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SUBMANIFOLDS WITH NONPARALLEL FIRST NORMAL BUNDLE

MARCOS DAJCZER AND RUY TOJEIRO

ABSTRACT. We provide a complete local geometric description of submanifolds of spaces with constant sectional curvature where the first normal spaces, that is, the subspaces spanned by the second fundamental form, form a vector subbundle of the normal bundle of low rank which is nonparallel in the normal connection. We also characterize flat submanifolds with flat normal bundle in Euclidean space satisfying the helix property.

1. **Introduction.** Let $f: M^n \longrightarrow Q_c^m$ be an isometric immersion of an n-dimensional connected Riemannian manifold into a complete and simply connected manifold where the subscript indicates constant sectional curvature c. Assume that the first normal spaces N_1^f, that is, the subspaces spanned by the second fundamental form, form a rank-k vector subbundle of the normal bundle. It is a standard fact (see [Sp] or [Da$_2$]) that $f(M^n)$ reduces codimension to k, *i.e.*, it is contained in an $(n+k)$-dimensional totally geodesic submanifold Q_c^{n+k} of Q_c^m, if and only if N_1^f is parallel in the normal connection of f.

Based on recent developments on the characterization of isometric immersions obtained through compositions (see [D-T]), we are able to provide a complete geometric description of submanifolds for which N_1^f is nonparallel and has low rank. Our results strongly improve previous achievements in [G-M], [R-T] and more recently in [Da$_1$].

In the rank-one case, we show that only a rather surprisingly simple situation is possible.

THEOREM 1. *Let $f: M^n \longrightarrow Q_c^m$, $n \geq 2$, be an isometric immersion with a nonparallel first normal bundle of rank one. Then M^n has constant sectional curvature c and there exist isometric immersions $i: M_c^n \longrightarrow L_c^{m-1}$ and $F: L_c^{m-1} \longrightarrow Q_c^m$, where i is a totally geodesic inclusion and F has no totally geodesic points, such that $f = F \circ i$.*

In other words, f is nothing else but the restriction of F to a totally geodesic M_c^n of L_c^{m-1} transversal to the codimension-one relative nullity foliation of F. The proof of Theorem 1 will be a consequence of a more general result, namely Theorem 8, where we allow the rank of N_1^f to be arbitrary but assume instead that the normal bundle is flat.

For the case $k = 2$, we use Theorem 1 to improve Theorem 4 of [G-M] and Theorem 1 in [Da$_1$].

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THEOREM 2. *Let $f: M^n \longrightarrow Q_c^m$, $n \geq 3$, be an isometric immersion with a nonparallel first normal bundle of rank 2. Then M^n contains an open and dense subset $M' = U_1 \cup U_2$, where the U_j's are open and disjoint, satisfying:*

i) $f(U_1)$ is foliated by $(n-2)$-dimensional totally geodesic leaves of relative nullity,

ii) For each component V_λ^n of U_2, there exist isometric immersions $g_\lambda: V_\lambda^n \longrightarrow L_c^{n+1}$ and $h_\lambda: L_c^{n+1} \longrightarrow Q_c^m$, where h_λ has a nonparallel first normal bundle of rank one, such that $f|_{V_\lambda} = h_\lambda \circ g_\lambda$.

For the case $k = 3$, besides submanifolds supporting a relative nullity foliation of codimension 3 or produced by a composition with one of the examples in Theorems 1 or 2, a new type of example may occur. The immersion may be $(n-2)$-ruled, that is, foliated by totally geodesic submanifolds of the ambient space, which are not necessarily of relative nullity.

THEOREM 3. *Let $f: M^n \longrightarrow Q_c^m$, $n \geq 4$, be an isometric immersion with a nonparallel first normal bundle of rank 3. Then M^n contains an open and dense subset $M' = \bigcup_{j=1}^4 U_j$, where the U_j's are open and disjoint, satisfying:*

i) $f(U_1)$ is $(n-2)$-ruled,

ii) $f(U_2)$ is foliated by $(n-3)$-dimensional totally geodesic leaves of relative nullity,

iii) For each component V_λ^n of U_3, there exist isometric immersions $g_\lambda: V_\lambda^n \longrightarrow L_c^{n+2}$ and $h_\lambda: L_c^{n+2} \longrightarrow Q_c^m$, where h_λ has a nonparallel first normal bundle of rank one, such that $f|_{V_\lambda} = f_\lambda \circ g_\lambda$,

iv) For each component W_β^n of U_4, there exist a Riemannian manifold L^{n+1} and isometric immersions $g_\beta: W_\beta^n \longrightarrow L^{n+1}$ and $h_\beta: L^{n+1} \longrightarrow Q_c^m$, where h_β is of type i) in Theorem 2, such that $f|_{W_\beta} = h_\beta \circ g_\beta$.

Following [D-N], we say that an isometric immersion $f: U \subset \mathbf{R}^r \longrightarrow \mathbf{R}^m$ satisfies the *helix property* with respect to a subspace $W \subset \mathbf{R}^m$ if any straight line in U is mapped by f onto a helix in the direction of any $w \in W$. Recall also that an isometric immersion $f: M^n \longrightarrow \mathbf{R}^m$ is said to be *cylindrical* with respect to a subspace $\mathbf{R}^l \subset \mathbf{R}^m$ if there exist an orthogonal decomposition $\mathbf{R}^m = \mathbf{R}^{m-l} \oplus \mathbf{R}^l$ and a Riemannian manifold L^{n-l} such that M^n is isometric to an open subset of $L^{n-l} \times \mathbf{R}^l$ and $f = f_1 \times \mathrm{Id}$ restricted to M^n, where $f_1: L^{n-l} \longrightarrow \mathbf{R}^{m-l}$ is an isometric immersion.

Simple examples of submanifolds satisfying the helix property can be constructed using cylinders.

EXAMPLES 4. 1) Let $f: U \subset \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be an isometric immersion cylindrical with respect to a subspace $\mathbf{R}^l \subset \mathbf{R}^m$. Then f trivially satisfies the helix property with respect to $\mathbf{R}^l$.

2) Let $F: U \subset \mathbf{R}^{m-r} \longrightarrow \mathbf{R}^m$ be an isometric immersion which is cylindrical with respect to the subspace $\mathbf{R}^l \subset \mathbf{R}^m$ and let $i: V \subset \mathbf{R}^n \longrightarrow U$ be a totally geodesic inclusion. Then $f = F \circ i$ satisfies the helix property with respect to $\mathbf{R}^l$. Moreover, if $i_*(V)$ is transversal to $\mathbf{R}^l$, then f is noncylindrical with respect to any subspace of $\mathbf{R}^l$.

Our next result shows that any noncylindrical isometric immersion $f: V \subset \mathbf{R}^n \longrightarrow \mathbf{R}^m$ with flat normal bundle which satisfies the helix property with respect to $W^l \subset \mathbf{R}^m$ is locally as in the second of the above examples.

THEOREM 5. *Let $f: V \subset \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be a noncylindrical isometric immersion with flat normal bundle of constant rank r satisfying the helix property with respect to $W^l \subset \mathbf{R}^m$. Then $m \geq n + r + l$ and there exist a flat Riemannian manifold L_0^{m-r} and isometric immersions $i: V \longrightarrow L_0^{m-r}$ and $F: L_0^{m-r} \longrightarrow \mathbf{R}^m$, where i is totally geodesic and F is cylindrical with respect to W^l, such that $f = F \circ i$*

Theorems 1 and 5 have the following nice consequence for curves in Euclidean space.

COROLLARY 6. *Any unit speed curve in $\mathbf{R}^m$ with nowhere vanishing curvature is a geodesic of a flat hypersurface of $\mathbf{R}^m$. The curve is a helix in the direction $v \in \mathbf{R}^m$ if and only if the hypersurface is cylindrical with respect to* $\mathbf{R} = \operatorname{span}\{v\}$.

2. **The results.** Given an isometric immersion $f: M^n \longrightarrow Q_c^m$, we define the *first normal space* $N_1^f(x)$ at $x \in M^n$ as the subspace of the normal space $T_xM^\perp$ given by

$$N_1^f(x) = \operatorname{span}\{\alpha_f(X, Y) : \forall X, Y \in T_xM\},$$

where $\alpha_f: T_xM \times T_xM \longrightarrow T_xM^\perp$ denotes the second fundamental form of f. We say that the first normal bundle N_1^f is *nonparallel* if in a neighborhood of each point there exists a vector field $\eta \in N_1^f$ such that $\nabla_Z^\perp \eta \notin N_1^f$ for some $Z \in TM$.

The *relative nullity foliation* is formed by the totally geodesic integral submanifolds of the smooth distribution $x \in U \longmapsto \Delta_f(x)$, where

$$\Delta_f(x) = N\big(\alpha_f(x)\big) = \{X \in T_xM : \alpha_f(X, Y) = 0,\ \forall Y \in T_xM\},$$

and $U \subset M^n$ is any open subset where the *index of relative nullity* $\nu_f(x) = \dim \Delta_f(x)$ is constant. We refer to [Da$_2$] for basic facts on those definitions.

Now suppose that the normal bundle of $f: M^n \longrightarrow Q_c^m$ splits smoothly as the orthogonal direct sum of two vector subbundles

$$T_fM^\perp = D \oplus E.$$

Assume that the subspaces

$$\triangle_E(x) = N(\pi_E \circ \alpha_f) = \{X \in T_xM : \pi_E \circ \alpha_f(X, Y) = 0,\ \forall Y \in T_xM\}$$

form a subbundle of the tangent bundle of rank $k > 0$, and that E is parallel along $\triangle_E$ in the normal connection of f. At each point $x \in M^n$, consider the subspace $R(x) \subset \Omega(x) = \triangle_E(x)^\perp \oplus D(x)$, defined by

$$\begin{aligned} R(x) &= \operatorname{span}\{(\tilde\nabla_X e)_{T_xM \oplus D(x)} : \forall X \in T_xM, e \in E\} \\ &= \operatorname{span}\{(\tilde\nabla_X e)_{\triangle_E^\perp(x) \oplus D(x)} : \forall X \in \triangle_E^\perp(x), e \in E\}, \end{aligned}$$

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where $\tilde{\nabla}$ denotes the connection in Q_c^m. Clearly,

$$\dim R(x) \geq \dim \Delta_E^\perp(x) = n - k.$$

Assume further that $\dim R(x) = n - k$ everywhere, and let Γ be the smooth vector subbundle of rank d, where $d = \operatorname{rank} D$, whose fibre $\Gamma(x)$ is the orthogonal complement to $R(x)$ in $\Omega(x)$. We claim that

$$\Gamma(x) \cap T_xM = \{0\}.$$

Given $Z \in \Gamma(x) \cap T_xM$, we have that $Z \in \Delta_E^\perp(x)$ and that

$$0 = \langle Z, \tilde{\nabla}_X e\rangle = -\langle \alpha_f(Z, X), e\rangle$$

for all $X \in T_xM$ and all $e \in E$. This implies $Z = 0$ and proves the claim. From the claim, the map $H: \Gamma \longrightarrow Q_c^m$ defined by

$$H\big(\gamma(x)\big) = \exp_{f(x)} \gamma(x), \quad \gamma(x) \in \Gamma(x),$$

is an immersion if it is restricted to a neighborhood L^{n+d} of the 0-section of M. Consider L^{n+d} endowed with the metric induced by $h = H|_{L^{n+d}}$.

PROPOSITION 7. *In the above conditions there exist isometric immersions $g: M^n \longrightarrow L^{n+d}$ and $h: L^{n+d} \longrightarrow Q_c^m$ such that $f = h \circ g$, where the following holds up to parallel identifications in Q_c^m:*

i) $TL = TM \oplus D$, *thus* $T_hL^\perp = E$,

ii) $\alpha_g = \pi_D \circ \alpha_f$,

iii) $N(\alpha_h)\big(\gamma(x)\big) = \Delta_E(x) \oplus \Gamma(x)$.

PROOF. A straightforward computation. ∎

The proof of Theorem 1 in the Introduction will be an easy consequence of the following general result.

THEOREM 8. *Let $f: M_c^n \longrightarrow Q_c^m$ be an isometric immersion with flat normal bundle and first normal bundle of rank r. Then there exist isometric immersions $i: M_c^n \longrightarrow L_c^{m-r}$ and $F: L_c^{m-r} \longrightarrow Q_c^m$, where i is totally geodesic, such that $f = F \circ i$.*

PROOF. We claim that on any simply connected open subset of M^n there exist a smooth orthonormal tangent frame $X_1, \ldots, X_n$, a smooth orthonormal normal frame $\xi_1, \ldots, \xi_{m-n}$ and smooth positive functions $\lambda_1, \ldots, \lambda_r$, such that:

i) $\alpha(X_i, X_k) = \lambda_i \delta_{ik} \xi_i$, $1 \leq i \leq r$, $1 \leq k \leq n$,

ii) $\nabla^\perp_{X_i} \xi_j = \lambda_i X_i(1/\lambda_j)\xi_i$, $1 \leq i \neq j \leq r$,

iii) $\nabla^\perp_{X_s} \xi_\beta = 0$, $r + 1 \leq s \leq n$, $1 \leq \beta \leq m - n$,

iv) $\nabla^\perp_{X_i} \xi_\alpha = g_i^\alpha \xi_i$, $1 \leq i \leq r$, $r + 1 \leq \alpha \leq m - n$,

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where the functions $\{g_i^\alpha : 1 \le i \le r, r+1 \le \alpha \le m-n\}$ are smooth.

Part i) of the claim follows from Proposition 7 of [D-T]. By the Codazzi equations, for any $\eta \in N_1^{f\perp}$, we have that

$$\langle \nabla^\perp_{X_i}\eta, \xi_j\rangle = 0, \quad i \neq j.$$

A straightforward calculation proves that the normal connection of f induces a flat connection on $N_1^{f\perp}$. So, we choose the ξ_α's, $1 \le \alpha \le m-n$, to be parallel in this connection. Conditions ii), iii) and iv) of the claim follow from the Codazzi equations.

Now take $E = N_1^f$ and $D = N_1^{f\perp}$ in Proposition 7. We have that

$$R(x) = \operatorname{span}\Big\{\lambda_j X_j + \sum_\alpha g_j^\alpha \xi_\alpha, 1 \le j \le r\Big\}.$$

In particular, $\dim R(x) = r = n - \dim \triangle_E$, since $\dim \triangle_E = n - r$. The proof follows easily from Proposition 7 using the above equations. ∎

PROOF OF THEOREM 1. An elementary argument (*cf.* part i) of Lemma 9 below) using the Codazzi equation shows that rank $A_\delta = 1$ everywhere, where $N_1^f = \operatorname{span}\{\delta\}$ and $A_\delta : TM \longrightarrow TM$ denotes the tangent valued second fundamental form in direction δ. Therefore, M^n has constant sectional curvature c and the proof follows from Theorem 8. ∎

PROOF OF THEOREM 5. First notice that if $\gamma : I \longrightarrow U$ is a unit-speed straight line and $\alpha = f \circ \gamma$, then

$$\langle \alpha''(s), w\rangle = \langle \alpha_f\big(\gamma'(s), \gamma'(s)\big).$$

From the assumption, we have that

$$N_1^f(x) \subset W^{l\perp}, \quad \forall x \in V. \tag{1}$$

Suppose that $m < n + r + l$. We claim that the subspaces $L_x = W \cap T_xV \neq 0$ are parallel in $\mathbf{R}^m$ along any open connected subset $U \subset V$ where they have constant dimension. Let Z be a vector field along U such that $Z(x) \in L_x$ everywhere. Then $\tilde\nabla_X Z \in W$ for all $X \in TV$, where $\tilde\nabla$ denotes the connection in $\mathbf{R}^m$. It follows from (1) that $\tilde\nabla_X Z \in L_x$ for any $X \in T_xV$, which proves our claim. But this contradicts our assumption because it easily implies that f must be cylindrical. Hence, we have $m \ge n+r+l$ and, by Theorem 8, there exist a flat Riemannian manifold L_0^{m-r} and isometric immersions $i : V \longrightarrow L_0^{m-r}$ and $F : L_0^{m-r} \longrightarrow \mathbf{R}^m$, where i is totally geodesic, such that $f = F \circ i$. Moreover, the proof of Theorem 8 and part i) of Proposition 7 show that F has $E = N_1^f$ as its normal space. It follows from (1) that F is cylindrical with respect to W^l. ∎

PROOF OF COROLLARY 6. By Theorem 1, given a smooth unit speed curve $c : I \subset \mathbf{R} \longrightarrow \mathbf{R}^{n+1}$ with nowhere vanishing curvature, there exists an isometric immersion $F : U \subset \mathbf{R}^n \longrightarrow \mathbf{R}^{n+1}$ such that $F(s, 0, \ldots, 0) = c(s)$. This proves the first statement. The later follows immediately from Theorem 5. ∎

REMARK. If $e_1, \dots, e_{n+1}$ is a Frenet frame for $c(s)$, then the proof of Theorem 8 actually provides F explicitly:

$$F(s, t_1, \dots, t_{n-1}) = c(s) + t_1\big(k_2(s)e_1(s) + k_1(s)e_3(s)\big) + t_2e_4(s) + \cdots + t_{n-1}e_{n+1}(s)$$

where $k_1(s)$, $k_2(s)$ are the first two curvatures.

Following [Da₁], we say that an isometric immersion $f: M^n \longrightarrow Q_c^m$ is *regular* if the first normal spaces form a normal subbundle which splits as the orthogonal sum of two subbundles

$$N_1^f = T_f \oplus S_f$$

where at each point $x \in M^n$, we define

$$S_f(x) = \operatorname{span}\{\operatorname{Im}\phi_\eta : \forall \eta \in N_1^{f\perp}\}.$$

Here, $\phi_\eta: T_xM \longrightarrow N_1^f(x)$ is the linear map defined by

$$\phi_\eta(Y) = (\nabla_Y^\perp \eta)_{N_1^f},$$

where $(V)_W$ denotes taking the orthogonal projection of V onto W.

Set

$$\triangle_{S_f}(x) = N(\pi_{S_f} \circ \alpha_f)(x), \quad \nu^s(x) = \dim \triangle_{S_f}(x).$$

LEMMA 9. ([Da₁]) *We have:*

i) $\bigcap_{\delta \in \operatorname{Im}\phi_\eta} \ker A_\delta = \ker \phi_\eta$, *thus* $\triangle_{S_f} = \bigcap_{\eta \in N_1^{f\perp}} \ker \phi_\eta$,

ii) S *is parallel in* Q_c^m *along* $\triangle_{S_f}$,

iii) T *is parallel in* $TM^\perp$ *along* $\triangle_{S_f}$.

PROPOSITION 10. *Let* $f: M^n \longrightarrow Q_c^m$ *be a regular isometric immersion with* $\dim T = t$ *and* $\nu^s(x) = k$ *everywhere. At* $x \in M^n$, *consider the subspace*

$$R(x) = \operatorname{span}\{(\tilde\nabla_X \eta)_{\triangle_{S_f}^\perp \oplus T_f} : \forall X \in \triangle_{S_f}^\perp(x), \eta \in S_f\}.$$

i) If $\dim R(x) = n - k + t$ *everywhere, then* $f(M^n)$ *is k-ruled.*

ii) If $\dim R(x) = n - k$ *everywhere, then there exist a Riemannian manifold* L^{n+t} *and isometric immersions* $g: M^n \longrightarrow L^{n+t}$ *and* $h: L^{n+t} \longrightarrow Q_c^m$ *so that*

$$f = h \circ g,$$

where up to parallel identification, $N_1^h = S_f = S_h$, $\alpha_g = \pi_{T_f} \circ \alpha_f$ *and h has constant index of relative nullity* $\nu_h = k + t$.

PROOF. From the proof of Theorem 4 in [Da₁], we have for all $Y, Z \in \triangle_{S_f}$, $X \in TM$ and $\delta \in S_f$ that

$$\langle \tilde\nabla_Y Z, \delta \rangle = \langle \tilde\nabla_Y Z, \tilde\nabla_X \delta \rangle = 0. \tag{2}$$

Set $D = T_f$, $E = S_f \oplus N_1^{f\perp}$. Then $\triangle_E = \triangle_{S_f}$ and E is parallel along $\triangle_E$ in Q_c^m from i) and ii) of Lemma 9. We conclude from (2) that $\tilde\nabla_Y Z \in \Gamma$ for all $Y, Z \in \triangle_E$.

CASE i). We have $\tilde{\nabla}_Y Z = 0$ for all $Y, Z \in \triangle_E$.

CASE ii). Follows from Proposition 7. ∎

REMARK. It is easy to see from the Codazzi equations for A_η, $\eta \in N_1^{f\perp}$, that if f has flat normal bundle ($R^\perp = 0$), or at least $R^\perp|_{S_f} = 0$, then condition ii) is verified.

Our next result improves Theorem 7 in [Da$_1$].

THEOREM 11. *Let $f: M^n \longrightarrow Q_c^m$, $n \geq 3$, be a regular isometric immersion with $\dim S_f(x) = 1$ and $\dim T_f(x) = t$ everywhere. Then there exist isometric immersions $g: M^n \longrightarrow P_c^{n+t}$, $i: P_c^{n+t} \longrightarrow L_c^{m-1}$ and $h: L_c^{m-1} \longrightarrow Q_c^m$, where $\alpha_g = \pi_{T_f} \circ \alpha_f$ and i is totally geodesic, such that $f = h \circ i \circ g$.*

PROOF. From Lemma 9 part i), we have $\nu^s(x) = n-1$ everywhere. Thus $\dim R(x) = 1$. The proof follows from Proposition 10 and Theorem 1. ∎

LEMMA 12. *Let $f: M^n \longrightarrow Q_c^m$ be a regular isometric immersion. We have:*

i) If $\dim S_f = 2$, then $\nu^s(x) = n - 2$ everywhere,

ii) If $\dim S_f = 3$, then $\nu^s(x) \geq n - 3$.

PROOF. We argue for case ii). If there exists $\eta \in N_1^{f\perp}$ such that $\operatorname{Im}\phi_\eta = S_f$, then $\triangle_{S_f} = \ker\phi_\eta$ by i) of Lemma 9. Hence $\nu^s = n - 3$. Otherwise, it is easy to see that there exist $\eta_1, \eta_2 \in N_1^{f\perp}$ so that $S_f = \operatorname{span}\{\operatorname{Im}\phi_{\eta_1}, \operatorname{Im}\phi_{\eta_2}\}$ and each $\operatorname{Im}\phi_{\eta_j}$ is two-dimensional. In particular, $\triangle_{S_f} = \ker\phi_{\eta_1} \cap \ker\phi_{\eta_2}$ by i) of Lemma 9. Also from there $\ker\phi_{\eta_j} \subset \ker A_\delta$ where δ spans $\operatorname{Im}\phi_{\eta_1} \cap \operatorname{Im}\phi_{\eta_2}$. Since $A_\delta \neq 0$, we conclude that $\nu^s \geq n-3$, as we wished. ∎

Next we prove Theorem 3 in the Introduction, being the proof of Theorem 2 similar and easier.

PROOF OF THEOREM 3. Let $V \subset M^n$ be the open subset where the index of relative nullity satisfies $\nu_f(x) < n - 3$, and define U_2 as the open and dense subset of points of $M^n - V$ where $\nu_f(x)$ is locally constant. From Lemma 12, $\dim S_f(x) = 1$ or 2 for any $x \in V$. Let $W \subset V$ the open subset where $\dim S_f(x) = 2$ and set $U_3 = V - W$. By Theorem 11, the statement holds on U_3. Finally, let $U_1 \subset W$ be the open subset where $\dim R(x) = 3$ and define $U_4 = W - U_1$. The remaining of the proof follows from Proposition 10. ∎

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IMPA
Estrada Dona Castorina 110
22460-320 Rio de Janeiro
Brasil
e-mail: MARCOS@impa.br

Universidade Federal de Uberlândia
38400-020 Uberlândia
Brasil
e-mail: DEMAT04@brufu.bitnet