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THE CHINESE REMAINDER THEOREM AND THE INVARIANT BASIS PROPERTY

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ABSTRACT. The Chinese Remainder Theorem states that if I and J are comaximal ideals of a ring R, then $A/(I\cap J)A$ is isomorphic to $A/IA\times A/JA$ for any left R-module A. In this paper we study the converse; when does $A/(I\cap J)A$ and $A/IA\times A/JA$ isomorphic imply that I and J are comaximal?

One of the most useful tools in ring theory is the Chinese Remainder Theorem (CRT): if I and J are ideals of a ring R (with 1) which are comaximal $(I+J=R)$, then the natural homomorphism $R\to R/I\times R/J$ induces an isomorphism f: $R/(I\cap J)\to R/I\times R/J$. f is both a ring and R-module isomorphism. More generally, if A is any left R-module, the natural homomorphism $A/(I\cap J)A\to A/IA\times A/JA$ is an isomorphism. We remark that CRT fails if I and J are only assumed to be comaximal left ideals.

A natural question arises: if $A/(I\cap J)A$ and $A/IA\times A/JA$ are isomorphic (not necessarily by the natural homomorphism), does $I+J=R$? We say that a R-module A satisfies CC1 if whenever $A/(I\cap J)A$ and $A/IA\times A/JA$ are isomorphic, then $I+J=R$. A module need not satisfy CC1; for example, if F is a free R-module of infinite rank, then $F\approx F\times F$, so CC1 fails for F with $I=J=0$. Also, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ does not satisfy CC1 with $I=J=3\mathbb{Z}$.

We first consider the case when R is commutative. Recall that a R-module A is locally finitely generated if A_M is a finitely generated R_M-module for all maximal ideals M of R. $J(R)$ will denote the Jacobson radical of R.

PROPOSITION 1. *Let R be a commutative ring and A a R-module.*

(1) *If A satisfies* CC1, *then A/MA is a finitely generated R-module for all maximal ideals M.*

(2) *Assume that A is locally finitely generated, then A satisfies* CC1 *iff $A_M\neq 0$ for all maximal ideals M.*

(3) *If A is locally finitely generated, then A satisfies* CC1 *implies $ann(A)\subset J(R)$. If A is finitely generated, then A satisfies* CC1 *iff $ann(A)\subset J(R)$.*

Proof. (1) If some $V=A/MA$ is not finitely generated, then V is an infinite dimensional vector space over $k=R/M$. Thus $V\approx V\times V$ as k-modules, and hence as R-modules. Thus CC1 fails for A with $I=J=M$.

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(2) Suppose there is an isomorphism $f: A/(I\cap J)A \to A/IA \times A/JA$ with $I+J\neq R$; then $I+J$ is contained in some maximal ideal M. Let $N = A_M/M_MA_M$, then f induces an isomorphism $\bar{f}: N\to N\times N$. Since N is a finitely generated R_M/M_M vector space, necessarily $N=0$. But thus $A_M=0$ by Nakayama's Lemma. Conversely, if some $A_M=0$, then $A/MA=0$, so $A/MA \approx A/MA\times A/MA$. Thus CC1 fails for A with $I=J=M$.

(3) This follows from (2) because $A_M\neq 0$ implies $\text{ann}(A)\subset M$. If A is finitely generated then $A_M\neq 0$ iff $\text{ann}(A)\subset M$. ■

(3) shows that the converse of (1) need not hold. Let P be the set of prime numbers, then $A=\mathbb{Q}\oplus\sum_{p\in P}\mathbb{Z}/p\mathbb{Z}$ is not locally finitely generated, but A satisfies CC1. Over a local ring any finitely generated module satisfies CC1. Any free R-module of finite rank satisfies CC1. Let Q be the set of odd prime numbers, then $A=\sum_{q\in Q}\mathbb{Z}/q\mathbb{Z}$ is locally finitely generated, has $\text{ann}(A)=0$, but does not satisfy CC1. Hence the converse of the first part of (3) does not hold.

A related question is: which rings R satisfy CC1 for all finitely generated free R-modules? Thus we say that a ring R satisfies CC2 if all finitely generated free left R-modules satisfy CC1. Proposition 1 shows that any commutative ring satisfies CC2.

We recall that a ring R satisfies the invariant basis property or invariant basis number (IBN) if $R^m\approx R^n$ implies $m=n$. Rings which satisfy IBN include commutative rings, division rings, and (left) noetherian rings. Let k be a field and V an infinite dimensional k vector space, then $R=\text{Hom}_k(V, V)$ does not satisfy IBN. An excellent reference on the invariant basis property is [1].

PROPOSITION 2. *A ring R satisfies* CC2 *iff every homomorphic image of R satisfies IBN.*

Proof. Suppose that some $\bar{R}=R/L$ does not satisfy IBN; then $\bar{R}^m\approx\bar{R}^n$ for some $m<n$. Choose $i, j\geq 0$ so that $i(n-m)=m+j$, then $\bar{R}^{m+i}\approx\bar{R}^{m+i(n-m)+j}=\bar{R}^{2(m+j)}$. Let $l=m+j$, then $\bar{R}^l\approx\bar{R}^l\times\bar{R}^l$; so CCl fails for R^l with $I=J=L$.

Conversely, suppose CC2 fails. Then there are ideals I and J with $I+J\neq R$ and a finitely generated free R-module F such that $F/(I\cap J)F$ and $F/IF\times F/JF$ are isomorphic, by say f. Let $L=I+J$, then f induces an isomorphism $\bar{f}$: $F/LF\to F/LF\times F/LF$. Thus $\bar{R}=R/L$ does not satisfy IBN. ■

Thus any ring which satisfies CC2 also satisfies IBN. However, the converse is not true. For there exists a ring R which satisfies IBN, but not all of its homomorphic images satisfy IBN [1, p. 221]. Thus the class of rings which satisfy CC2 lies strictly between the class of commutative rings and the class of rings which satisfy IBN.

REFERENCES

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