which leads to the energy integral $T + V = constant$, when the power of the nonconservative generalised forces, given by the right hand side of (1.3), is zero. The second consequence follows for a variable q_r which is cyclic, that is, T and V are both independent of q_r, leading to

$$F_r = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right), \tag{1.4}$$

or in integrated form

$$\int_{t_1}^{t_2} F_r\, dt = \left.\frac{\partial T}{\partial \dot{q}_r}\right|_{t_2} - \left.\frac{\partial T}{\partial \dot{q}_r}\right|_{t_1}. \tag{1.5}$$

If the impulse of the nonconservative force corresponding to the cyclic variable, given by the left hand side of (1.5), is zero, then the change in the generalised momentum corresponding to this variable, given by the right hand side of (1.5), is also zero. It is the two properties expressed by (1.3) and (1.4) which the discretisations are forced to preserve.

2. Partial Differences

Consider $f = f(q) = f(q_1, \ldots, q_n)$ with

$$df = \sum_{r=1}^{n} \frac{\partial f(q)}{\partial q_r}\, dq_r. \tag{2.1}$$

If f is independent of q_r, then

$$\partial f / \partial q_r = 0. \tag{2.2}$$

If the time t is discretised at time instants $t(k), k = 0, 1, \ldots$ then the present aim is to choose a discrete analogue of the partial derivative, which we shall denote by

$$\frac{\Delta_r f(k)}{\Delta q_r(k)} = \frac{\Delta_r f(k)}{q_r(k+1) - q_r(k)}, \tag{2.3}$$

so that the discrete analogues of (2.1) and (2.2) are satisfied. In (2.3), Δ is the usual forward difference operator and Δ_r will be called the (forward) partial difference operator. The discrete analogue of (2.1) is

$$\Delta f = f[q(k+1)] - f[q(k)] = \sum_{r=1}^{n} \frac{\Delta_r f(k)}{\Delta q_r(k)} \Delta q_r(k) \tag{2.4}$$

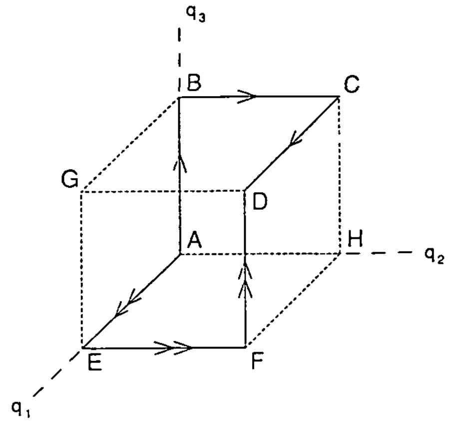

Figure 1. For a function $f = f[q_1(k), q_2(k), q_3(k)]$ the six paths along the edges of the cube from vertex A (for k) to D ($k+1$) lead to possible formulae for the partial difference operator $\Delta_r f$. The paths $ABCD$ and $AEFD$ correspond to expressions (2.7) and (2.8) with $n = 3$.

or simply

$$\Delta f = \sum_{r=1}^{n} \Delta_r f(k) \tag{2.5}$$

and the analogue of (2.2) is

$$\Delta_r f(k) = 0, \qquad f \text{ independent of } q_r. \tag{2.6}$$

In summary, we seek an expression for the partial difference $\Delta_r f(k)$ which satisfies (2.5) and (2.6).

It is easy to verify that the form chosen by Neuman and Tourassis [**3**], namely

$$\begin{aligned}\Delta_r f(k) = f[&q_1(k), \dots, q_{r-1}(k), q_r(k+1), q_{r+1}(k+1), \dots, q_n(k+1)]\\ &- f[q_1(k), \dots, q_{r-1}(k), q_r(k), q_{r+1}(k+1), \dots, q_n(k+1)]\end{aligned} \tag{2.7}$$

satisfies (2.5) and (2.6), and so also does that chosen by Gotusso [**1**], which is essentially the average of the right hand side of (2.7) and a similar term

$$\begin{aligned}&f[q_1(k+1), \dots, q_{r-1}(k+1), q_r(k+1), q_{r+1}(k), \dots, q_n(k)]\\ &\quad - f[q_1(k+1), \dots, q_{r-1}(k+1), q_r(k), q_{r+1}(k), \dots, q_n(k)].\end{aligned} \tag{2.8}$$

https://doi.org/10.1017/S0004972700026769 Published online by Cambridge University Press

The proof follows by summing (3.5) from $r = 1$ to $r = n$ giving, with the use of (2.5),

$$\sum_r F_r(k)\Delta q_r(k) = \sum_r [\Delta q_r(k)/\Delta t(k)]\Delta\left[\sum_j a_{rj}(k)\dot{q}_j(k)\right]$$

$$-\frac{1}{2}\sum_{i,j}[\Delta a_{ij}(k)]Q(\dot{q}_i,\dot{q}_j) \quad + \quad \Delta V(k). \tag{3.6}$$

Comparison with (3.3) shows that what is required is that

$$\begin{aligned}\Delta T &= \frac{1}{2}\Delta\sum_{i,j} a_{ij}(k)\dot{q}_i(k)\dot{q}_j(k)\\ &= \frac{1}{2}\sum_{i,j}\big[[\Delta q_i(k)/\Delta t(k)]\Delta[a_{ij}(k)\dot{q}_j(k)] + [\Delta q_j(k)/\Delta t(k)]\Delta[a_{ij}(k)\dot{q}_i(k)]\big]\\ &\quad -\frac{1}{2}\sum_{i,j}[\Delta a_{ij}(k)]Q(\dot{q}_i,\dot{q}_j)\end{aligned} \tag{3.7}$$

where use has been made of the symmetry $a_{ij} = a_{ji}$. Equating coefficients of $a_{ij}(k+1)$ and $a_{ij}(k)$ in turn gives

$$[\Delta q_i(k)/\Delta t(k)]\dot{q}_j(k+1) + [\Delta q_j(k)/\Delta t(k)]\dot{q}_i(k+1) - Q(\dot{q}_i,\dot{q}_j) = \dot{q}_i(k+1)\dot{q}_j(k+1) \tag{3.8}$$

$$[\Delta q_i(k)/\Delta t(k)]\dot{q}_j(k) + [\Delta q_j(k)/\Delta t(k)]\dot{q}_i(k) - Q(\dot{q}_i,\dot{q}_j) = \dot{q}_i(k)\dot{q}_j(k). \tag{3.9}$$

Putting $j = i$ and subtracting (3.9) from (3.8) gives

$$2[\Delta q_i(k)/\Delta t(k)][\dot{q}_i(k+1) - \dot{q}_i(k)] = \dot{q}_i(k+1)^2 - \dot{q}_i(k)^2 \tag{3.10}$$

from which the smoothing formula

$$\Delta q_i(k)/\Delta t(k) = [\dot{q}_i(k+1) + \dot{q}_i(k)]/2 \tag{3.11}$$

follows immediately The expression for Q is obtained by substituting (3.11) into (3.9) :

$$Q(\dot{q}_i,\dot{q}_j) = [\dot{q}_i(k)\dot{q}_j(k+1) + \dot{q}_i(k+1)\dot{q}_j(k)]/2. \tag{3.12}$$

Note that when Q is inserted into (3.5) it can be replaced by $\dot{q}_i(k)\dot{q}_j(k+1)$ because $a_{ij} = a_{ji}$.

https://doi.org/10.1017/S0004972700026769 Published online by Cambridge University Press

4. Conclusion

It has been shown that the general formula (2.11) for the partial difference operator Δ_r satisfies the requirements expressed by (2.5) and (2.6). With these requirements satisfied it is then shown that the discretisation of Lagrange's equations of motion naturally lead to the smoothing formula (3.1).

Recent work [4] has shown that the discretisations discussed in this paper are very effective in the modelling and planning of the intricate motion of robots.

References

[1] L. Gotusso, 'On the energy theorem for the Lagrange equations in the discrete case', *Appl. Math. and Comput.* **17** (1985), 120-136.

[2] D. Greenspan, *Discrete Models* (Addison–Wesley, Mass. U.S.A., 1973).

[3] C.P. Neumann and V.D. Tourassis, 'Discrete dynamic robot models', *IEEE Trans. Systems Man, Cybernet* **SMC-15** (1985), 193-204.

[4] H.H. Tan and R.B. Potts, 'Minimum time trajectory planner for the discrete dynamic robot model with dynamic constraints', (to be published), *IEEE J. of Robot. and Autom.*.

Applied Mathematics Department
The University of Adelaide
Adelaide, South Australia. 5001 Australia.