THE BROWN McCOY RADICAL OF SEMIGROUP RINGS OF COMMUTATIVE CANCELLATIVE SEMIGROUPS

by E. JESPERS,† J. KREMPA and P. WAUTERS‡

(Received 1 October, 1983; revised 10 May, 1984)

1. We give a complete description of the Brown–McCoy radical of a semigroup ring $R[S]$, where R is an arbitrary associative ring and S is a commutative cancellative semigroup; in particular we obtain the answer to a question of E. Puczyłowski stated in [**11**].

Throughout this note all rings R are associative with unity 1; all semigroups S are commutative and cancellative with unity. Note that the condition that R and S have a unity can be dropped (cf. [**8**]). The quotient group of S is denoted by $Q(S)$. We say that S is torsion free (resp. has torsion free rank n) if $Q(S)$ is torsion free (resp. has torsion free rank n). The Brown–McCoy radical (i.e. the upper radical determined by the class of all simple rings with unity) of a ring R is denoted by $\mathcal{U}(R)$. We refer to [**2**] for further detail on radicals and in particular on the Brown–McCoy radical.

First we state some well-known results and a preliminary lemma. Let R and T be rings with the same unity such that $R \subset T$. Then T is said to be a *normalizing extension* of R if $T = Rx_1 + \ldots + Rx_n$ for certain elements $x_1, \ldots, x_n$ of T and $Rx_i = x_iR$ for all i such that $1 \leq i \leq n$. If all x_i are central in T, then we say that T is a *central normalizing extension* of R.

PROPOSITION 1.1. *Let R and T be rings such that T is a normalizing extension of R. Then $\mathcal{U}(R) = \mathcal{U}(T) \cap R$.*

Proof. cf. [**9**] or [**11**].

PROPOSITION 1.2. (1) *Let G be a finite abelian group of order n and let R be a G-graded ring. If $\alpha = \sum_{g \in G} \alpha_g \in \mathcal{U}(R)$, then $n\alpha_g \in \mathcal{U}(R)$ for all $g \in G$.*

(2) *If S is a torsion free commutative semigroup and if R is an S-graded ring, then $\mathcal{U}(R)$ is homogeneous, i.e. if $\sum_s r_s \in \mathcal{U}(R)$, then $r_s \in \mathcal{U}(R)$ for all s.*

Proof. G. M. Bergman has proved this for the Jacobson radical [**1**], but the result remains valid for the Brown–McCoy radical (cf. [**11**]). Since G. M. Bergman's result is only available in preprint, we refer to [**9**] for an account of his results in the $\mathbb{Z}$-graded case.

Let H and S be semigroups with $H \subset S$. We say that H is a

† Part of this work was done while the first named author visited the Department of Mathematics of the University of Warsaw. This author was partially supported by an N.F.W.O.-grant.

‡ The third named author is a research assistant at the National Fund for Scientific Research (Belgium).

grouplike subset of S (cf. [**12**]) if $a, ab \in H$ imply $b \in H$ $(a, b \in S)$. Note that H contains the unity of S.

LEMMA 1.3. *If S is a commutative cancellative semigroup and if H is a grouplike subset of S, then*

$$\mathcal{U}(R[S]) \cap R[H] \subset \mathcal{U}(R[H]),$$

for each ring R.

Proof. It suffices to show that $\mathcal{U}(R[S]) \cap R[H]$ is a $\mathcal{U}$-radical ideal of $R[H]$ (in the sense of [**2**]). Let $\alpha \in \mathcal{U}(R[S]) \cap R[H]$. Because $\alpha \in \mathcal{U}(R[S])$ there exist $\beta_i, \gamma_i \in R[S]$, $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^{n} \beta_i(1+\alpha)\gamma_i = 1$. We may suppose that all $\beta_i \in R$; for if $\beta_i = \sum_{j=1}^{m} r_{ij}s_{ij}$, $r_{ij} \in R$, $s_{ij} \in S$, then

$$1 = \sum_i \beta_i(1+\alpha)\gamma_i = \sum_{i,j} r_{ij}(1+\alpha)s_{ij}\gamma_i$$

because all s_{ij} are central. Write each $\gamma_i = \gamma_{i,0} + \gamma_{i,1}$ such that supp $\gamma_{i,0} \subset H$ and supp $\gamma_{i,1} \subset S \backslash H$. Then

$$1 = \sum_i \beta_i(1+\alpha)\gamma_{i,0} + \sum_i \beta_i(1+\alpha)\gamma_{i,1}.$$

The first summation belongs to $R[H]$ and $\operatorname{supp}\left(\sum_i \beta_i(1+\alpha)\gamma_{i,1}\right) \cap H = \varnothing$, because H is a grouplike subset of S. Therefore $1 = \sum_i \beta_i(1+\alpha)\gamma_{i,0}$. Since α is arbitrary, this shows that $\mathcal{U}(R[S]) \cap R[H]$ is a $\mathcal{U}$-radical ideal of $R[H]$.

2. Torsion free semigroups.

PROPOSITION 2.1 (cf. [**5, 7**]). *If S is a free semigroup of rank n (finite or infinite), then*

$$\mathcal{U}(R[S]) = \mathcal{U}_n(R)[S]$$

for each ring R, where $\mathcal{U}_n(R) = \mathcal{U}(R[S]) \cap R$. Moreover

$$\mathcal{U}(R) \supset \mathcal{U}_1(R) \supset \mathcal{U}_2(R) \supset \ldots \supset \bigcap_{n=1}^{\infty} \mathcal{U}_n(R) = \mathcal{U}_\infty(R)$$

and if n is infinite, then $\mathcal{U}_n(R) = \mathcal{U}_\infty(R)$.

LEMMA 2.2. *Let S be a free group of rank n. Then*

$$\mathcal{U}(R[S]) = \mathcal{U}_n(R)[S]$$

for each ring R.

Proof. Because the Brown–McCoy radical satisfies Lemma 2.1 and behaves well with respect to normalizing extensions (i.e. satisfies Proposition 1.1), the proof is similar to the proof of Lemma 2.2 of [**8**].

LEMMA 2.3. *Let S be a semigroup of torsion free rank n. Then*

$$\mathcal{U}_n(R)[S] \subset \mathcal{U}(R[S])$$

for each ring R.

Proof. Suppose first that S is a group. By the definition of rank there exists a free subgroup F of rank n such that S/F is a torsion group. Hence by Lemma 2.2, $\mathcal{U}_n(R) \subset \mathcal{U}_n(R)[F] = \mathcal{U}(R[F])$. So it suffices to prove that $\mathcal{U}(R[F]) \subset \mathcal{U}(R[S])$, i.e. we will show that for each $\alpha \in \mathcal{U}(R[F])$ and $\beta_i, \gamma_i \in R[S]$, $1 \leqslant i \leqslant m$, $\sum_i \beta_i \alpha \gamma_i$ is G-regular in $R[S]$ (in the sense of [2]). Let H be the subgroup generated by the set $F \cup \left(\bigcup_i \operatorname{supp} \beta_i\right) \cup \left(\bigcup_i \operatorname{supp} \gamma_i\right)$. Then H/F is finite since S/F is an abelian torsion group. Hence $R[H]$ is a normalizing extension of $R[F]$. Therefore, $\sum_i \beta_i \alpha \gamma_i \in \mathcal{U}(R[H])$ (Proposition 1.1); in particular $\sum_i \beta_i \alpha \gamma_i$ is G-regular in $R[H]$ and thus also in $R[S]$.

Suppose now that S is a semigroup. Let $a \in \mathcal{U}_n(R)$; we show that a belongs to each maximal ideal of $R[S]$. Let M be a maximal ideal of $R[S]$ and let T denote $\{x \in S \mid x \notin M\}$. Then,

(i) T is a semigroup because M is a prime ideal,

(ii) if $x \in S \backslash T$, then $xy \in S \backslash T$ for all $y \in S$.

Define $\pi : R[S] \to R[T]$: $\sum_{s \in S} r_s s \mapsto \sum_{s \in T} r_s s$. By (ii) π is a ring epimorphism. Clearly $\ker \pi \subset M$. Therefore $\pi(M)$ is a maximal ideal of $R[T]$. Moreover $\pi(M) \cap T = \varnothing$. So $\pi(M)R[Q(T)] \neq R[Q(T)]$. Since $R[Q(T)]$ is a localisation of $R[T]$, ideals of $R[Q(T)]$ are generated by their intersection with $R[T]$. In particular, $\pi(M)R[Q(T)]$ is a maximal ideal of $R[Q(T)]$. Clearly $\pi(M) = \pi(M)R[Q(T)] \cap R[T]$ by maximality of $\pi(M)$. Now, $Q(T)$ is a group of torsion free rank m and $m \leqslant n$. Since $a \in \mathcal{U}_n(R) \subset \mathcal{U}_m(R)$, the first part of the proof shows that $a = \pi(a) \in \pi(M)R[Q(T)]$. So $\pi(a) \in \pi(M)R[Q(T)] \cap R[T] = \pi(M)$ and hence $a \in M$.

LEMMA 2.4. *If S is a torsion free semigroup such that $Q(S)$ is a finitely generated free group with free generators $x_1 = s_1 t^{-1}, \ldots, x_n = s_n t^{-1}$ and $s_i, t \in S$, then $s_1, \ldots, s_n$ or $s_1 t, \ldots, s_n t$ is a set of free generators of a subsemigroup of S.*

Proof. cf. [**3**].

THEOREM 2.5. *Let S be a torsion free semigroup of rank n, then*

$$\mathcal{U}(R[S]) = \mathcal{U}_n(R)[S]$$

for each ring R.

Proof. Once the statement is known for finite rank n it follows from Lemma 1.3, Proposition 2.1 and by using the same method as in the proof of Theorem 2.3 of [**8**] that the result is also valid for infinite rank.

So we assume that n is finite. By Lemma 2.3 it suffices to prove that $\mathcal{U}(R[S]) \subset \mathcal{U}_n(R)[S]$. Moreover, Proposition 1.2 implies that we only have to prove that if $rs \in \mathcal{U}(R[S])$, where $r \in R$ and $s \in S$, then $r \in \mathcal{U}_n(R)$. Since $Q(S)$ has torsion free rank n, $Q(S)$ contains a free subgroup F of rank n. Note that F is the quotient group of $S \cap F$ (since $Q(S)/F$ is torsion). By Lemma 2.4 S contains a free subsemigroup X of rank n, so $Q(X)$ is a free group of rank n and $Q(S)/Q(X)$ is torsion. Clearly $S' = S \cap Q(X)$ is a grouplike subset of S. Hence $rs^l \in \mathcal{U}(R[S'])$ for some $l > 0$ (Lemma 1.3). Write $s^l = y^{-1}x$, $y, x \in X$ (since $s^l \in Q(X)$). So $rx \in \mathcal{U}(R[S'])$. Let $x_1, \ldots, x_n$ be a set of free generators of X. Then $x = x_{i_1}^{k_1} \ldots x_{i_m}^{k_m}$, where $k_i \in \mathbb{N}$ and $i_j \in \{1, \ldots, n\}$. Since

$$(R[S']rx_{i_1} \ldots x_{i_m}R[S'])^{k_1 + \cdots + k_m} \subset R[S']rxR[S'] \subset \mathcal{U}(R[S'])$$

and $\mathcal{U}(R[S'])$ is a semiprime ideal, it follows that $rx_{i_1} \ldots x_{i_m} \in \mathcal{U}(R[S'])$. By multiplying by those $x_j \in \{x_1, \ldots, x_n\} \setminus \{x_{i_1}, \ldots, x_{i_m}\}$, we obtain that $\alpha = rx_1 \ldots x_n \in \mathcal{U}(R[S']) \cap R[X]$.

To complete the proof it suffices to show that $\alpha = rx_1 \ldots x_n \in \mathcal{U}(R[X]) = \mathcal{U}_n(R)[X]$. Let M be a maximal ideal of $R[X]$. If $M \cap X \neq \emptyset$, M contains one of the generators x_i (M is prime). Because of the form of $\operatorname{supp} \alpha$ it follows that $\alpha \in M$. If $M \cap X = \emptyset$, then $R[Q(X)]M$ is a maximal ideal of $R[Q(X)]$. This follows as in Lemma 2.3. Since $R[X] \subset R[S'] \subset R[Q(X)]$ we have $M = M' \cap R[X]$ where $M' = R[Q(X)]M \cap R[S']$. We claim that M' is a maximal ideal of $R[S']$. Let N' be an ideal of $R[S']$ such that $M' \subset N' \subsetneqq R[S']$. Then $M = M' \cap R[X] = N' \cap R[X]$ by maximality of M. Therefore $R[Q(X)]M = R[Q(X)](N' \cap R[X])$ and clearly $R[Q(X)](N' \cap R[X]) = R[Q(X)]N'$. Combining these, we obtain $N' \subset R[Q(X)]N' \cap R[S'] = R[Q(X)]M \cap R[S'] = M'$ and hence M' is maximal. From $\alpha \in \mathcal{U}(R[S']) \cap R[X]$ we deduce that $\alpha \in M' \cap R[X] = M$. Since M was arbitrary this proves that $\alpha \in \mathcal{U}(R[X])$ and thus $r \in \mathcal{U}_n(R)$.

3. The main theorem. Before proving the main theorem we need a lemma about the Brown–McCoy radical of a group ring of a finite abelian group. In the case where the coefficient ring is commutative, this result has been proved by G. Karpilovsky [**5**].

LEMMA 3.1. *Let R be a ring and G a finite abelian group. Then*

$$\mathcal{U}(R[G]) = \mathcal{U}(R)[G] + \left\{ \sum_i r_i(x_i - y_i) \mid r_i \in R,\ x_i, y_i \in G,\ x_i^{p_i^k} = y_i^{p_i^k} \text{ for some } k \geq 0,\ p_i \text{ a prime number and } p_i r_i \in \mathcal{U}(R) \right\}.$$

Proof. By Proposition 1.1, $\mathcal{U}(R)[G] \subset \mathcal{U}(R[G])$, so $\mathcal{U}(R[G])/\mathcal{U}(R)[G] \cong \mathcal{U}(R/\mathcal{U}(R)[G])$. By replacing R by $R/\mathcal{U}(R)$, we may assume that $\mathcal{U}(R) = \{0\}$. In particular, we may assume that R is a semiprime ring.

Let I be the torsion part of R for the additive structure. Because I is an ideal of R, we obtain that $\mathcal{U}(I[G]) \subset \mathcal{U}(R[G])$ (cf. [**2**]). For the converse inclusion, let $\alpha = \sum r_g g \in \mathcal{U}(R[G])$. If n is the order of the group G, then, by Proposition 1.2, $nr_g g \in \mathcal{U}(R[G])$ for all $g \in \operatorname{supp} \alpha$. So $nr_g \in \mathcal{U}(R[G]) \cap R \subset \mathcal{U}(R) = \{0\}$ (Lemma 1.3), i.e. $r_g \in I$. Therefore $\alpha \in \mathcal{U}(R[G]) \cap I[G] = \mathcal{U}(I[G])$ and consequently $\mathcal{U}(R[G]) = \mathcal{U}(I[G])$. By Bezout's theorem $I = \bigoplus_p I_p$ where p runs through the set of all prime numbers and where $I_p = \{x \in I \mid \exists n \geq 0\ p^n x = 0\}$. But R is semiprime, and hence so is I. Therefore $I_p = \{x \in I \mid px = 0\}$. Hence

$$\mathcal{U}(R[G]) = \mathcal{U}(I[G]) = \bigoplus_p \mathcal{U}(I_p[G]) \qquad (*).$$

The last equality holds by standard results on radicals (cf. [**2**]). Now, for any prime p, we can write $G = G_p \times G_{p'}$, where G_p is the p-torsion part of G and $G_{p'}$ is a p'-group, i.e. it has no elements of order p. Because $I_p[G] = (I_p[G_p])[G_{p'}]$, $I_p[G]$ is graded by $G_{p'}$ in a natural way. Let m be the order of $G_{p'}$. Then as above, if $\alpha = \alpha_1 g_1 + \cdots + \alpha_m g_m \in \mathcal{U}(I_p[G])$, where $g_1, \ldots, g_m \in G_{p'}$ and every $\alpha_i \in I_p[G_p]$, we obtain that $m\alpha_i \in \mathcal{U}(I_p[G_p])$ for all $1 \leq i \leq m$. Since $p\alpha_i = 0$ and because m and p are relatively prime it follows that $\alpha_i \in \mathcal{U}(I_p[G_p])$. Thus $\mathcal{U}(I_p[G]) \subset \mathcal{U}(I_p[G_p])[G]$ and the converse inclusion follows from Proposition 1.1. Now, similarly as in the proof of Lemma 3.1.6 of [**10**], we obtain that

$$\mathcal{U}(I_p[G_p]) = \omega(I_p[G_p]),$$

where $\omega(I_p[G_p])$ is the augmentation ideal of $I_p[G_p]$. Therefore

$$\begin{aligned}\mathcal{U}(I_p[G]) &= \mathcal{U}(I_p[G_p])[G] = \omega(I_p[G_p])[G] \\ &= \left\{\sum_i r_i(x_i - y_i) \mid x_i, y_i \in G,\ x_i^{p^k} = y_i^{p^k} \text{ for some } k \geq 0,\ pr_i = 0\right\}.\end{aligned}$$

The result follows now from equality $(*)$.

Remark 3.2. If $x, y \in G$, $x^{p^k} = y^{p^k}$, $k \geq 0$, p prime and $r \in R$ such that $pr \in \mathcal{U}(R)$, then $r(x-y)$ generates a nilpotent ideal modulo $\mathcal{U}(R)[G]$. Note that this remains true if G is a semigroup and if we replace $\mathcal{U}(R)$ by $\mathcal{U}_n(R)$.

Theorem 3.3. *Let S be an arbitrary commutative cancellative semigroup of torsion free rank n (finite or infinite). Then for each ring R*

$$\mathcal{U}(R[S]) = \mathcal{U}_n(R)[S] + \left\{\sum_i r_i(s_i - t_i) \mid r_i \in R,\ s_i, t_i \in S,\ s_i^{p_i^k} = t_i^{p_i^k} \text{ for some } k \geq 0,\ p_i \text{ a prime number and } p_i r_i \in \mathcal{U}_n(R)\right\}.$$

Proof. By Lemma 2.3 $\mathcal{U}_n(R)[S] \subset \mathcal{U}(R[S])$ and, by Remark 3.2, $\mathcal{N} = \left\{\sum_i r_i(s_i - t_i) \mid r_i \in R,\ s_i, t_i \in S,\ s_i^{p_i^k} = t_i^{p_i^k} \text{ for some } k \geq 0,\ p_i \text{ a prime number and } p_i r_i \in \mathcal{U}_n(R)\right\} \subset \mathcal{U}(R[S])$. So it suffices to prove that $\mathcal{U}(R[S]) \subset \mathcal{U}_n(R)[S] + \mathcal{N}$. As in the proof of Lemma 3.1 we may suppose that $\mathcal{U}_n(R) = \{0\}$. Let $\alpha \in \mathcal{U}(R[S])$. Let $\langle \operatorname{supp} \alpha \rangle$ be the subgroup of $Q(S)$ generated by $\operatorname{supp} \alpha$. Then $\langle \operatorname{supp} \alpha \rangle = G_1 \times G_2'$, the direct product of a finite group G_1 and a free group G_2'. Since $Q(S)$ and $Q(S)/G_1$ have the same torsion free rank, we can add free generators to G_2' such that we obtain a free group of rank n. This holds also if n is infinite. So $\operatorname{supp} \alpha$ is contained in a subgroup $G = G_1 \times G_2$ of $Q(S)$ where G_1 is finite and G_2 is free of rank n. Let $H = G \cap S$; since $Q(S)/G$ is torsion (because rank G = rank $Q(S)$), we have $Q(H) = G$. Because H is a grouplike subset of S, $\alpha \in \mathcal{U}(R[H])$. Let H' be the subsemigroup of G generated by $H \cap G_1$. Then $H' = G_1 \times H''$, where $H'' = H' \cap G_2$. Let $x \in H'$; then $x = g_1 h$ where $g_1 \in G_1$, $h \in H$. Write $h = h_1 h_2$, $h_1 \in G_1$, $h_2 \in G_2$. So $h_2 = (g_1 h_1)^{-1} x \in H' \cap G_2$ since G_1 is a group. Note that H'' is a torsion free semigroup of rank n (because $Q(H'') = G_2$). Now $R[H']$ is a normalizing extension of $R[H]$ and thus $\alpha \in \mathcal{U}(R[H'])$. Note that $R[H'] = (R[H''])[G_1]$. Because $\mathcal{U}_n(R) = \{0\}$, Theorem 2.5 implies that $\mathcal{U}(R[H'']) = \{0\}$. By Lemma 3.1,

$$\alpha \in \mathcal{U}(R[H']) = \mathcal{U}((R[H''])[G_1]) = \Big\{ \sum_i \alpha_i (x_i - y_i) \mid \alpha_i \in R[H''],$$
$$x_i, y_i \in G_1,\ x_i^{p_i^k} = y_i^{p_i^k} \text{ for some } k \geq 0,\ p_i \text{ a prime number and } p_i \alpha_i = 0 \Big\}.$$

Write $\alpha_i = \sum_j r_{ij} h_j$ with $r_{ij} \in R$ and $h_j \in H''$ for all i, j. Then $\alpha = \sum_{i,j} r_{ij}(h_j x_i - h_j y_i)$ and clearly $(h_j x_i)^{p_i^k} = (h_j y_i)^{p_i^k}$ and $p_i r_{ij} = 0$ for all i and j. This finishes the proof.

4. On other radicals. In [**3**] the analogue of Theorem 2.5 is proved for the Jacobson radical. In the general case the authors of [**3**] obtained the analogue of Theorem 3.3 only for algebras over a field. Now, if in our situation we replace maximal ideals by maximal right ideals and G-regularity by quasi-regularity, we obtain the full analogue of Theorem 3.3 for the Jacobson radical.

Note that Theorem 2.5 also remains valid for the upper nil radical (cf. [**3**]). If one wants to extend this result to arbitrary semigroups then there will appear problems which are related to the unsolved Köethe problem.

For the prime and locally nilpotent radicals the results of G. Bergman are also true (cf. [**11**]) and therefore one can easily obtain the analogue of Theorem 3.3 for these two radicals (see also [**4**] for the torsion free case).

REFERENCES

1. G. M. Bergman, On Jacobson radicals of graded rings, *preprint.*

2. N. J. Divinsky, *Rings and radicals* (University of Toronto Press, London, 1965).

3. E. Jespers and E. Puczyłowski, The Jacobson radical of semigroup rings of commutative cancellative semigroups, *Comm. in Algebra.* **11** (1984), 1115–1124.

4. E. Jespers, J. Krempa and E. Puczyłowski, On radicals of graded rings, *Comm. in Algebra,* **10** (17) (1982), 1849–1854.

5. G. Karpilovsky, The Jacobson radical of a commutative group ring, *Arch. Math.* **39** (1982), 428–430.

6. J. Krempa, Radicals of semigroup rings, *Fund. Math.* **85** (1974), 54–71.

7. J. Krempa, On radical properties of polynomial rings, *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.* **20** (1972), 545–548.

8. J. Krempa and A. Sierpinska, The Jacobson radical of certain group and semigroup rings, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **26** (1978), 963–967.

9. C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory* (North-Holland, 1982).

10. D. S. Passman, *The algebraic structure of group rings* Wiley Interscience, New York, 1977).

11. E. R. Puczyłowski, Behaviour of radical properties of rings under some algebraic constructions, *to appear.*

12. H. Schneider and J. Weissglass, Group rings, semigroup rings and their radicals, *J. of algebra* **5** (1967), 1–15.

THE UNIVERSITY OF STELLENBOSCH
DEPARTMENT OF MATHEMATICS
STELLENBOSCH
7600 – SOUTH AFRICA

UNIVERSITY OF WARSAW
INSTITUTE OF MATHEMATICS
UNIVERSITY
PKiN, 00-901 WARSAW – POLAND

KATHOLIEKE UNIVERSITEIT LEUVEN
DEPARTEMENT WISKUNDE
CELESTIJNENLAAN 200 B
B-3030 LEUVEN, BELGIUM

https://doi.org/10.1017/S0017089500005863 Published online by Cambridge University Press