τ, then D_1 and D_2 are isospectral.*

DEFINITION 6 (REPRESENTATION τ_p). We denote by τ_p the representation of K on the fibre E_x over $x \in X$ where G_x is isomorphic to K for the special case where D is Δ_p and E is the bundle of differential p-forms over X.

THEOREM 5 (NECESSITY AND SUFFICIENCY ON FORMS [20]). *Let (X, m) be a compact Riemannian manifold, G a group of isometries of (X, m) and Γ_1 and Γ_2 discrete subgroups such that $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are compact. If the real p-eigenspaces of (X, m) are irreducible under the action of G, then $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are isospectral on p-forms if and only if they are τ_p-equivalent.*

COROLLARY 1. *The examples described in Section 3, Table 3, are τ_p-equivalent for the given value of p but not τ_q-equivalent for $q \neq p$.*

Proof. The p-form eigenspaces of the canonical sphere are irreducible [16]. Thus, by the previous theorem, since they are isospectral on p-forms, the bundle of differential p-forms must be τ_p-equivalent. Note that τ^p is equivalent to the natural representation of $K = O(m)$ on $\wedge^p \mathbb{C}^m$, where m is the dimension of the lens space. □

REMARK 4. The proof of sufficiency is given in [20], using transplantation. We include the proof using Frobenius reciprocity, which was mentioned in [20]. The necessity of the condition is likewise mentioned in the last remark in [20]; thus the proof given below is due to Pesce, and we provide more details here for completeness of the literature.

REMARK 5. In the case of lens spaces, $\tau_p = \tau_{pc} \oplus \tau_{pe}$, where τ_{pc} is τ_p restricted to closed forms, and τ_{pe} is τ_p restricted to exact forms. The examples in Section 3 that are half-p-isospectral are τ_{pc}-equivalent but not τ_{pe}-equivalent.

The following is an application of the examples of Section 3 to representation theory.

COROLLARY 2. *There exist (X, m) a compact Riemannian manifold and Γ_1, Γ_2 discrete groups of isometries acting on X that, with definitions as in Theorem 4, are τ_{pc}-equivalent but not τ_{pe}-equivalent.*

Proof of Theorems 4 and 5 using Frobenius reciprocity. Assume that (X, m) is a compact Riemannian manifold, G a group of isometries of (X, m), and K the generic stabilizer of the action of G on X. Let Γ_1 and Γ_2 be discrete subgroups of G such that $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are compact.

Since Δ^p on (X, m) has discrete spectrum, let spec^p denote the spectrum of Δ^p without multiplicity. If $\lambda \in \mathrm{spec}^p$, then let V_λ^p denote the eigenspace of λ. The multiplicity of the eigenvalue λ is the dimension of V_λ^p; that is, $\dim_{\mathbb{C}} V_\lambda^p = \mathrm{mult}(\lambda)$.

Any closed group of isometries H of (X, m) acts on V_λ^p by

$$\pi_\lambda^H(h)\omega = (h^{-1})^*\omega$$

for all $\omega \in V_\lambda^p$ and all $h \in H$.

The p-eigenforms of $(\Gamma_i\backslash X, m_i)$ correspond to eigenforms of (X, m) that are Γ_i-invariant with the same eigenvalue, $i = 1, 2$. Thus, $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are isospectral on p-forms if and only if for all $\lambda \in \mathrm{spec}^p$, we have

$$\left[\mathbf{1}_{\Gamma_1} : \pi_\lambda^{\Gamma_1}\right] = \left[\mathbf{1}_{\Gamma_2} : \pi_\lambda^{\Gamma_1}\right];$$

that is, an eigenform ω is left Γ_i-invariant if and only if for all $\gamma \in \Gamma_i$, $\gamma \cdot \omega = \omega$ if and only if $\mathrm{span}_{\mathbb{C}}\{\omega\}$ is an invariant subspace of $\pi_\lambda^{\Gamma_i}$, $i = 1, 2$.

Now

$$\begin{aligned}\left[\mathbf{1}_{\Gamma_i} : \pi_\lambda^{\Gamma_i}\right] &= \left[\mathbf{1}_{\Gamma_i} : \mathrm{Res}_{\Gamma_i}^G(\pi_\lambda^G)\right] \\ &= \left[\mathbf{1}_{\Gamma_i} : \mathrm{Res}_{\Gamma_i}^G\left(\sum_{\rho\in\hat{G}}\left[\rho : \pi_\lambda^G\right]\rho\right)\right] \\ &= \sum_{\rho\in\hat{G}}\left[\rho : \pi_\lambda^G\right]\left[\mathbf{1}_{\Gamma_i} : \mathrm{Res}_{\Gamma_i}^G(\rho)\right].\end{aligned}$$

Using Frobenius reciprocity, we have

$$\begin{aligned}\left[\mathbf{1}_{\Gamma_i} : \pi_\lambda^{\Gamma_i}\right] &= \sum_{\rho\in\hat{G}}\left[\rho : \pi_\lambda^G\right]\left[\mathrm{Ind}_{\Gamma_i}^G(\mathbf{1}_{\Gamma_i}) : \rho\right] \\ &= \sum_{\rho\in\hat{G}}\left[\rho : \pi_\lambda^G\right]\left[\pi_{\Gamma_i}^G : \rho\right].\end{aligned}$$

Now, let

$$\hat{G}_p = \{\rho \in \hat{G} : \rho_{\tau_p} \neq 0\}.$$

By Theorem 6 below, the value $\left[\rho : \pi_\lambda^G\right] \neq 0$ if and only if $\rho \in \hat{G}_p$. We thus have

$$\left[\mathbf{1}_{\Gamma_i} : \pi_\lambda^{\Gamma_i}\right] = \sum_{\rho\in\hat{G}_p}\left[\rho : \pi_\lambda^G\right]\left[\pi_{\Gamma_i}^G : \rho\right].$$

Now, note that $[\rho : \pi_\lambda^G]$ is independent of Γ_i. Clearly, if $\pi_{\Gamma_1}^G$ and $\pi_{\Gamma_2}^G$ are τ_p-equivalent, then $(\Gamma_1\backslash X, m_1)$ and $(\Gamma_2\backslash X, m_2)$ are isospectral on p-forms.

To see the converse using the hypothesis of the theorem, note that if π is a real, irreducible representation of G, then its complexification $\pi_{\mathbb{C}}$ satisfies exactly one of the following conditions:

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(i) $\pi_{\mathbb{C}}$ is an irreducible complex representation, (π is real type), or

(ii) $\pi_{\mathbb{C}} = \sigma \oplus \sigma^*$, where σ^* is the contragredient representation of σ (π is complex type), or

(iii) $\pi_{\mathbb{C}} = \sigma \oplus \sigma$ (π is quaternionic type).

Assume that the eigenspaces $(\pi_\lambda^G, V_\lambda^p)$ are real and irreducible. If π_λ^G is real or quaternionic, then $\pi_\lambda^G = \sigma$ or $\pi_\lambda^G = \sigma \oplus \sigma$ for some $\sigma \in \hat{G}_p$. Thus

$$\begin{aligned}\left[\mathbf{1}_{\Gamma_i} : \pi_\lambda^{\Gamma_i}\right] &= \sum_{\rho \in \hat{G}_p} \left[\rho : \pi_\lambda^G\right]\left[\pi_{\Gamma_i} : \rho\right] \\ &= \left[\sigma : \pi_\lambda^G\right]\left[\pi_{\Gamma_i} : \sigma\right].\end{aligned}$$

If $(\Gamma_1 \backslash X, m_1)$ and $(\Gamma_2 \backslash X, m_2)$ are isospectral on p-forms, then $[\pi_{\Gamma_1} : \sigma] = [\pi_{\Gamma_2} : \sigma]$. If π_λ^G is complex, then $\pi_\lambda^G = \sigma \oplus \sigma^*$, so

$$\begin{aligned}\left[\mathbf{1}_{\Gamma_i} : \pi_\lambda^{\Gamma_i}\right] &= \sum_{\rho \in \hat{G}_p} \left[\rho : \pi_\lambda^G\right]\left[\pi_{\Gamma_i} : \rho\right] \\ &= \left[\sigma : \pi_\lambda^G\right]\left[\pi_{\Gamma_i} : \sigma\right] + \left[\sigma^* : \pi_\lambda^G\right]\left[\pi_{\Gamma_i} : \sigma^*\right].\end{aligned}$$

Now note that a subspace $W_\lambda \subset V_\lambda^p$ is equivalent to σ if and only if $\bar{W}_\lambda \subset V_\lambda^p$ is equivalent to σ^*. That is, $\Delta\omega = \lambda\omega$ if and only if $\Delta\bar{\omega} = \lambda\bar{\omega}$. We thus have $[\sigma : \pi_\lambda^G] = [\sigma^* : \pi_\lambda^G]$ and $[\sigma : \pi_{\Gamma_i}^G] = [\sigma^* : \pi_{\Gamma_i}^G]$. Therefore, if $(\Gamma_1 \backslash X, m_1)$ and $(\Gamma_2 \backslash X, m_2)$ are isospectral on p-forms, then $[\pi_{\Gamma_1} : \sigma] = t[\pi_{\Gamma_2} : \sigma]$. □

THEOREM 6. *An irreducible representation ρ of G appears as an eigenspace of Δ_p if and only if* $\mathrm{Res}_K^G(\rho)$ *is composed of irreducible representations isomorphic to τ_p.*

Proof. The proposition is proved carefully and completely in the case of functions in [5]. We briefly comment on the extension of this proof to the Laplace operator on differential forms, and note that the idea extends naturally to any natural fiber bundle, as stated in [20].

With notation as in [5],

$$\begin{aligned}\sum_\lambda \eta^p(\lambda, \rho) e^{-t\lambda} &= \sum_\lambda e^{-t\lambda} \frac{1}{\operatorname{vol} G} \int_G \chi_\lambda^p(g)\overline{\chi_\rho(g)}\, dg \\ &= \sum_\lambda \frac{1}{\operatorname{vol} G} \int_G e^{-t\lambda} \operatorname{trace}(g_\lambda)\overline{\chi_\rho(g)}\, dg.\end{aligned}$$

Here, g_λ is the action of g on the Eigenspace of λ; that is,

$$g \cdot \omega = (g^{-1})^* \omega.$$

Now, by the definition of trace,

$$\operatorname{trace}(g_\lambda) = \sum_{i=1}^{\operatorname{mult}(\lambda)} \int_M (\omega_i(x), g \cdot (\omega_i)(x))\, dx$$

where this is a finite sum over a basis of eigenforms of λ.

Thus

$$\sum_{\lambda} \eta^p(\lambda, \rho) e^{-t\lambda} = \frac{1}{\operatorname{vol} G} \int_G \overline{\chi_\rho(g)} \int_M \sum_{\lambda} (\omega_x, (g \cdot \omega_\lambda)_x)\, dx\, dg$$
$$= \frac{1}{\operatorname{vol} G} \int_G \overline{\chi_\rho(g)} \int_M \operatorname{trace}(K(t, x, gx) \circ g)\, dx\, dg.$$

Now we may plug in the asymptotics (see [5]) and obtain as a leading term, just as in [5],

$$a_0 = \operatorname{vol} \bar{M}[\rho_K : \tau_p],$$

as desired.

The rest of the argument follows from [5] without comment. □

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Appendix A. Mathematica *code*

Below is the Mathematica code for generating all examples when $k = 2$. The code is identical for other values of k, except for the initialization. We note that a simple modification of this code allows one to print the subsets R and S that give the generating functions, which would allow one (if one had the time) to check the calculations by hand.

Links to Mathematica notebooks with code to generate the examples cited above can be found in Appendix B. For people without Mathematica, PDF files with the code and computations can be found in Appendix C. Finally, for historical comparison, our original Maple code can be found in Appendix D.

```
 /* This portion computes the
generating functions */

For[q = 1, q <= 100, q++,
  Print["q= ", q];
  qlist = Table[i, {i, 1, q - 1}];
q0 = EulerPhi[q]/2; k = 2; n = q0 - k; pmax = n - 1; relprime =
{}; For[i = 2, i <= Floor[q/2], i++,
    If[GCD[i, q] == 1, relprime = Append[relprime, i]]
    ];
schoices = {}; subsetlist = Subsets[relprime, {k - 1}]; For[i = 1,
i <= Length[subsetlist], i++,
    schoices = Append[
    schoices, Join[subsetlist[[i]], q - subsetlist[[i]], {1}, {q - 1}]]];
valid = Table[i, {i, Length[schoices]}]; numgoodtups =
Length[schoices]; For[sx = 1, sx <= Length[schoices], sx++,
    If[MemberQ[valid, sx],
      For[mult = 2, mult <= q0, mult++,
        tset = Mod[mult*schoices[[sx]], q];
        m = 1;
        dn = False;
        While[dn  True && m <= Length[schoices],
          If[MemberQ[valid, m],
            tset2 = schoices[[m]];
            If[Sort[tset2] == Sort[tset],
              dn = True;
              valid = Complement[valid, {m}];
              numgoodtups--;
```

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```
                ]
              ]
            m++;
            ]
          ]
        ]
      ];
complete = Union[relprime, q - relprime, {1}, {q - 1}];
ReducePower = Function[mono,
  Coefficient[mono, x, Exponent[mono, x]]*x^Mod[Exponent[mono, x], q]];
Q = Table[q, {k}]; eta = ZeroMatrix[Length[valid], n + 1]; For[m =
1, m <= Length[valid], m++,
    s = schoices[[valid[[m]]]];
    r = Complement[complete, s];
    sizes = 2*k;
    sizer = 2*n;
    sumofs = Plus @@ s;
    sumofr = Plus @@ r;
    S = ZeroMatrix[sizes + 1, q];
    R = ZeroMatrix[q, sizer + 1];
    Gs = Expand[Product[(1 + x^s[[i]]*y), {i, sizes}]];
    Gr = Expand[Product[(1 + x^r[[i]]*y), {i, sizer}]];
    Gs = Map[ReducePower, Gs];
    Gr = Map[ReducePower, Gr];
    For[pick = 0, pick <= sizes, pick++,
      p = Coefficient[Gs, y, pick];
      For[sidx = 1, sidx <= q, sidx++,
        S[[pick + 1, sidx]] = Coefficient[p, x, (sidx - 1)];
        ];
      ];
    For[pick = 0, pick <= sizer, pick++,
      p = Coefficient[Gr, y, pick];
      For[ridx = 1, ridx <= q, ridx++,
        R[[q - Mod[ridx - 2, q], pick + 1]] = Coefficient[p, x, (ridx - 1)];
        ];
      ];
    Cs = q*(S.R);
    For[p = 0, p <= n, p++,
      eta[[m, p + 1]] =
    Sum[Sum[(-1)^(t + a)*(x^(a - t) - x^(a + t + 2))*Cs[[a +1, p - t + 1]],
          {t, 0, p}], {a, 0, 2*k}];
      ];
    ];

    /* Now we look for examples by comparing the generating functions */

rows = Dimensions[eta][[1]]; eta = Simplify[eta];
PositionsOfRunsZero[x_List] :=
  {First[#], Last[#]} & /@ DeleteCases[Map[Last, Split[Transpose[
  {x,Range[Length[x]]}], First[#1] === First[#2] && First[#1] == 0 &], {2}],
  \{_}];
matchsets = ZeroMatrix[Length[valid], n + 1]; For[pidx = 1, pidx
<= n + 1, pidx++,
    matchpolys = Union[eta[[All, pidx]]]; Print[Length[matchpolys]];
    For[rowidx = 1, rowidx <= Length[valid],
      rowidx++,matchsets[[rowidx, pidx]] =
      Position[matchpolys, eta[[rowidx, pidx]]];
      ]
    ];
Print[matchsets // MatrixForm]; For[i = 1, i <= rows, i++,
```


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```
For[j = i + 1, j <= rows, j++,
  diffs = eta[[i]] - eta[[j]];
  zeroruns = PositionsOfRunsZero[diffs];
  If[Length[zeroruns] > 1,
  If[diffs[[1]] == 0, Print["Match on Functions"]];
    Print["i= ", i, "j = ", j, " ", zeroruns];,
    If[Length[zeroruns] == 1,
      If[zeroruns[[1, 1]] > 1,
         If[diffs[[1]] == 0, Print["Match on Functions"]];
        Print["i= ", i, "j = ", j, " ", zeroruns]
        ]
      ]
    ]
  ]
]]
```

Appendix B. *Links to* MATHEMATICA *notebooks*

Links to MATHEMATICA notebooks with code to generate the examples cited above can be found at:

http://www.lms.ac.uk/jcm/9/lms2006-001/appendix-b.

Appendix C. *Links to PDF files of the code*

For people without MATHEMATICA, PDF files with the code and computations can be found at:

http://www.lms.ac.uk/jcm/9/lms2006-001/appendix-c.

Appendix D. *Link to the original* MAPLE *code*

Finally, for historical comparison, our original MAPLE code can be found at:

http://www.lms.ac.uk/jcm/9/lms2006-001/appendix-d.

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