DERIVATIVES AND INTEGRALS WITH RESPECT TO A BASE FUNCTION OF GENERALIZED BOUNDED VARIATION

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1. Introduction. In this paper we consider measures determined by arbitrary functions $G(x)$ for which finite right and left limits exist everywhere and indicate how some of these measures permit the definition of generalized integrals of constructive or Denjoy type. These definitions are related to corresponding descriptive definitions based on the Perron approach as given by Ward **(6)** and Henstock **(2)**. An exposition of the introductory theory is given in **(1)**.

2. We shall denote by $\mathfrak{F}$ the space of functions $G(x)$ that are defined on $X = (-\infty, \infty)$ such that:

(1) $G(x^+)$ and $G(x^-)$ exist and are finite everywhere.

(2) For every x, either $G(x^-) \leqslant G(x) \leqslant G(x^+)$ or $G(x^+) \leqslant G(x) \leqslant G(x^-)$. In **(1)**, (2) is called the Intermediate Value Property (IVP) and this space is called $\check{\mathfrak{F}}$. $\mathfrak{F}_{BV}$ and $\mathfrak{F}'_{BV}$ will denote the subsets of $\mathfrak{F}$ consisting of functions of bounded variation and functions that are of bounded variation on every finite interval, respectively.

Following Munroe let $\mathbf{C}$ denote the covering class of finite open intervals, $\mathbf{C}_d$ for arbitrary $d > 0$ the covering class of open intervals of length less than d. Defining $\tau(\emptyset) = 0$, $\tau(a, b) = G(b^-) - G(a^+)$ leads to Method I outer measures $\mu^*_{G,\infty}$, $\mu^*_{G,d}$ corresponding to $\mathbf{C}$ and $\mathbf{C}_d$. For any sequence $d_i \downarrow 0$ (i.e. $d_1 \geqslant d_2 \ldots$, $\lim d_i = 0$) define

$$\mu^*(A) = \mu^*_{G,0}(A) = \lim_{d_i \to 0} \mu^*_{G,d_i}(A) \leqslant \infty$$

for every subset A of X ($A \in \mathbf{P}(X)$, the collection of all subsets of X). Then μ^* is a Method II outer measure independent of the sequence chosen. If G is monotone, then all of the above outer measures coincide.

As shown in **(1)**, Condition 2 ensures that if $G \in \mathfrak{F}$ and $|G|(x)$, $G^+(x)$, $G^-(x)$ denote the total, positive, and negative variation functions of $G(x) - G(0)$ **(1**, §2**)**, then

$$\mu^* = \mu^*_{G,0} = \mu^*_{|G|} = \mu^*_{G^+} + \mu^*_{G^-},$$

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and the μ^*-measurable sets $\mathbf{S}$ coincide with the intersection $\mathbf{S}_{G^+} \cap \mathbf{S}_{G^-}$ of the $\mu^*_{G^+}$- and $\mu^*_{G^-}$-measurable sets. There exists $X_0 \in \mathbf{S}$ with

$$\mu^*(A) = \begin{cases} \mu^*_{G^+}(A), & A \subset X_0, \\ \mu^*_{G^-}(A), & A \subset CX_0 \end{cases} \qquad \text{(Hahn decomposition).}$$

The signed measure μ_s is then defined on $\mathbf{S}$ by

$$\mu^*_s = \mu^*_{G^+} - \mu^*_{G^-}.$$

Since for each $G \in \mathfrak{F}$, μ^* is a Method II outer measure:

I. μ^* is a metric outer measure (**5**, Theorem 13.3).

II. Every Borel set is Carathéodory measurable for μ^* (**5**, Corollary 13.2.1).

III. If $A_n \uparrow A$, then $\mu^*(A_n) \uparrow \mu^*(A) \leqslant \infty$ (**5**, Corollary 12.1.1). If $A_n \downarrow A$, $A_n \in S$, and there exists n with $\mu(A_n) < \infty$, then $\mu(A_n) \downarrow \mu(A)$.

IV. Given A there exists a G_δ set B, $B \supset A$, with $\mu(B) = \mu^*(A)$, i.e. μ^* is a regular outer measure (**5**, p. 108).

From III and IV we obtain

V. If there exists an open set U containing A with $\mu(U) < \infty$, then, given $\epsilon > 0$, there exists an open set $U' \supset A$ with $\mu^*(A) < \mu(U') + \epsilon$.

The existence of right and left limits everywhere leads to a simple analogue of the Vitali Covering Theorem.

THEOREM 2.1 (the μ^*-Vitali Covering Theorem). *Let A be any subset of a finite interval (a, b) with $\mu(a, b) < \infty$. Suppose that each $x \in A$ is the left end of a sequence of intervals $v_{x,i} = [x, x + h_{x,i}]$ with $\lim_{i\to\infty} h_{x,i} = 0$. Let V denote this family of intervals $\{v_{x,i}, x \in A, i = 1, 2, \ldots\}$ which cover A in the Vitali sense. Then, given $\epsilon > 0$, there exists a finite disjoint subcollection $\{v_i\}$ with*

$$\sum \mu^*(A \cap v_i) > \mu^*(A) - \epsilon, \qquad \sum \mu(v_i) < \mu^*(A) + \epsilon.$$

Proof. Given $\epsilon > 0$ there exists (by V) an open set U, $(a, b) \supset U \supset A$, with

$$\mu^*(A) \leqslant \mu(U) < \mu^*(A) + \epsilon.$$

Writing

$$U = \bigcup_1^\infty (a_i, b_i),$$

let

$$U_d = \bigcup_{(i: b_i - a_i > d)} (a_i, b_i - d),$$

$$A_d = \{x \in A: \text{there exists } h_{x,i} > d \text{ with } [x, x + h_{x,i}] \subset U_d\}.$$

Since $A_d \uparrow A$ as $d \downarrow 0$, then $\mu^*(A_d) \uparrow \mu^*(A)$ as $d \downarrow 0$ by III and, for d sufficiently small,

$$\mu^*(A) - \epsilon < \mu^*(A_d) < \mu^*(A).$$

Note also that for each x $(x, x_i) \downarrow 0$ if $x_i \to x$ and by (III)

$$\mu(x, x_i) \downarrow 0.$$

Let $x'_1 = \inf\{x \in A_d\}$. If $x'_1 \in A_d$, then set $x_1 = x'_1$ and fix $h_1 = h_{x_i,j}$ with $h_1 > d$ and $v_1 = [x_1, x_1 + h_1] \subset U_d$. If $x'_1 \notin A_d$, then there exists a sequence $\{x'_{1,i}\}$ in A_d with $x'_{1,i} \downarrow x'_1$; we fix $x_1 = x'_{1,i}$ with i sufficiently large so that $\mu(x'_1, x_1) < \epsilon/4$ and choose h_1 and v_1 as before.

Let $x'_2 = \inf\{x \in A_d \cap (x_1 + h_1, \infty)\}$. If $x'_2 = x_1 + h_1 \in A_d$, then we can enlarge v_1 to an interval with $h_1 > 2d$. Otherwise we proceed as for x_1 and obtain $x_2 \geqslant x'_2$, $h_2 > d$, $v_2 \subset U_d$, and $\mu(x'_2, x_2) < \epsilon/8$. We can continue this process to obtain a sequence of disjoint intervals $\{v_i = [x_i, x_i + h_i]\}$, all in the Vitali covering V and contained in U_d with each $h_i > d$. Since $U_d \subset (a, b)$, the process terminates in a finite number of steps. The points of A_d not contained in these intervals are contained in the intervals (x'_i, x_i) and thus in a set with μ-measure not exceeding $\epsilon/2$.

Since the intervals v_i are disjoint and all contained in U_d,

$$\sum \mu(v_i) \leqslant \mu(U_d) \leqslant \mu(U) < \mu^*(A) + \epsilon.$$

On the other hand

$$\sum \mu^*(A \cap v_i) \geqslant \mu^*(A_d \cap (\bigcup v_i)) \geqslant \mu^*(A_d) - \epsilon/2 > \mu^*(A) - \epsilon.$$

COROLLARY. *The theorem extends to arbitrary* $A \in \mathbf{P}(X)$ *if* $G \in \mathfrak{F}_{\mathrm{BV}}$ *or* $\mathfrak{F}'_{\mathrm{BV}}$.

It can be shown that if

$$A_n = A \cap (-n, n), \qquad A_n \uparrow A, \qquad \text{and} \qquad \mu^*(A - A_n) \downarrow 0,$$

the theorem applies to each A_n and can be established for A by first approximating A by A_n.

3. Derived numbers and derivatives with respect to a function $G \in \mathfrak{F}$. For $F, G \in \mathfrak{F}$, $x < y$, define

$$D_G F(x, y) = \begin{cases} \dfrac{F(y^+) - F(x^-)}{G(y^+) - G(x^-)} & \text{if } G(x^-) \neq G(y^+), \\ 0, & \text{otherwise,} \end{cases}$$

$$D_G F^+(x) = \lim_{y \to x} D_G F(x, y), \qquad D_G F^-(y) = \lim_{x \to y} D_G F(x, y)$$

when these limits exist. When $D_G F^+(x) = D_G F^-(x)$ with their common value finite, we write $D_G F(x)$ and call it the derivative of F with respect to G at x. Using upper and lower right and left limits, there are similar definitions of upper and lower right and left derivatives

$$\underline{D}_G F^+(x), \qquad \bar{D}_G F^+(x), \qquad \underline{D}_G F^-(x), \qquad \bar{D}_G F^-(x).$$

Remarks. If $G(x)$ is non-decreasing, then $G(y^+) - G(x^-) = \mu[x, y]$ and we sometimes write $D_\mu F$ for $D_G F$.

If x is a point of discontinuity of $G(x)$, then the limits exist and

$$D_G F(x) = [F(x^+) - F(x^-)]/[G(x^+) - G(x^-)].$$

If $F \in \mathfrak{F}_{BV}$ and $G(x)$ is non-decreasing, arguments similar to those used for Lebesgue measure $(G(x) = x)$ show that $D_G F(x)$ exists and is finite almost everywhere (μ^*) and is measurable (μ^*).

Given $G \in \mathfrak{F}$, let $E^+ = \{x: \mu[x, x+h] < \infty$ for some $h > 0\}$. Then if $x \in E^+$, $\mu[x, x+h] \downarrow \mu(\{x\})$ as $h \downarrow 0$. Let $E_0{}^+ = \{x \in E^+: \mu[x, x+h'] = 0$ for some $h' > 0\}$. It is not difficult to show that $\mu(E_0{}^+) = 0$. For $x \in E^+ - E_0{}^+$ a set A is said to have right G-density $D^+(A, x)$ at x if

$$D^+(A, x) = \lim_{h \to 0+} \mu^*(A \cap [x, x+h])/\mu[x, x+h]$$

exists (necessarily $\leqslant 1$). With $[x, x+h]$ replaced by $[x-h, x]$, $h > 0$, there are similar definitions of left G-density $D^-(A, x)$ of A at x. When

$$D^+(A, x) = D^-(A, x)$$

we denote the common value by $D(A, x)$ and call it the density of A at x. Approximate derivatives with respect to G, $AD_G F^+$, $AD_G F^-$, $AD_G F$, may now be defined in terms of G-density by analogy with the classical case $(G(x) = x)$.

We note that if $\mu(a, b) < \infty$, then $(a, b) \subset E^+ \cap E^-$, and if $G \in \mathfrak{F}_{BV}$ or $\mathfrak{F}'_{BV}$, then $E^+ = E^- = X$.

THEOREM 3.1. *If $G \in \mathfrak{F}'_{BV}$, then at almost all (μ^*) points of an arbitrary set A the G-density of A is* 1. *If A is measurable (μ^*), then at almost all points of A the density of CA is* 0.

The argument for the Lebesgue case $(G(x) = x)$ (**4**, §5.2) applies with minor changes, using Theorem 2.1.

THEOREM 3.2. *Let $G \in \mathfrak{F}'_{BV}$ and let* $\mathbf{S}$ *denote the μ^*-measurable sets. Then if $f \in L^1(X, \mathbf{S}, \mu)$ and*

$$F(x) = \int_{(-\infty, x]} f\, d\mu,$$

$D_{|G|} F(x) = D_\mu F(x) = f(x)$ *almost everywhere* (μ^*).

Proof. (i) Let $f(x)$ be simple, i.e.

$$f(x) = \sum_1^n c_i \chi_{e_i}, \qquad e_i \in \mathbf{S},\ \mu(e_i) < \infty.$$

Then, if $x \in e_i$,

$$\begin{aligned} D_\mu F(x, y) &= [F(y^+) - F(x^-)]/\mu[x, y] \\ &= \sum c_i \int_{[x,y]} \chi_{e_i}\, d\mu/\mu[x, y] \\ &= \sum c_i\, \mu(e_i \cap [x, y])/\mu[x, y] \end{aligned}$$

if $\mu[x, y] \neq 0$; $D_\mu F(x, y) = 0$ otherwise. Thus $D_\mu F^+(x) = c_i$ and $D_\mu F^-(x) = c_i$ a.e. in e_i; they vanish a.e. in $C(\bigcup_1^n e_i)$ by Theorem 3.1, and Theorem 3.2 is a direct consequence of the density theorem if $f(x)$ is a simple function.

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(ii) Assume next that $f(x) \geqslant 0$. There is then a sequence $\{f_n\}$ of simple functions with $f_n \uparrow f$ and, for each $[x, y]$,

$$\int_{[x,y]} f_n \, d\mu \uparrow_{n=1}^{\infty} \int_{[x,y]} f \, d\mu.$$

Since

$$D_\mu F(x, y) = \int_{[x,y]} f \, d\mu / \mu[x, y] \geqslant \int_{[x,y]} f_n \, d\mu / \mu[x, y], \qquad n = 1, 2, \ldots,$$

it follows that

$$\underline{D}_\mu F^{\pm}(x) \geqslant f_n(x) \text{ a.e.}, \qquad n = 1, 2, \ldots,$$

and thus $\underline{D}_\mu F^{\pm}(x) \geqslant f(x)$ a.e.

(iii) Assume next that $0 \leqslant f(x) < M < \infty$, i.e. that $f(x)$ in (ii) is bounded. We can then assume that

$$0 \leqslant f(x) - f_n(x) < 1/n, \qquad n = 1, 2, \ldots.$$

Then

$$\int_{[x,y]} f \, d\mu / \mu[x, y] \leqslant \int_{[x,y]} (f_n + 1/n) \, d\mu / \mu[x, y]$$
$$= \int_{[x,y]} f_n \, d\mu / \mu[x, y] + 1/n;$$

and it follows that

$$\bar{D}_\mu F^{\pm}(x) \leqslant f_n(x) + 1/n \leqslant f(x) + 1/n, \qquad n = 1, 2, \ldots \text{ a.e.},$$
$$\bar{D}_\mu F^{\pm}(x) \leqslant f(x) \text{ a.e.},$$

completing the proof of the theorem when $f(x)$ is bounded and non-negative.

(iv) When $f(x)$ is non-negative but unbounded it is sufficient with (ii) to show that $\bar{D}_\mu F^{\pm}(x) \leqslant f(x)$ a.e. Let $E_n = \{x : f(x) \geqslant n\}$. Then $f(x) = f_n(x)$ in CE_n and $\mu(E_n) \to 0, \int_{E_n} f \, d\mu \to 0$ as $n \to \infty$.

Let x be a point of CE_n at which the density of E_n is zero. Then

$$\frac{\int_{[x,y]} f \, d\mu}{\mu[x, y]} \leqslant \frac{\int_{[x,y]} f_n \, d\mu}{\mu[x, y]} + \frac{\int_{E_n[x,y]} f \, d\mu}{\mu[x, y]}$$

and the limit of the first term on the right as $y \to x$ is $f_n(x) = f(x)$ at almost all such points x. Now

$$\int_{E_n[x,y]} f \, d\mu = \int_{E_{n+i}[x,y]} f \, d\mu + \int_{E_n \cap CE_{n+i}[x,y]} f \, d\mu,$$

and for each i,

$$0 \leqslant \frac{\int_{E_n \cap CE_{n+i}[x,y]} f \, d\mu}{\mu[x, y]} \leqslant (n + i) \frac{\mu(E_n \cap [x, y])}{\mu[x, y]} \to 0 \qquad \text{as } y \to x.$$

Let

$$\delta_i(x) = \overline{\lim_{y \to x}} \int_{E_{n+i}[x,y]} f \, d\mu / \mu[x, y].$$

Then $\delta_i(x) \geqslant \delta_{i+1}(x)$, $i = 1, 2, \ldots$. If $\delta_i(x) \downarrow 0$, given $\epsilon > 0$, we can first fix i with $\delta_i < \epsilon$, and then obtain

$$\bar{D}_\mu F^{+}(x) \leqslant f(x) + \epsilon,$$

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for almost all x, with similar arguments giving the same inequality for $\bar{D}_\mu F^-(x)$ at almost all x.

Assume that there exists a set e' with $\mu^*(e') > 0$ in which

$$\lim_{i \to n} \delta_i(x) = \delta(x) > 0.$$

There then exists $d > 0$ and a subset $e \subset e'$ with $\delta(x) > d$ for all $x \in e$ and with $\mu^*(e) > 0$. For $x \in e$ there exists a sequence $y_j(x) \downarrow x$ with

$$\int_{E_{n+i}[x,y_j]} f \, d\mu \geqslant d \,.\, \mu[x, y_j].$$

These intervals for all x, $y_j(x)$ cover e in the Vitali sense and thus there is a finite disjoint subset

$$[x_j, y_j], \qquad j = 1, 2, \ldots, n,$$

with

$$\sum \mu[x,_j y_j] \geqslant \mu^*(e) - \epsilon.$$

Thus

$$\int_{E_{n+i}} f \, d\mu \geqslant \sum_{i=1}^{n} \int_{E_{n+i}[x_j,y_j]} f \, d\mu \geqslant d \sum_{1}^{n} \mu[x_j, y_j] \geqslant d(\mu^*(e) - \epsilon),$$

giving a contradiction for i sufficiently large.

(v) Finally the general case is obtained by considering $F(x)$ as the difference between the integrals of the positive and negative parts of $f(x)$.

Assume that $G \in \mathfrak{F}_{BV}$, define $\mu_s = \mu_{G^+} - \mu_{G^-}$ on $\mathbf{S}$, the μ^*-measurable sets, and let X_0 be a measurable set as given by the Hahn decomposition (§2). Set $L^1(X, \mathbf{S}, \mu_s) = L^1(X, \mathbf{S}, \mu)$ and, if f is measurable ($\mathbf{S}$), write

$$F(x) = \int_{(-\infty,x]} f \, d\mu_s = \int_{(-\infty,x]} f \, d\mu_{G^+} - \int_{(-\infty,x]} f \, d\mu_{G-},$$

where it is defined (in particular for all x if $f \in L^1(\text{X}, \mathbf{S}, \mu_s)$).

THEOREM 3.3.

$$D_{|G|}\, G(x) = D_\mu\, G(x) = \begin{cases} 1 & \text{a.e. in } X_0, \\ -1 & \text{a.e. in } CX_0. \end{cases}$$

Note that

$$D_{|G|}\, G(x) = \lim_{x' \to x} [G(x'^+) - G(x^-)]/[|G|(x'^+) - G(x^-)]$$

$$= \lim_{x' \to x} [\int_{[x,x']} \chi_{X_0} \, d\mu - \int_{[x,x']} \chi_{CX_0} \, d\mu]/\mu[x, x']$$

and apply Theorem 3.2.

THEOREM 3.4. *If $f \in L^1(X, \mathbf{S}, \mu_s)$ and $F(x) = \int_{(-\infty,x]} f \, d\mu_s$, then*

$$D_G\, F(x) = f(x) \text{ a.e. } (\mu^*).$$

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Proof.

$$D_G F^+(x) = \lim_{x' \to x^+} [\int_{[x,x']} f\, d\mu_{G^+} - \int_{[x,x']} f\, d\mu_{G^-}]/[G(x'^+) - G(x^-)]$$

$$= \lim_{x' \to x^+} \frac{\int_{[x,x']} f\, d\mu_{G^+} - \int_{[x,x']} f\, d\mu_{G^-}}{\mu[x, x']} \Big/ \frac{G(x'^+) - G(x^-)}{\mu[x, x']}$$

$$= \begin{cases} f/1 = f & \text{a.e. in } X_0, \\ -f/-1 = f & \text{a.e. in } CX_0. \end{cases}$$

Analogous theorems hold if $G \in \mathfrak{F}'_{BV}$.

4. Derivatives with respect to $G \in \mathfrak{F}$ where G is BV or BV* on a closed set E. A function $G(x)$ is of bounded variation (BV) on a set E if there exists $M < \infty$ with

$$\sum |G(y_i) - G(x_i)| < M$$

for every finite collection $\{(x_i, y_i)\}$ of non-overlapping intervals with end points in E.

$G(x)$ is BV* on a closed set E if it is BV on E and if, in addition, where

$$CE = \bigcup_1^\infty (a_i, b_i), \qquad \sum_1^\infty \theta_i < \infty \qquad \text{where } \theta_i = \sup_{a_i < x < y < b_i} |F(y) - F(x)|$$

(θ_i is called the oscillation of $G(x)$ on (a_i, b_i)).

Let E be an arbitrary closed set,

$$a = \inf\{x \in E\} \geqslant -\infty, \qquad b = \sup\{x \in E\} \leqslant \infty.$$

Assume that $G \in \mathfrak{F}$ and that G is BV on E. Then define

$$\bar{G}(x) = \begin{cases} G(x) & \text{in } E, \\ G(a^-) & \text{if } x < a, \\ G(b^+) & \text{if } x > b. \end{cases}$$

and define $\bar{G}(x)$ to be linear and given by the line segment joining $(a_i, G(a_i^+))$ and $(b_i, G(b_i^-))$, $i = 1, 2, \ldots$, in CE. Then $\bar{G} \in \mathfrak{F}_{BV}$ and determines finite positive measures $\bar{\mu}^* = \mu^*_{\bar{G},0}$, $\bar{\mu}^{+*} = \mu^*_{\bar{G}^+}$, and $\bar{\mu}^{-*} = \mu_{\bar{G}^-}{}^*$ as above.

THEOREM 4.1. *Assume that $G \in \mathfrak{F}$ and that G is* BV *on the closed set E. Then, defining $\bar{G}$ and $\bar{\mu}^*$ as above, $\bar{\mu}(E) \geqslant \mu(E)$.*

Proof. We observe that for every $x \in E$,

$$\mu(\{x\}) = \bar{\mu}(\{x\}) = |G(x^+) - G(x^-)|.$$

Thus there is no loss of generality in assuming that E has no isolated points. Now $E = E \cap [a, b]$ is closed. Let $\bigcup_i (a_i, b_i)$ denote the complement of E relative to $[a, b]$, $A = \{a, b, a_i, b_i, i = 1, 2, \ldots\}$. Then $A \subset E$ and

$$\bar{\mu}(A) = \mu(A) = \sum_i \mu(\{a_i\}) + \sum_i \mu(\{b_i\}) + \mu(\{a\}) + \mu(\{b\}) \leqslant \bar{\mu}(E) < \infty.$$

Let $[\alpha_i, \beta_i]$, $i = 1, 2, \ldots, n+1$ denote the closed intervals on $[a, b]$ complementary to $\bigcup_1^n (a_i, b_i)$. Then

$$\bar{\mu}(E) = \bar{\mu}[a, b] - \sum_1^{\infty} \bar{\mu}(a_i, b_i)$$

$$= \mu(A) + \sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i) - \sum_{n+1}^{\infty} \mu(\{a_i\}) - \sum_{n+1}^{\infty} \mu(\{b_i\}) - \sum_{n+1}^{\infty} \bar{\mu}(a_i, b_i),$$

$$\bar{\mu}(E - A) - \sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i) \to 0 \qquad \text{as } n \to \infty.$$

Let $B = \{x_i\}$ denote the points of E that are points of discontinuity of $G(x)$. Then $\mu(B) = \bar{\mu}(B) \leqslant \bar{\mu}(E) < \infty$ and

$$\sum_n^{\infty} \mu(\{x_i\}) = \sum_n^{\infty} |G(x_i^+) - G(x_i^-)| \to 0 \qquad \text{as } n \to \infty.$$

Note also that

$$\sum_n^{\infty} \bar{\mu}(a_i, b_i) \to 0 \qquad \text{as } n \to \infty.$$

Let $\epsilon > 0$ be fixed and fix $\delta > 0$ with
(i) $\mu^*_{G,\delta}(E) > \mu(E) - \epsilon$.
Fix n sufficiently large such that

(ii) $$b_i - a_i < \delta/4, \qquad i > n$$

(iii) $$\sum_{n+1}^{\infty} \mu(\{x_i\}) < \epsilon,$$

(iv) $$\sum_{n+1}^{\infty} \bar{\mu}(a_i, b_i) < \epsilon,$$

(v) $$\left|\bar{\mu}(E - A) - \sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i)\right| < \epsilon.$$

By (**1**, Theorem 4.1), $\bar{\mu}(\alpha_i, \beta_i) = V\bar{G}(\alpha_i, \beta_i)$. Thus there exist partitions $\{\alpha_i < x_{i1} < \ldots < x_{ik} < \beta_i \ (k = k(i))\}$ with

$$\bar{\mu}(\alpha_i, \beta_i) > \sum_{j=1}^{k-1} |\bar{G}(x_{i,j+1}) - \bar{G}(x_{ij})| > V\bar{G}(\alpha_i, \beta_i) - \epsilon_i = \bar{\mu}(\alpha_i, \beta_i) - \epsilon_i,$$

$$\sum \epsilon_i < \epsilon,$$

and the same inequality holds for any refinement of this partition. We replace this partition by a doubly infinite partition for which the intervals $[x_{ir}, x_{i,r+1}]$ cover $E \cap (\alpha_i, \beta_i)$ for each i, and assume that $x_{i,r+1} - x_{ir} < \delta/4$ for every r and i.

Using an argument similar to (**1**, Theorem 5), we can replace the points of x_{ir} that coincide with points x_j, $j \leqslant n$, by points of continuity without changing the sums by more than ϵ. Thus the points x_{ir} which are points of discontinuity are included in $\{x_j, j > n\}$ and, using (iii),

$$(*) \qquad \sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i) \geqslant \sum_i \sum_r |\bar{G}(x_{i,r+1}) - \bar{G}(x_{ir})| - \epsilon$$
$$\geqslant \sum_i \sum_r |\bar{G}(x^-_{i,r+1}) - \bar{G}(x^+_{ir})| - 3\epsilon.$$

Our next step is to replace $\bar{G}$ in (*) by G. This can be done at once for the points $x_{ir} \in E$. Assume that $x_{ir} \notin E$. Then $a_j < x_{ir} < b_j$ for some $j \geqslant n$. If $x_{i,r+1} < b_j$, omit the interval $(x_{ir}, x_{i,r+1})$.

Assuming $x_{i,r+1} > b_j$, let $y_k \downarrow a_j$, y_k a point of continuity of $G(x)$, $k = 1, 2, \ldots$. Then

$$|G(y_k) - \bar{G}(y_k)| \leqslant |G(y_k) + G(a_j{}^+)| + |\bar{G}(y_k) - \bar{G}(a_j{}^+)| \to 0 \qquad \text{as } j \to \infty.$$

Then

$$|\bar{G}(x^-{}_{i,r+1}) - G(y_k)| \leqslant |\bar{G}(x^-{}_{i,r+1}) - \bar{G}(x^+{}_{i,r})| + \bar{\mu}(a_i, b_i) + |\bar{G}(y_k) - G(y_k)|,$$

$x_{i,r+1} - y_k < \frac{1}{2}\delta$. We can make $|\bar{G}(y_k) - G(y_k)|$ arbitrarily small by choice of k. We can modify intervals with right end point not in E in a similar manner. Denoting the modified intervals by $(t_{ir}, t_{i,r+1})$, we have $t_{i,r+1} - t_{i,r} < \delta/2$,

$$\sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i) \geqslant \textstyle\sum_i \sum_j |\bar{G}(x^-{}_{i,r+1}) - \bar{G}(x^+{}_{i,r})| - 3\epsilon$$
$$\geqslant \textstyle\sum_i \sum_j |G(t^-{}_{i,r+1}) - G(t^+{}_{i,r})| - 4\epsilon - 2 \displaystyle\sum_{n+1}^{\infty} \bar{\mu}(a_i, b_i)$$
$$\geqslant \textstyle\sum_i \sum_j |G(t^-{}_{i,r+1}) - G(t^+{}_{i,r})| - 6\epsilon, \qquad \text{by (iv)},$$

and suitable choices of the points y_k. Then

$$\cup_{i,r} [t_{i,r}, t_{i,r+1}] \supset \cup_i [E \cap (\alpha_i, \beta_i)].$$

We next show that if $t_{i,r} \in E$, we can modify one of $(t_{i,r-1}, t_{i,r})$, $(t_{i,r}, t_{i,r+1})$ to cover $t_{i,r}$. If $t_{i,r} \in E$, it is a limit on the right or left of points of E. Assume that there is a sequence $s_j \downarrow t_{i,r}$ with $s_j \in E$. Then

$$|G(s_j{}^+) - G(t^-{}_{i,r-1})| \leqslant |G(t^+{}_{i,r}) - G(t^-{}_{i,r-1})| + |G(s_j{}^+) - G(t^+{}_{i,r})|,$$

where the last term can be made arbitrarily small by taking j sufficiently large. Then $t_{i,r}$ is covered by the interval $(t_{i,r-1}, s_j)$ and we can assume that

$$s_j - t_{i,r-1} < \tfrac{3}{4}\delta.$$

Having modified the intervals to cover the points of E that are limit points on the right of points of E we can apply a similar process to the points that are points of E on the left keeping the length of the intervals $<\delta$. Denoting the final intervals by $(t'_{ir}, t'_{i,r+1})$, their union covers

$$\bigcup_1^n [E \cap (\alpha_i, \beta_i)]$$

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and

$$\sum_1^{n+1} \bar{\mu}(\alpha_i, \beta_i) \geqslant \sum_i \sum_j |G(t^-_{i,r+1}) - G(t^+_{ir}| - 6\epsilon$$

$$\geqslant \sum_i \sum_j |G(t'^-_{i,r+1}) - G(t'^+_{ir})| - 7\epsilon$$

$$\geqslant \sum_1^{n+1} \mu^*_{G,\delta}[E \cap (\alpha_i, \beta_i)] - 7\epsilon$$

$$\geqslant \sum_1^{n+1} \mu[E \cap \cup(\alpha_i, \beta_i)] - 8\epsilon \qquad \text{by (i)},$$

$$\bar{\mu}(E - A) \geqslant \sum_1^{n+1} \mu[E \cap \cup(\alpha_i, \beta_i)] - 9\epsilon \qquad \text{by (v)}.$$

Letting $n \to \infty$,

$$\bigcup_1^{n+1} [E \cap (\alpha_i \ \beta_i)] \downarrow E - A$$

whence, by III,

$$\bar{\mu}(E - A) \geqslant \mu(E - A) - 9\epsilon.$$

Since ϵ is arbitrary, $\bar{\mu}(E - A) \geqslant \mu(E - A)$, whence $\bar{\mu}(E) \geqslant \mu(E)$.

THEOREM 4.2. *If $G \in \mathfrak{F}$ is* BV* *on a closed set E, then $\bar{\mu}(E) = \mu(E)$.*

Proof. Using Theorem 4.1, we need only prove that $\bar{\mu}(E) \leqslant \mu(E)$, and III shows that we may assume that (a, b) is finite.

Given $\epsilon > 0$, there exists δ with

$$\mu(E) > \mu^*_d(E) - \epsilon/2, \qquad d < \delta \ (\mu^*_d = \mu^*_{G,d}).$$

Since E is compact, there is a finite covering of E by intervals (c_j, d_j) in $\mathbf{C}d$ with

$$(^{**}) \qquad \mu^*_d(E) > \sum |G(d_j^-) - G(c_j^+)| - \epsilon/2$$

and, as in (**1**, Theorem 4.1), we can assume that each point of X is contained in at most two of these intervals.

Fix n sufficiently large that

$$\sum_{n+1}^{\infty} \theta_i < \epsilon/4.$$

For the points c_j, d_j in E we can replace G by $\bar{G}$ in (**). The remaining points fall in the intervals (a_i, b_i), complementary to E, and no such interval can intersect more than four of the (c_j, d_j) intervals assuming that each of the latter intersects E.

Assume that $a_i < c_j < b_i < d_j$, $i \leqslant n$. Then, since $G(b_i^-) = \bar{G}(b_i^-)$,

$$|G(c_j^+) - \bar{G}(c_j^+)| \leqslant |G(b_i^-) - G(c_j^+)| + |\bar{G}(b_i^-) - \bar{G}(c_j^+)|$$

and this is near zero if d has been fixed sufficiently small. It follows that d can be fixed small enough that G can be replaced by $\bar{G}$ for the c_j, d_j falling in the intervals (a_i, b_i), $i \leqslant n$, without increasing the sum in (**) by more than $\epsilon/2$.

If $a_i < c_j < b_i$ or $a_i < d_j < b_i$, $i > n$,

$$|G(c_j^+) - \bar{G}(c_j^+)| \leqslant 2\theta_i \quad \text{or} \quad |G(d_j^-) - \bar{G}(d_j^-)| \leqslant 2\theta_i.$$

Thus for $d < \delta$, d sufficiently small, c_j and/or d_j in

$$\bigcup_{n+1}^{\infty} (a_i, b_i),$$

$$|\sum|G(d_j^-) - G(c_j^+)| - \sum|\bar{G}(d_j^-) - \bar{G}(c_j^+)|| < 4 \sum_{n+1}^{\infty} \theta_i < \epsilon,$$

$$\mu(E) \geqslant \mu_d^+ (E) - \epsilon/2 \geqslant \sum|\bar{G}(a_j^-) - \bar{G}(c_j^+)| - 2\epsilon$$

$$\geqslant \mu_{\bar{G},d}{}^*(E) - 2\epsilon \geqslant \bar{\mu}(E) - 3\epsilon.$$

THEOREM 4.3. *If G is* BV *on the closed set E and $A \subset E$, then $\bar{\mu}^*(A) \geqslant \mu^*(A)$. If G is* BV* *on E and $A \subset E$, then $\bar{\mu}^*(A) = \mu^*(A)$.*

Proof. A minor modification of the argument of Theorem 4.1 gives $\bar{\mu}(A) = \mu(A)$ if A is closed. We show that if U is open, A closed, then $\bar{\mu}(A \cap U) = \mu(A \cap U)$. Let

$$U = \bigcup_{1}^{\infty} (a_i, b_i).$$

Then

$$\bar{\mu}(A \cap (a_i, b_i)) = \bar{\mu}(A \cap [a_i, b_i]) - \bar{\mu}(\{a_i\}) - \bar{\mu}(\{b_i\})$$
$$\geqslant \mu(A \cap [a_i, b_i]) - \mu(\{a_i\}) - \mu(\{b_i\}) = \mu(A \cap (a_i, b_i))$$

and $\bar{\mu}(A \cap U) \geqslant \mu(A \cap U)$ is obtained from the countable additivity of μ and $\bar{\mu}$. If B is an arbitrary subset of E there exists (using §2, IV) a sequence of open sets U_n, $U_n \downarrow U' \supset B$ with $\bar{\mu}(U') = \bar{\mu}^*(B)$. Now $U_n \cap E \downarrow U' \cap E$ with $U' \supset U' \cap E \supset B$ so that $\bar{\mu}(U' \cap E) = \bar{\mu}^*(B)$. Now $\mu(U_n \cap E) \leqslant \mu(E) < \infty$ so that, by III, $\mu(U_n \cap E) \downarrow \mu(U' \cap E) \geqslant \mu^*(B)$, whence

$$\bar{\mu}^*(B) = \lim_{n\to\infty} \bar{\mu}(U_n \cap E) = \lim_{n\to\infty} \mu(U_n \cap E) \geqslant \mu^*(B).$$

To prove the last part let $E' \subset E$ be closed, $CE' = \bigcup_i (a'_i, b'_i)$. Then $E = E' \cap (\bigcup_i E \cap (a'_i, b'_i))$ and

$$\mu(E) = \bar{\mu}(E) = \mu(E') + \textstyle\sum_i \mu(E \cap (a'_i, b'_i)) = \bar{\mu}(E') + \sum_i \bar{\mu}(E \cap (a'_i, b'_i)).$$

With the first part this implies that

$$\mu(E') = \bar{\mu}(E'), \qquad \mu(E \cap (a'_i, b'_i)) = \bar{\mu}(E \cap (a'_i, b'_i)).$$

It is then easy to show that for any $A \subset E$,

$$\bar{\mu}^*(A) = \bar{\mu}(E \cap U') = \mu(E \cap U') \geqslant \mu^*(A)$$

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as in the first part. If A is measurable ($\bar{\mu}^*$),

$$\mu(E \cap (U' - A)) \leqslant \bar{\mu}(E \cap (U' - A)) = 0$$

and $\bar{\mu}(A) = \mu(A)$. Thus $\mu = \bar{\mu}$ on the Borel sets. If A is an arbitrary subset of E, IV implies that there exists a G_δ set B containing A with $\mu(B) = \mu^*(A)$. We can assume that $B \subset E$ so that $\bar{\mu}(B) = \mu(B)$. Then

$$\bar{\mu}^*(A) \leqslant \bar{\mu}(B) = \mu(B) = \mu^*(A).$$

It follows that the μ^*- and $\bar{\mu}^*$-null and measurable subsets of E coincide. In the following almost everywhere in E, measurable subset of E will refer to both μ^* and $\bar{\mu}^*$.

THEOREM 4.4. *Assume that G is* BV* *on the closed set E. Then $D_{\bar{G}}\, G(x) = 1$ a.e. in E.*

Proof. At a point of discontinuity in E

$$[G(x') - G(x^-)]/[\bar{G}(x') - \bar{G}(x^-)] \to [G(x^+) - G(x^-)]/[\bar{G}(x^+) - \bar{G}(x^-)] = 1,$$

since $G(x^+) = \bar{G}(x^+)$, $G(x^-) = \bar{G}(x^-)$. The end points of the intervals complementary to E that are points of continuity of G form a null set. At an interior point of E the result is trivial. Thus we need only consider the set E^* of points of E that are points of continuity of G, limit points of both E and CE on the right and left, and points of $\bar{G}$-density of E. It follows that $AD_{\bar{G}}\, G(x) = 1$ a.e. in E. This result remains valid if the BV* condition is replaced by the weaker BV condition on E. To prove the theorem we need

LEMMA 4.1. *For almost points $x \in E^*$, where $CE = \bigcup_i (a_i, b_i)$,*

$$(\#) \qquad \overline{\lim_{a_i \to x^+}}\ \theta_i/\bar{\mu}(E \cap (x, a_i]) = 0.$$

Proof. For $\eta > 0$ let

$$E_\eta = \{x \in E^*: \overline{\lim}_{a_i \to x^+}\, \theta_i/\bar{\mu}(E \cap (x, a_i]) > \eta\}$$

and assume that there exists η with $\bar{\mu}^*(E_\eta) > 0$. Fix $\epsilon > 0$ and δ such that

$$(\#\#) \qquad \sum_{(i: b_i - a_i < \delta)} \theta_i < \epsilon.$$

For each $x \in E_\eta$ there is a sequence $\{a_j\} = \{a_{j(x)}\}$ with $a_j \to x^+$, $b_j - a_j < \delta$, and

$$\theta_j/\bar{\mu}(E \cap (x, a_j]) > \eta.$$

By Theorem 2.1 there is a finite set of disjoint closed intervals $v_i = [x_j, a'_{j'}]$, $x_j \in E_\eta$ for which

$$\sum \bar{\mu}^*(E_\eta \cap v_j) > \bar{\mu}^*(E_\eta) - \epsilon,$$

$$\sum_{(i: b_i - a_i < \delta)} \theta_i \geqslant \textstyle\sum_j \theta_j > \eta \sum_j \bar{\mu}(E \cap v_j) \geqslant \eta \sum_j \bar{\mu}^*(E_\eta \cap v_j) > \eta(\bar{\mu}^*(E_\eta) - \epsilon).$$

This contradicts (**) if ϵ is sufficiently small.

To complete the proof of the theorem, let X_0 denote the Hahn decomposition set for $\bar{G}$ and set $E_0 = E \cap X_0$. By Theorem 3.3,

$$D_{|\bar{G}|}\,\bar{G}(x) = \begin{cases} 1 & \text{a.e. in } E_0, \\ -1 & \text{a.e. in } E - E_0. \end{cases}$$

Thus

$$D_{\bar{G}}\,G(x) = D_{|\bar{G}|}\,G(x)/D_{|\bar{G}|}\,\bar{G}(x) = \begin{cases} D_{|\bar{G}|}\,G(x) & \text{a.e. in } E_0, \\ -D_{|\bar{G}|}\,G(x) & \text{a.e. in } E - E_0. \end{cases}$$

Let $x \in E^* \cap E_0$ be a point where (*) holds. Then if $a_i < x' < b_i$, x' is a point of continuity of $\bar{G}$ and

$$D_{|\bar{G}|}\,G(x, x') = \frac{G(a_i^+) - G(x)}{|\bar{G}|(x') - |\bar{G}|(x)} + \frac{G(x') - G(a_i^+)}{|\bar{G}|(x') - |\bar{G}|(x)}.$$

Now the first term on the right is

$$\frac{G(a_i^+) - G(x)}{\bar{G}(a_i^+) - \bar{G}(x)} \cdot \frac{\bar{G}(a_i^+) - \bar{G}(x)}{|\bar{G}|(a_i^+) - |\bar{G}|(x)} \cdot \frac{\bar{\mu}(x, a_i)}{\bar{\mu}(x, x')} = P.Q.R, \text{ say.}$$

Now $P = 1$ since $a_i, x \in E$; $Q \to 1$ as $a_i \to x$, $x \in E_0$, $Q \to -1$, $x \in E - E_0$, and $R = \bar{\mu}(E \cap (x, x'))/\bar{\mu}(x, x') \to 1$ since x is a point of $\bar{G}$-density of E. Now

$$\left| \frac{G(x') - G(a_i^+)}{|\bar{G}|(x') - |\bar{G}|(x)} \right| = \left| \frac{G(x') - G(a_i^+)}{\bar{\mu}(E \cap (x, a_i])} \right| \left| \frac{\bar{\mu}(E \cap (x, a_i])}{\bar{\mu}(x, x')} \right|$$

$$\leqslant \theta_i/\bar{\mu}(E \cap (x, a_i]) \to 0 \text{ as } x' \to x^+.$$

Write $f_E(x) = f(x)\chi_E$, where χ_E denotes the characteristic function of E. For $f_E \in L^1(X, \mathbf{S}, \mu)$ set

$$F_E(x) = \int_{(-\infty, x]} f_E\, d\bar{\mu}_s = \int_{(-\infty, x]} f_E\, d\bar{\mu}^+ - \int_{(-\infty, x]} f_E\, d\bar{\mu}^-.$$

COROLLARY 4.1. *Almost everywhere in E, $D_G\, F_E(x) = f(x)$.*

Proof. By Theorem 3.2, $D_{\bar{G}}\, F_E(x) = f(x)$ a.e. in E. By Theorem 4.4, $D_{\bar{G}}\, G(x) = 1$ a.e. in E. Thus a.e. in E,

$$D_G\, F_E(x) = D_{\bar{G}}\, F_E(x)/D_{\bar{G}}\, G(x) = f(x).$$

5. Integrals with respect to base functions $G \in \mathfrak{F}$ which are BVG^*. A function G is BVG^* on X if

$$X = \bigcup_{n=1}^{\infty} E_n$$

with each set E_n closed and with $G(x)$ BV* on each E_n.

To each E_n corresponds $\bar{G}_n \in \mathfrak{F}_{BV}$, finite positive measures $\bar{\mu}_n$, $\bar{\mu}_n^+$, $\bar{\mu}_n^-$, $\bar{\mu}_{n,s}$ and Hahn decomposition sets $E_{n0} \subset E_n$ as in §4 with

$$\bar{\mu}_n(A) = \mu(A) = \begin{cases} \bar{\mu}_n^+(A), & A \subset E_{n0}, \\ 0 & \text{if } A \subset E_n - E_{n0}, \end{cases}$$

$$= \begin{cases} \bar{\mu}_n^-(A), & A \subset E_n - E_{n0}, \\ 0 & \text{if } A \subset E_{n0}. \end{cases}$$

Let $X_0 = \bigcup_n E_{n0}$. Note that if $A \subset E_{n0} \cap E_{m0}$, $\bar{\mu}_n(A) = \bar{\mu}_m(A) = \mu(A)$ with similar relations if $A \subset (E_n - E_{n0}) \cap (E_m - E_{m0})$. Suppose that

$$A = E_{n0} \cap (E_m - E_{m0}).$$

Then a.e. in A, $D_{\bar{G}_n} G(x) = 1$, $D_{\bar{G}_m} G(x) = -1$ by Theorem 4.3. Thus a.e. in A

$$D_{\bar{G}_n} \bar{G}_m(x) = D_{\bar{G}_n} G(x)/D_{\bar{G}_m} G(x) = -1.$$

However, this is false at a point of discontinuity of G and at any point x that is a limit point of points of A. In the latter case there is a sequence $x_i \to x$ with $D_{\bar{G}_n} \bar{G}(x_i, x)$ or $D_{\bar{G}_n} \bar{G}_m(x, x_i) = 1$. It follows that if $\mu(A) = 0$, then

$$\bar{\mu}_n(A) = \bar{\mu}_m(A) = 0;$$

$$\mu(X_0 \cap (E_n - E_{n0})) = 0, \qquad \mu(E_{n0} \cap CX_0) = 0; \qquad n = 1, 2, \ldots .$$

Thus, if $A \subset E_n$,

$$\begin{aligned} \bar{\mu}_{n,s}(A) &= \bar{\mu}_n(A \cap E_{n0}) - \bar{\mu}_n(A \cap (E_n - E_{n0})) \\ &= \bar{\mu}_n(A \cap X_0) - \bar{\mu}_n(A \cap CX_0) \\ &= \mu(A \cap X_0) - \mu(A \cap CX_0). \end{aligned}$$

We use this expression to extend the definition of signed measures by setting

$$\mu_s(A) = \mu(A \cap X_0) - \mu(A \cap CX_0)$$

for the sets A in $\mathbf{S}$ for which the right side is defined in the extended reals, that is except for the case where both terms on the right are infinite. In particular the right side will be defined and finite if A can be covered by a finite number of the sets E_n.

We call an $\mathbf{S}$-measurable function $f(x)$ absolutely integrable if $f \in L^1(X, \mathbf{S}, \mu)$, which implies that $|f| \in L^1(X, \mathbf{S}, \mu)$. We denote by $L^1(X, \mathbf{S}, \mu_s)$ the space of absolutely integrable functions considered with respect to μ_s rather than μ and set

$$F_s(x) = \int_{(-\infty, x]} f \, d\mu_s = \int_{(-\infty, x]} f_{X_0} \, d\mu - \int_{(-\infty, x]} f_{CX_0} \, d\mu.$$

The last expression can be written in terms of sums of the positive and negative measures $\bar{\mu}_n^+$ and $\bar{\mu}_n^-$.

THEOREM 5.1. *If $f \in L^1(X, \mathbf{S}, \mu_s)$, $D_G F_s(x) = f(x)$ almost everywhere (μ^*).*

Proof. Since for non-overlapping intervals (x_i, x'_i)

$$\sum |F_s(x'_i) - F_s(x_i)| = \sum |\int_{(x'_i, x_i]} f\, d\mu_s| < \sum \int_{(x'_i, x_i]} |f|\, d\mu \leqslant \int |f|\, d\mu,$$

F_s is BV on X and therefore BV* on every E_n. If $x \in E_n$,

$$F_s(x) = \int_{(-\infty, x]} f_{E_n}\, d\mu_s - \int_{(-\infty, x]} f_{CE_n}\, d\mu_s.$$

By Corollary 4.1, the derivative with respect to G of the first expression on the right is $f(x)$ and of the second 0 almost everywhere in E_n.

To conclude this paper we consider Denjoy type extensions of the absolute integrals with respect to μ and μ_s when these measures are determined by functions G in $\mathfrak{F}$ that are BVG*. The generalized integral of an $\mathbf{S}_G$-measurable function with respect to μ_s will be defined in a finite or transfinite number of steps and will be denoted by $G(f, e)$ for suitable measurable sets e.

Definition 1. If $e \subset E_n$, $G(f, e) = \int f_e\, d\mu_s$.

Since each point is in some set E_n, $G(f, \{x\})$ is defined and finite for every point if $f(x)$ is finite at every point of discontinuity of G. As is usual for non-absolutely convergent integrals, $G(f, e)$ may fail to be defined or may be defined but not finite over some measurable sets. We require it to be defined and finite over intervals and additive over disjoint intervals and points, i.e. we require

(A) $G(f,(a, b)) = G(f,(a, x]) + G(f,(x, b)) = G(\{,(a, x)) + G(f,[x, b)),$
$G(f,(a, x]) = G(f,(a, x)) + G(f,\{x\})$, etc.

Definition 2. Suppose that $G(f,(a,' b'))$ has been defined over every interval $(a', b') \subset (a, b)$ (and satisfies (A)). Then if for $a < x_0 < b$,

$$\lim_{a' \downarrow a} G(f, (a', x_0)) = \alpha, \qquad \lim_{b' \uparrow b} G(f, (x_0, b')) = \beta$$

exist, we define

$$G(f,(a, b)) = \alpha + \beta + G(f, \{x_0\}).$$

It is then easy to verify that $G(f)$ satisfies (A) on (a, b).

Definition 3. Let (l, m) be an interval such that for some n with

$$\bigcup (a_i, b_i) = CE_n \cap (l, m),$$

$G(f, (a_i, b_i))$ has been defined for every i and

$$\sum_i \theta_i < \infty, \qquad \theta_i = \sup_{a_i < x < y < b_i} |G(f, (a_i, x]) - G(f, (a_i y])|.$$

Then define

$$F(f, (l, m)) = \int_{E_n \cap (l,m)} f\, d\mu_s + \sum_i G(f, (a_i, b_i)).$$

Again it is easy to verify that $G(f)$ satisfies (A) on (l, m).

A measurable function f will be called G-totalizable if

I. $f_{E_n} \in L^1(X, \mathbf{S}, \mu_s)$, $n = 1, 2, \ldots$.

II. The limits in (2) always exist.

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III. f is such that if A is any closed set for which $G(f,(a_i, b_i))$ has been determined for all open intervals (a_i, b_i), $CA = \bigcup_i (a_i, b_i)$, there exists (l, m) with $A \cap (l, m)$ non-empty and $\sum_{(l,m)} \theta_i < \infty$. (If $a_i < l < b_i$ for some i, then θ_i will mean the oscillation over (l, b_i). A similar convention will hold if $a_i < m < b_i$.)

THEOREM 5.2. *If f is G-totalizable, $G(f,(-\infty, \infty))$ can be determined in a countable number of steps and $G(f)$ is additive over finite sets of disjoint intervals and points.*

Proof. Let A_1 denote the set of points for which there is no neighbourhood over which f is absolutely integrable. Clearly A_1 is closed. If (a, b) is any interval, Baire's Theorem implies that there is a subinterval (l, m) and an integer n with $(l, m) = E_n \cap (l, m)$. By (I), f is absolutely integrable over (l, m). Thus A_1 is nowhere dense.

Let $CA_1 = \bigcup_i (a'_i, b'_i)$. Then associated with each $x \in CA_1$ is an open interval containing x over which f is absolutely integrable. If $a'_i < \alpha < \beta < b'_i$, then $[\alpha, \beta]$ is covered by a finite set of such open intervals and thus f is absolutely integrable over (α, β). Condition II and Definition 2 then extend $G(f)$ to $G(f, (a'_i, b'_i))$ for every i.

Let A_2 denote the points x for which there is no neighbourhood (α, β) of x over which f_{A_1} is absolutely integrable and $\sum_{(\alpha,\beta)} \theta_i < \infty$. Again A_2 is nowhere dense in A_1 and closed. If $CA_2 = \bigcup_i (a_i^2, b_i^2)$ and $a_i^2 < \alpha < \beta < b_i^2$, there are a finite number of points in $[\alpha, \beta]$ for which the corresponding neighbourhoods cover $[\alpha, \beta]$. It follows that Definition 3 applies to determine $G(f, (\alpha, \beta))$. Then Condition II and Definition 2 determine $G(f, (a_i^2, b_i^2))$ for each i. Standard procedures apply to give $G(f, (-\infty, \infty))$ in a finite or countable number of steps; cf. (**4**, Theorem 5.6).

If f is G-totalizable, then $f_x = f\chi_{(-\infty,x]}$ is G-totalizable and thus

$$F(x) = G(f, (-\infty, x]) = G(f_x, (-\infty, \infty))$$

is defined and finite for every x.

THEOREM 5.3. *If f is G-totalizable, $D_G F(x) = f(x)$ almost everywhere.*

Proof. With the notation of the preceding theorem consider

$$(\alpha, \beta), \qquad a'_i < \alpha < \beta < b'_i.$$

Then $f\chi_{(\alpha,\beta)} \in L^1(X, \mathbf{S}, \mu_s)$ and a.e. in (α, β), writing

$$f^* = f\chi_{(\alpha,\beta)}, \qquad F^*(x) = G(f^*, (\alpha, x]),$$
$$D_G F(x) = D_G F^*(x) = f(x),$$

by Theorem 5.1. It follows that $D_G F(x) = f(x)$ a.e. in CA_1.

Consider $[\alpha, \beta]$, $a_i^2 < \alpha < \beta < b_i^2$. Set $f^* = f\chi_{[\alpha,\beta]}$, $F^*(x) = G(f^*,(-\infty, x])$. Then in (α, β) $F(x) - F^*(x) = G(f, (-\infty, \alpha))$ so that $D_G F(x) = D_G F^*(x)$. Now, for $\alpha < x < \beta$,

$$F^*(x) = \int_{A_1 \cap [\alpha,\beta]} f \, d\mu_s + \sum_{[\alpha,x]} \theta_i,$$

with $\sum_{[\alpha,\beta]} \theta_i < \infty$. For any sequence $\{(x_i, y_i)\}$ of non-overlapping intervals with $x_i, y_i \in A_1$,

$$\sum_i |F^*(y_i) - F^*(x_i)| \leqslant \int_{A_1 \cap [\alpha,\beta]} |f|\, d\mu + \sum_{[\alpha,\beta]} \theta_i.$$

It follows that $F^*(x)$ is BV* on $A_1 \cap [\alpha, \beta]$. As in Theorem 4.4 and Corollary 4.1, $D_G F^*(x) = f(x)$ a.e. in $A_1 \cap (\alpha, \beta)$. It now follows easily that $D_G F(x) = f(x)$ a.e. in CA_2. Similar arguments apply at each extension stage and, since only a countable number of extensions are required to exhaust X, the final exceptional set is μ-null.

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