

## Chebyshev Sets

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### Abstract

A Chebyshev set is a subset of a normed linear space that admits unique best approximations. In the first part of this paper we present some basic results concerning Chebyshev sets. In particular, we investigate properties of the metric projection map, sufficient conditions for a subset of a normed linear space to be a Chebyshev set, and sufficient conditions for a Chebyshev set to be convex. In the second half of the paper we present a construction of a nonconvex Chebyshev subset of an inner product space.

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### 1. Introduction

Given a normed linear space  $(X, \|\cdot\|)$ , we shall say that a subset of  $X$  is a Chebyshev set if every point outside of it admits a unique nearest point. In this paper we will review some of the literature, and a little, a very little, of the history of the question of whether every Chebyshev set is convex (for a brief summary of the history of this problem, see [25, page 307]). The volume of research invested in this topic is vast, and we will not even pretend to present even a small amount of what is known. Instead, we will be content to present some of the most basic results concerning Chebyshev sets (see Section 2) and then present what we believe are among some of the most interesting partial solutions to the following question, which henceforth will be referred to as the ‘Chebyshev set problem’:

‘Is every Chebyshev set in a Hilbert space convex?’

In the latter part of this paper we will present, in full gory detail, an example, due to Balaganskiĭ and Vlasov [7], of a nonconvex Chebyshev set in an infinite-dimensional inner product space. While this result appears tantalisingly close to a solution, in the negative, of the Chebyshev set problem, it is not at all clear how to transfer this

construction of Balaganskiĭ and Vlasov (which is based upon an earlier construction of Johnson [46], later corrected by Jiang [45]) into the realm of Hilbert spaces.

So at the time of writing this paper, the problem of whether every Chebyshev set in a Hilbert space is convex is still wide open. We hope that the gentle introduction presented here might serve to inspire some of the next generation of mathematicians to tackle, and perhaps even solve, this long-standing open problem.

This paper is of course not the first survey article ever written on the Chebyshev set problem. Prior to this there have been many such articles, among the most prominent of these are the survey articles of [5, 7, 11, 24, 59, 79, 87], each of which had something different to offer.

In the present paper, we believe that our point of difference from the aforementioned articles is that our approach is more naive and elementary and hence, hopefully, more accessible to students and researchers entering this area of research.

**REMARK 1.1.** Throughout this paper all normed linear spaces/Hilbert spaces will be assumed to be nontrivial, that is, contain more than just the zero vector, and be over the field of real numbers.

The structure of the remainder of this paper is as follows. Section 2 contains preliminary results, which comprise basic results (Section 2.1), strict convexity (Section 2.2) and continuity of the metric projection (Section 2.3). Section 3, presents four distinct proofs of the convexity of Chebyshev sets under various hypotheses: proof by fixed point theorem (Section 3.1), proof using inversion in the unit sphere (Section 3.2), a proof using convex analysis (Section 3.3) and Vlasov's theorem (Section 3.4). The final section, Section 4, is concerned with a nonconvex Chebyshev set in an inner product space. The paper concludes with five appendices which contain the proofs of several technical results that are called upon within the main text of the paper and an index of notation and assumed knowledge.

## 2. Preliminary results

**2.1. Basic facts.** We begin this section by formally defining how we measure the distance of a point to a subset of a normed linear space. From this we define the metric projection mapping, which in turn is used to formulate the concepts of proximal sets and Chebyshev sets. A few simple examples of such sets are then given to familiarise the reader with these objects.

**DEFINITION 2.1.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $K$  be a nonempty subset of  $X$ . For any point  $x \in X$  we define  $d(x, K) := \inf_{y \in K} \|x - y\|$  and call this the distance from  $x$  to  $K$ . We will also refer to the map  $x \mapsto d(x, K)$  as the distance function for  $K$ .

The following simple result regarding the distance function will be used repeatedly throughout the remainder of the paper.

**PROPOSITION 2.2.** *Let  $K$  be a nonempty subset of a normed linear space  $(X, \|\cdot\|)$ . Then the distance function for  $K$  is nonexpansive (and hence continuous).*

**PROOF.** Let  $x, y \in X$  and  $k \in K$ . By the triangle inequality and the definition of the distance function, we get that  $d(x, K) \leq \|x - k\| \leq \|x - y\| + \|y - k\|$ . By rearranging this equation, we obtain  $d(x, K) - \|x - y\| \leq \|y - k\|$ . Since  $k \in K$  was arbitrary and the left-hand side of the previous expression is independent of  $k$ , we see that  $d(x, K) - \|x - y\| \leq d(y, K)$ . By symmetry,  $d(y, K) - \|x - y\| \leq d(x, K)$ . Thus,  $-\|x - y\| \leq d(x, K) - d(y, K) \leq \|x - y\|$ , and so  $|d(x, K) - d(y, K)| \leq \|x - y\|$ . Therefore, we have the nonexpansive property.  $\square$

The fundamental concept behind the definition of a proximal set, or a Chebyshev set, is that of ‘nearest points’. The following definition makes this idea precise.

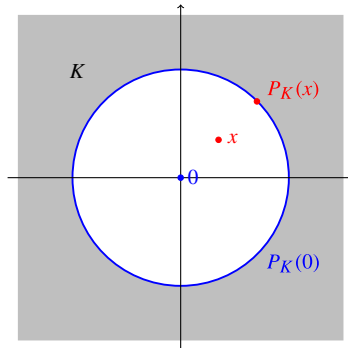
**DEFINITION 2.3.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $K$  be a subset of  $X$ . We define a set-valued mapping  $P_K : X \rightarrow \mathcal{P}(K)$  by  $P_K(x) := \{y \in K : \|x - y\| = d(x, K)\}$  if  $K$  is nonempty, and by  $P_K(x) = \emptyset$  if  $K$  is the empty set. We refer to the elements of  $P_K(x)$  as the best approximations of  $x$  in  $K$  (or the nearest points to  $x$  in  $K$ ). We say that  $K$  is a proximal set if  $P_K(x)$  is nonempty for each  $x \in X$  and that  $K$  is a Chebyshev set if  $P_K(x)$  is a singleton for each  $x \in X$ . For a Chebyshev set we define the map  $p_K : X \rightarrow K$  as the map that assigns to each  $x \in X$  the unique element of  $P_K(x)$ . We will refer to both  $P_K$  and  $p_K$  as the metric projection mapping (for  $K$ ).

*Note.* It should be immediately clear from the definition that every nonempty compact set in a normed linear space is proximal. Moreover,  $P_K(x) = \{x\}$  for any  $x \in K$  and

$$P_K(X \setminus K) \subseteq \text{Bd}(K) = \text{Bd}(X \setminus K).$$

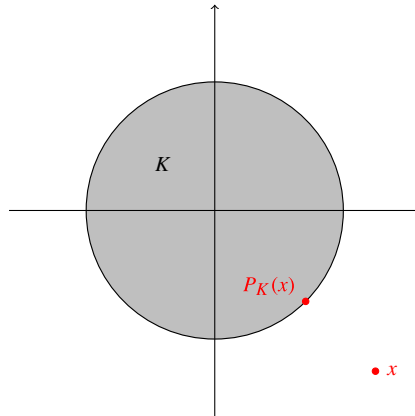
To familiarise ourselves with these notions we now present a few examples.

**EXAMPLE 2.4.** Consider  $K := \mathbb{R}^2 \setminus B(0; 1) \subseteq \mathbb{R}^2$  equipped with the Euclidean norm.



It is easy to check that for any  $x \in B(0; 1) \setminus \{0\}$ ,  $P_K(x) = \{x/\|x\|\}$ , whilst  $P_K(0) = S_{\mathbb{R}^2}$ . Hence,  $K$  is proximal, but not a Chebyshev set.

**EXAMPLE 2.5.** Consider  $K := B[0; 1] \subseteq \mathbb{R}^2$  equipped with the Euclidean norm.



It is straightforward to check that for any  $x \in \mathbb{R}^2 \setminus K$ ,  $P_K(x) = \{x/\|x\|\}$ . Therefore,  $K$  is a Chebyshev set.

**EXAMPLE 2.6.** Let  $n \in \mathbb{N}$ . Consider  $K := B(0; 1) \subseteq \mathbb{R}^n$  equipped with the Euclidean norm. Choose  $x \in S_{\mathbb{R}^n}$ . Clearly,  $d(x, K) = 0$ , but since  $x \notin K$ ,  $P_K(x) = \emptyset$ , and so  $K$  is not proximal.

In general it is difficult to deduce any structure concerning the behaviour of the metric projection mapping. However, one exception to this is when the set  $K$  is a subspace.

**LEMMA 2.7.** Let  $K$  be a subspace of a normed linear space  $(X, \|\cdot\|)$ . Then for each  $x \in X$ ,  $k \in K$  and  $\lambda \in \mathbb{R}$ ,  $P_K(\lambda x + k) = \lambda P_K(x) + k$ . Note: if  $P_K(x) = \emptyset$ , then  $\lambda P_K(x) + k = \emptyset$ .

**PROOF.** Let  $x \in X$ ,  $k \in K$  and  $\lambda \in \mathbb{R}$ . Firstly, observe that

$$\begin{aligned} d(\lambda x + k, K) &= \inf_{y \in K} \|(\lambda x + k) - y\| \\ &= |\lambda| \inf_{z \in K} \|x - z\| \quad \text{since } K \text{ is a subspace} \\ &= |\lambda| d(x, K). \end{aligned}$$

If  $\lambda = 0$ , then the result holds trivially. So suppose that  $\lambda \neq 0$ . Therefore,

$$\begin{aligned} y \in P_K(\lambda x + k) &\Leftrightarrow \|y - (\lambda x + k)\| = d(\lambda x + k, K) \\ &\Leftrightarrow |\lambda| \left\| \left( \frac{y - k}{\lambda} \right) - x \right\| = |\lambda| d(x, K) \\ &\Leftrightarrow \left\| \left( \frac{y - k}{\lambda} \right) - x \right\| = d(x, K) \\ &\Leftrightarrow \frac{y - k}{\lambda} \in P_K(x) \\ &\Leftrightarrow y \in \lambda P_K(x) + k. \end{aligned}$$

□

**EXAMPLE 2.8 [3].** Let  $(X, \|\cdot\|)$  be a normed linear space and  $x^* \in X^* \setminus \{0\}$ . Then  $\ker(x^*)$  is a proximal set if and only if  $x^*$  attains its norm (that is,  $\|x^*\| = |x^*(x)|$  for some  $x \in S_X$ ).

**PROOF.** Suppose that  $M := \ker(x^*)$  is a proximal set. Then there exists an  $x \in X$  such that  $d(x, M) = \|x\| = 1$ . To see this, choose  $z \in X \setminus M$ . This is possible since  $x^* \neq 0$ . Since  $M$  is proximal, we can find  $m \in M$  such that  $\|z - m\| = d(z, M)$ . Let

$$x := \frac{z - m}{d(z, M)} = \frac{z - m}{\|z - m\|}.$$

Clearly,  $\|x\| = 1$  and from Lemma 2.7, it follows that

$$d(x, M) = d\left(\frac{z - m}{d(z, M)}, M\right) = d\left(\frac{z}{d(z, M)} + \frac{-m}{d(z, M)}, M\right) = \frac{d(z, M)}{d(z, M)} = 1.$$

Let  $y \in X$ . Then  $y = (x^*(y)/x^*(x))x + m$  for some  $m \in M$ . Now,

$$\begin{aligned} \frac{|x^*(y)|}{|x^*(x)|} &= \frac{|x^*(y)|}{|x^*(x)|} d(x, M) \\ &= \frac{|x^*(y)|}{|x^*(x)|} \inf_{z \in M} \|x - z\| \\ &= \inf_{z \in M} \left\| \frac{x^*(y)}{x^*(x)}x + m - z \right\| \quad \text{since } M \text{ is a subspace and } m \in M \\ &= \inf_{z \in M} \|y - z\| \\ &= d(y, M) \\ &\leq \|y\| \quad \text{since } 0 \in M. \end{aligned}$$

Thus,  $|x^*(y)| \leq |x^*(x)| \|y\|$  for all  $y \in X$ . Hence,  $\|x^*\| \leq |x^*(x)| \leq \|x^*\|$  since  $\|x\| = 1$ , and so  $\|x^*\| = |x^*(x)|$ .

Conversely, suppose that  $x^* \in X^* \setminus \{0\}$  attains its norm at  $x \in S_X$ , that is,  $\|x^*\| = x^*(x)$ . Let  $M := \ker(x^*)$  and  $m \in M$ . Then

$$1 = \frac{\|x^*\|}{\|x^*\|} = \frac{x^*(x)}{\|x^*\|} = \frac{x^*(x - m)}{\|x^*\|} \leq \frac{\|x^*\| \|x - m\|}{\|x^*\|} = \|x - m\|.$$

Therefore,  $1 \leq d(x, M) \leq \|x - 0\| = \|x\| = 1$ . Thus,  $0 \in P_M(x)$ . Now let  $y \in X$ . We can write  $y = \lambda x + m$  for some  $\lambda \in \mathbb{R}$  and  $m \in M$ . By Lemma 2.7, it follows that  $P_M(y) = P_M(\lambda x + m) = \lambda P_M(x) + m \neq \emptyset$ . Hence,  $M$  is proximal.  $\square$

We are particularly interested in finding necessary and sufficient conditions for a set to be a proximal set or a Chebyshev set. Our first necessary condition, as suggested by Example 2.6, is that a proximal set must be closed.

**PROPOSITION 2.9.** *Let  $K$  be a proximal set in a normed linear space  $(X, \|\cdot\|)$ . Then  $K$  is nonempty and closed.*

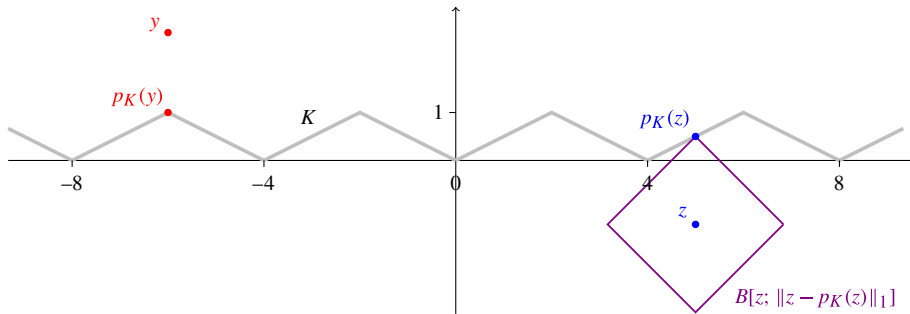
**PROOF.** To see that  $K$  is nonempty we simply note that  $\emptyset \neq P_K(0) \subseteq K$ . To show that  $K$  is closed it is sufficient to show that  $\overline{K} \subseteq K$ . To this end, let  $x \in \overline{K}$ . Since  $K$  is proximal, there exists  $k \in K$  such that  $\|x - k\| = d(x, K) = 0$ . Therefore,  $x = k \in K$ .  $\square$

Of particular importance in the study of Chebyshev sets is the notion of convexity.

**DEFINITION 2.10.** Let  $X$  be a vector space and  $C$  be a subset of  $X$ . We shall say that  $C$  is convex if, for any  $a, b \in C$  and any  $\lambda \in [0, 1]$ , we have  $[a, b] \in C$ . We say that  $C$  is concave if  $X \setminus C$  is convex. Furthermore, we say that  $C$  is midpoint convex if for any  $a, b \in C$ ,  $(a + b)/2 \in C$ .

Motivated by Example 2.4, one might be tempted to conjecture that all Chebyshev sets are convex. However, the following example demonstrates that this is not the case.

**EXAMPLE 2.11.** Define the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) := \frac{1}{2}d(x, 4\mathbb{Z})$  for all  $x \in \mathbb{R}$ .



Then  $K := \text{Graph}(f)$ , viewed as a subset of  $(\mathbb{R}^2, \|\cdot\|_1)$ , is a (nonconvex) Chebyshev set. Indeed, for any  $z := (a, b) \in \mathbb{R}^2$ ,  $p_K(z) = (a, f(a))$ .

The following result, in combination with Proposition 2.9, shows that when considering the convexity of a proximal set (or a Chebyshev set) we need only check that it is midpoint convex.

**PROPOSITION 2.12.** A closed subset of a normed linear space  $(X, \|\cdot\|)$  is convex if and only if it is midpoint convex.

**PROOF.** Clearly, a convex set is midpoint convex, so suppose that  $C \subseteq X$  is a closed midpoint convex set. Let  $x, y \in C$ . Consider the mapping  $T : [0, 1] \rightarrow X$  defined by  $T(\lambda) := \lambda x + (1 - \lambda)y$ . Let  $U := \{\lambda \in [0, 1] : T(\lambda) \notin C\}$ . Since  $T$  is continuous and  $C$  is closed, it follows that  $U$  is open. If  $U = \emptyset$ , then  $[x, y] \subseteq C$  and we are done, so suppose otherwise. Thus, there exist distinct  $\lambda_1, \lambda_2 \in [0, 1] \setminus U$  such that  $(\lambda_1, \lambda_2) \subseteq U$ . Therefore,  $(\lambda_1 + \lambda_2)/2 \in (\lambda_1, \lambda_2) \subseteq U$ , and so

$$\frac{T(\lambda_1) + T(\lambda_2)}{2} = T\left(\frac{\lambda_1 + \lambda_2}{2}\right) \notin C;$$

but this is impossible since  $T(\lambda_1), T(\lambda_2) \in C$  and  $C$  is midpoint convex. Hence, it must be the case that  $U = \emptyset$ , which implies that  $C$  is convex.  $\square$

**REMARK 2.13.** Having closedness in the hypotheses of Proposition 2.12 is essential as  $\mathbb{Q} \subseteq \mathbb{R}$  is midpoint convex, but not convex.

There is also a corresponding notion of a function being convex and/or concave:

**DEFINITION 2.14.** Let  $K$  be a nonempty convex subset of a vector space  $X$ . We say that a function  $f : K \rightarrow \mathbb{R}$  is convex if  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$  for all  $x, y \in K$  and all  $\lambda \in [0, 1]$ , and concave if  $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$  for all  $x, y \in K$  and all  $\lambda \in [0, 1]$ .

The following results show that the convexity (concavity) of a set is closely related to the convexity (concavity) of its distance function.

**PROPOSITION 2.15.** *Let  $K$  be a nonempty closed subset of a normed linear space  $(X, \|\cdot\|)$ . Then  $K$  is convex if and only if the distance function for  $K$  is convex.*

**PROOF.** Firstly, suppose that  $K$  is convex. Fix  $x, y \in X$ ,  $\lambda \in [0, 1]$  and  $\varepsilon > 0$ . By the definition of the distance function for  $K$ , there exist  $k_x, k_y \in K$  such that

$$\|x - k_x\| < d(x, K) + \varepsilon \quad \text{and} \quad \|y - k_y\| < d(y, K) + \varepsilon.$$

Since  $K$  is convex, it follows that

$$\begin{aligned} d(\lambda x + (1 - \lambda)y, K) &\leq \|\lambda x + (1 - \lambda)y - (\lambda k_x + (1 - \lambda)k_y)\| \\ &= \|\lambda(x - k_x) + (1 - \lambda)(y - k_y)\| \\ &\leq \lambda\|x - k_x\| + (1 - \lambda)\|y - k_y\| \\ &< \lambda d(x, K) + (1 - \lambda)d(y, K) + \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, we conclude that the distance function for  $K$  is convex.

Conversely, suppose that the distance function for  $K$  is convex. Let  $x, y \in K$  and  $\lambda \in [0, 1]$ . Then

$$0 \leq d(\lambda x + (1 - \lambda)y, K) \leq \lambda d(x, K) + (1 - \lambda)d(y, K) = 0.$$

As  $K$  is closed, this forces  $\lambda x + (1 - \lambda)y \in K$  and so  $K$  is convex. □

In order to expedite the proof of the next result, let us recall the following easily provable fact. If  $(X, \|\cdot\|)$  is a normed linear space,  $x, y \in X$ ,  $0 < r, R$  and  $\lambda \in [0, 1]$ , then

$$B(\lambda x + (1 - \lambda)y; \lambda r + (1 - \lambda)R) = \lambda B(x; r) + (1 - \lambda)B(y; R).$$

**PROPOSITION 2.16 [12, Lemma 3.1].** *Let  $K$  be a nonempty proper closed subset of a normed linear space  $(X, \|\cdot\|)$ . Then  $K$  is concave if and only if the distance function for  $K$ , restricted to  $\text{co}(X \setminus K)$ , is concave.*

**PROOF.** Suppose that  $K$  is concave. Then  $\text{co}(X \setminus K) = X \setminus K$ . Let  $x, y \in X \setminus K$  and  $\lambda \in [0, 1]$ . Clearly,

$$B(x; d(x, K)), \quad B(y; d(y, K)) \subseteq X \setminus K.$$

Since  $X \setminus K$  is convex, it follows that

$$B(\lambda x + (1 - \lambda)y; \lambda d(x, K) + (1 - \lambda)d(y, K)) = \lambda B(x; d(x, K)) + (1 - \lambda)B(y; d(y, K)) \subseteq X \setminus K.$$

Therefore,  $\lambda d(x, K) + (1 - \lambda)d(y, K) \leq d(\lambda x + (1 - \lambda)y, K)$ . Conversely, suppose that the distance function for  $K$ , restricted to  $\text{co}(X \setminus K)$ , is concave. Let  $x, y \in X \setminus K$  and  $\lambda \in [0, 1]$ . Then, since  $K$  is closed,  $d(\lambda x + (1 - \lambda)y, K) \geq \lambda d(x, K) + (1 - \lambda)d(y, K) > 0$ . Thus,  $\lambda x + (1 - \lambda)y \in X \setminus K$  as required.  $\square$

The next result is useful when working with nearest points. It will be used repeatedly throughout the remainder of the paper.

**PROPOSITION 2.17** [25, Lemma 12.1]. *Let  $K$  be a nonempty subset of a normed linear space  $(X, \|\cdot\|)$ . Suppose that  $x \in X \setminus K$  and  $z \in P_K(x)$ . Then, for any  $y \in [x, z)$ , we have  $z \in P_K(y)$ .*

**PROOF.** Since  $y \in [x, z)$ ,  $\|x - y\| + \|y - z\| = \|x - z\|$ . Now,  $z \in K \cap B[y, \|y - z\|]$  and so  $d(y, K) \leq \|y - z\|$ . On the other hand, by the triangle inequality,

$$B(y, \|y - z\|) \subseteq B(x, \|x - y\| + \|y - z\|) = B(x, \|x - z\|) \subseteq X \setminus K$$

and so  $\|y - z\| \leq d(y, K)$ . Thus,  $\|y - z\| = d(y, K)$ , which shows that  $z \in P_K(y)$ .  $\square$

It is perhaps natural to speculate that the metric projection mapping of a Chebyshev set is always continuous. This possibility is refuted by Example 2.54. However, the following weaker property is possessed by all metric projection mappings.

**LEMMA 2.18.** *Let  $K$  be a nonempty closed subset of a normed linear space  $(X, \|\cdot\|)$ . If  $(x_n, y_n)_{n=1}^\infty$  is a sequence in  $X \times X$ , with  $y_n \in P_K(x_n)$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} (x_n, y_n) = (x, y)$ , then  $y \in P_K(x)$ .*

**PROOF.** Let  $(x_n, y_n)_{n=1}^\infty$  be a sequence in  $X \times X$  converging to some  $(x, y)$ , with  $y_n \in P_K(x_n)$  for all  $n \in \mathbb{N}$ . Firstly,  $y \in K$ , since  $K$  is closed. Furthermore,

$$\|x - y\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} d(x_n, K) = d(x, K),$$

since the distance function for  $K$  is continuous by Proposition 2.2. Thus,  $y \in P_K(x)$ .  $\square$

**COROLLARY 2.19.** *If  $K$  is a Chebyshev set in a normed linear space  $(X, \|\cdot\|)$ , then the metric projection mapping for  $K$  has a closed graph.*

The following curious result can be, and will be, used to establish the convexity of Chebyshev sets in certain normed linear spaces.

**PROPOSITION 2.20.** *Let  $K$  be a nonempty closed subset of a normed linear space  $(X, \|\cdot\|)$ . Suppose that  $x \in X \setminus K$  and  $z \in P_K(x)$ . Define  $y(\lambda) := z + \lambda(x - z)$  for each  $\lambda \in \mathbb{R}$ . Then  $I := \{\lambda \geq 0 : z \in P_K(y(\lambda))\}$  is a nonempty closed interval.*



**PROOF.** By Proposition 2.17, if  $\alpha \in I$  and  $\beta \in [0, \alpha]$ , then  $\beta \in I$ , which establishes that  $I$  is an interval. Obviously,  $1 \in I$ , and so  $I$  is nonempty. Finally, suppose that  $(\lambda_n)_{n=1}^\infty$  is a sequence in  $I$  converging to some  $\lambda \geq 0$ . Clearly,  $\lim_{n \rightarrow \infty} y(\lambda_n) = y(\lambda)$ . Since  $z \in P_K(y(\lambda_n))$  for all  $n \in \mathbb{N}$ , Lemma 2.18 says that  $z \in P_K(y(\lambda))$ . Therefore,  $\lambda \in I$ , and so  $I$  is closed.  $\square$

The next definition and lemma concern a boundedness property possessed by all metric projection mappings.

**DEFINITION 2.21.** Let  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|')$  be normed linear spaces and  $\Phi : X \rightarrow \mathcal{P}(Y)$  a function. We say that  $\Phi$  is locally bounded on  $X$  if for every  $x_0 \in X$  there exist  $r, M > 0$  such that  $\Phi(B(x_0; r)) \subseteq MB_Y$ .

**LEMMA 2.22.** Let  $K$  be a nonempty subset of a normed linear space  $(X, \|\cdot\|)$ . Then the metric projection mapping  $x \mapsto P_K(x)$  is locally bounded on  $X$ .

**PROOF.** Fix  $x_0 \in X$  and let  $r := 1$  and  $M := d(x_0, K) + \|x_0\| + 2$ . Suppose that  $y \in P_K(B(x_0; r))$ , that is,  $y \in P_K(x)$  for some  $x \in B(x_0; 1)$ . Using the triangle inequality (twice) and the fact that the distance function for  $K$  is nonexpansive, we have that

$$\begin{aligned} \|y\| &= \|y - 0\| \leq \|y - x_0\| + \|x_0 - 0\| \\ &\leq \|y - x\| + \|x - x_0\| + \|x_0 - 0\| \\ &= d(x, K) + \|x - x_0\| + \|x_0\| \\ &\leq (d(x_0, K) + \|x - x_0\|) + \|x - x_0\| + \|x_0\| \\ &< d(x_0, K) + 2 + \|x_0\| \\ &= M. \end{aligned}$$

Hence,  $P_K(B(x_0; r)) \subseteq MB_Y$ , and so the metric projection mapping for  $K$  is locally bounded on  $X$ .  $\square$

We now proceed to establish some more properties of the metric projection mapping. After defining what it means for a multivalued map to be continuous, we derive sufficient conditions for the continuity of the metric projection mapping. As an immediate corollary, we deduce a fundamental result—in finite-dimensional spaces the metric projection mapping for a Chebyshev set is continuous.

**DEFINITION 2.23.** Let  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|')$  be normed linear spaces and  $\Phi : X \rightarrow \mathcal{P}(Y)$  a function. We say that  $\Phi$  is continuous at  $x \in X$  if  $\Phi(x)$  is a singleton and for any sequence  $(x_n, y_n)_{n=1}^\infty$  in the graph of  $\Phi$ , with  $\lim_{n \rightarrow \infty} x_n = x$ , we have  $\lim_{n \rightarrow \infty} y_n = z$ , where  $\Phi(x) = \{z\}$ .

**REMARK 2.24.** It is clear that when  $\Phi(x)$  is a singleton for each  $x \in X$  this definition agrees with the standard definition of continuity when we view  $\Phi$  as a function into  $Y$ .

**DEFINITION 2.25.** Let  $K$  be a subset of a normed linear space  $(X, \|\cdot\|)$ . We say that  $K$  is boundedly compact if for any  $r > 0$ ,  $rB_X \cap K$  is compact.

It follows almost immediately from this definition that all nonempty boundedly compact sets are proximal and hence closed, see Proposition 2.9.

Our reason for considering boundedly compact sets is revealed in the next theorem.

**THEOREM 2.26** [50, Proposition 2.3]. *Let  $K$  be a boundedly compact subset of a normed linear space  $(X, \|\cdot\|)$ . If  $P_K(x)$  is a singleton for some  $x \in X$ , then the metric projection mapping,  $y \mapsto P_K(y)$ , is continuous at  $x$ .*

**PROOF.** Let  $x \in X$  and suppose that  $P_K(x) = \{z\}$  for some  $z \in K$ . To show that  $P_K$  is continuous at  $x$ , it suffices to show that for each sequence  $(x_n, y_n)_{n=1}^\infty$  in the graph of  $P_K$  with  $\lim_{n \rightarrow \infty} x_n = x$ , there exists a subsequence  $(x_{n_k}, y_{n_k})_{k=1}^\infty$  of  $(x_n, y_n)_{n=1}^\infty$  such that  $\lim_{k \rightarrow \infty} y_{n_k} = z$ . To this end, let  $(x_n, y_n)_{n=1}^\infty$  be a sequence in the graph of  $P_K$  with  $\lim_{n \rightarrow \infty} x_n = x$ . Since  $P_K$  is locally bounded (see Lemma 2.22) and  $K$  is boundedly compact, there exists a subsequence  $(x_{n_k}, y_{n_k})_{k=1}^\infty$  of  $(x_n, y_n)_{n=1}^\infty$  such that  $\lim_{k \rightarrow \infty} y_{n_k} = y$  for some  $y \in K$ . Since  $\lim_{k \rightarrow \infty} (x_{n_k}, y_{n_k}) = (x, y)$ , Lemma 2.18 tells us that  $y \in P_K(x)$ . Thus,  $y = z$  as required.  $\square$

A special case of the previous theorem is the following folklore result.

**COROLLARY 2.27** [78, page 251]. *The metric projection mapping for a Chebyshev set in a finite-dimensional normed linear space is continuous.*

**2.2. Strictly convex and reflexive Banach spaces.** In this subsection we will define strict convexity of the norm and present some equivalent formulations of this notion. We will then examine the single-valuedness implications of strict convexity to the metric projection mapping. In the latter part of this subsection we will look at the implications of reflexivity to the study of proximal sets.

**DEFINITION 2.28** [22]. A normed linear space  $(X, \|\cdot\|)$  is said to be strictly convex (or rotund) if for any  $x, y \in S_X$ ,

$$\left\| \frac{x + y}{2} \right\| = 1 \quad \text{if and only if } x = y.$$

**REMARK 2.29.** From this definition it follows that in a strictly convex normed linear space  $(X, \|\cdot\|)$ :

- (i) if  $x \neq y$  and  $\|x\| = \|y\|$ , then  $\|(x + y)/2\| < (\|x\| + \|y\|)/2$ ;
- (ii) the only nonempty convex subsets of  $S_X$  are singletons.

Many of the classical Banach spaces are strictly convex, for example all the  $L_p$  and  $\ell_p$  spaces are strictly convex for any  $1 < p < \infty$ , [22, 39]. It is also known that every separable [22] or reflexive space [82] admits an equivalent norm that is strictly convex. On the other hand,  $C(K)$ -spaces are rarely strictly convex.

**EXAMPLE 2.30.** Let  $K$  be a compact Hausdorff space. Then  $(C(K), \|\cdot\|_\infty)$  is strictly convex if and only if  $K$  is a singleton set.

**PROOF.** Clearly, if  $K$  is a singleton set, then  $(C(K), \|\cdot\|_\infty)$  is strictly convex. On the other hand, if  $K$  is not a singleton set, then there exist distinct elements  $x, y \in K$ . Since  $K$  is completely regular, there exists a continuous function  $f : K \rightarrow [0, 1]$  such that  $f(x) = 1$  and  $f(y) = 0$ . Let  $g : K \rightarrow [0, 1]$  be defined by  $g(k) := 1$  for all  $k \in K$ . Clearly,  $f, g \in S_{C(K)}$  and  $f \neq g$ . However,  $(f + g)(x) = 2$ , and so  $\|f + g\|_\infty = 2$ , which shows that  $(C(K), \|\cdot\|_\infty)$  is not strictly convex.  $\square$

In practice, it is often convenient to apply the following equivalent characterisations of strict convexity.

**LEMMA 2.31.** *A normed linear space  $(X, \|\cdot\|)$  is strictly convex if and only if, for all  $x, y \in X$  and all  $r, R > 0$  such that  $\|x - y\| = r + R$ , we have  $B[x; r] \cap B[y; R] = \{z\}$ , where*

$$z := \left(\frac{R}{r + R}\right)x + \left(\frac{r}{r + R}\right)y.$$

**PROOF.** Suppose that  $(X, \|\cdot\|)$  is strictly convex,  $x, y \in X$ ,  $r, R > 0$  and  $\|x - y\| = r + R$ . Since

$$\|z - x\| = \left\| \left(\frac{R}{r + R}\right)x + \left(\frac{r}{r + R}\right)y - x \right\| = \left(\frac{r}{r + R}\right)\|x - y\| = r$$

and

$$\|z - y\| = \left\| \left(\frac{R}{r + R}\right)x + \left(\frac{r}{r + R}\right)y - y \right\| = \left(\frac{R}{r + R}\right)\|x - y\| = R,$$

we have  $z \in B[x; r] \cap B[y; R]$ . Let  $w$  be an arbitrary element of  $B[x; r] \cap B[y; R]$ . Since  $\|x - w\| \leq r$  and  $\|y - w\| \leq R$  and  $r + R = \|x - y\| \leq \|x - w\| + \|y - w\|$ , we must have that  $\|x - w\| = r$  and  $\|y - w\| = R$ . However, if  $u, v \in B[x; r] \cap B[y; R]$ , then

$$\frac{u + v}{2} \in B[x; r] \cap B[y; R]$$

since  $B[x; r] \cap B[y; R]$  is convex. On the other hand, if  $u \neq v$ , that is,  $(x - u) \neq (x - v)$ , then

$$\left\| x - \left(\frac{u + v}{2}\right) \right\| = \left\| \frac{(x - u) + (x - v)}{2} \right\| < \frac{\|x - u\| + \|x - v\|}{2} = r,$$

which is impossible. Therefore,  $B[x; r] \cap B[y; R] = \{z\}$ .

For the converse, suppose that  $(X, \|\cdot\|)$  is not strictly convex. Hence, there exist distinct  $x, y \in X$  such that

$$\|x\| = \|y\| = \left\| \frac{x + y}{2} \right\| = 1.$$

Let  $r := 1 =: R$ . Then  $x, y \in B[0; r] \cap B[x + y; R]$  and  $\|(x + y) - 0\| = r + R$ .  $\square$

From Lemma 2.31 we can obtain another useful characterisation of strict convexity.

**LEMMA 2.32.** *A normed linear space  $(X, \|\cdot\|)$  is strictly convex if and only if, for any nonzero  $x, y \in X$ , if  $\|x + y\| = \|x\| + \|y\|$ , then  $x = \alpha y$  for some  $\alpha > 0$ .*

**PROOF.** Suppose that  $(X, \|\cdot\|)$  has the property given above. Let  $x, y \in X$  with

$$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1.$$

Then  $\|x+y\| = 2 = \|x\| + \|y\|$ , so  $x = \alpha y$  for some  $\alpha > 0$ . Since  $\|x\| = \|y\| = 1$ ,  $\alpha = 1$ , and so  $x = y$ . Thus,  $(X, \|\cdot\|)$  is strictly convex.

Conversely, suppose that  $(X, \|\cdot\|)$  is strictly convex and let  $x, y \in X \setminus \{0\}$  be such that

$$\|x+y\| = \|x\| + \|y\|.$$

Since  $x \neq -y$  and  $\|x - (-y)\| = \|x\| + \|y\|$ , it follows by Lemma 2.31 that

$$B[x; \|x\|] \cap B[-y; \|y\|] = \left\{ \left( \frac{\|y\|}{\|x\| + \|y\|} \right) x - \left( \frac{\|x\|}{\|x\| + \|y\|} \right) y \right\}.$$

However, it is also clear that  $0 \in B[x; \|x\|] \cap B[-y; \|y\|]$ . Therefore,

$$\left( \frac{\|y\|}{\|x\| + \|y\|} \right) x - \left( \frac{\|x\|}{\|x\| + \|y\|} \right) y = 0,$$

which rearranges to give  $x = (\|x\|/\|y\|)y$ . □

Next we present our first application of the notion of strict convexity to the study of Chebyshev sets.

**PROPOSITION 2.33.** *Let  $K$  be a nonempty closed and convex subset of a strictly convex normed linear space  $(X, \|\cdot\|)$ . Then for each  $x \in X$ ,  $P_K(x)$  contains at most one element.*

**PROOF.** The result is clearly true if  $x \in K$ , so let us suppose that  $x \in X \setminus K$ . Looking for a contradiction, suppose that there exist distinct  $y_1, y_2 \in P_K(x)$ . Then

$$\|x - y_1\| = \|x - y_2\| = d(x, K) > 0.$$

Now, since  $K$  is convex,  $(y_1 + y_2)/2 \in K$ , and so

$$\begin{aligned} d(x, K) &\leq \left\| x - \left( \frac{y_1 + y_2}{2} \right) \right\| = \left\| \frac{(x - y_1) + (x - y_2)}{2} \right\| \\ &< \frac{\|x - y_1\| + \|x - y_2\|}{2} \quad \text{since } \|\cdot\| \text{ is strictly convex} \\ &= d(x, K), \end{aligned}$$

which is impossible. Hence,  $y_1 = y_2$ , and so  $P_K(x)$  is at most a singleton. □

**COROLLARY 2.34.** *Every convex proximal set in a strictly convex normed linear space  $(X, \|\cdot\|)$  is a Chebyshev set.*

**REMARK 2.35.** It is easy to show that in every normed linear space that is not strictly convex, there exists a convex proximal set that is not a Chebyshev set.

**THEOREM 2.36** (Support theorem, [3], [28, Theorem V.9.5]). *Let  $C$  be a closed and convex subset of a normed linear space  $(X, \|\cdot\|)$  with nonempty interior. Then for every  $x \in \text{Bd}(C)$  there exists  $x^* \in X^* \setminus \{0\}$  such that  $x \in \text{argmax}(x^*|_C)$ .*

**DEFINITION 2.37.** Let  $K$  be a subset of a normed linear space  $(X, \|\cdot\|)$  and let  $x \in K$ . We say that  $x^* \in X^* \setminus \{0\}$  supports  $K$  if  $\text{argmax}(x^*|_K) \neq \emptyset$  and we say that  $x^* \in X^* \setminus \{0\}$  supports  $K$  at  $x$  if  $x \in \text{argmax}(x^*|_K)$ .

**LEMMA 2.38.** *Let  $(X, \|\cdot\|)$  be a strictly convex normed linear space and  $x^* \in X^* \setminus \{0\}$ . If  $x \in X$  and  $r > 0$ , then  $\text{argmax}(x^*|_{B[x,r]})$  is at most a singleton set.*

**PROOF.** Since  $\text{argmax}(x^*|_{B[x,r]})$  is the intersection of a hyperplane and a closed ball, it is convex. On the other hand,  $\text{argmax}(x^*|_{B[x,r]})$  must be a subset of  $S[x,r]$ . Therefore, from the strict convexity of the norm,  $\text{argmax}(x^*|_{B[x,r]})$  must be at most a singleton.  $\square$

Using the previous definitions and results we can prove the following interesting result which will be used in Section 4.

**LEMMA 2.39.** *Let  $(X, \|\cdot\|)$  be a strictly convex normed linear space and let  $K$  be a closed concave subset of  $X$ . Suppose that  $x \in P_K(y) \cap P_K(z)$  for some  $y, z \in X$ . Then  $x, y, z$  are collinear.*

**PROOF.** If either  $y$  or  $z$  is a member of  $K$ , then the result holds trivially, so suppose otherwise. To show that  $x, y, z$  are collinear it is sufficient to show that

$$\frac{x - y}{\|x - y\|} = \frac{x - z}{\|x - z\|}.$$

Let  $C := \overline{X \setminus K}$ . Then, by Corollary 3.41,  $\text{Bd}(C) = \text{Bd}(X \setminus K) = \text{Bd}(K)$ . Thus,  $x \in \text{Bd}(C)$ . Therefore, by Theorem 2.36, there exists  $x^* \in X^* \setminus \{0\}$  that supports  $C$  at  $x$ . Now since  $x \in B[y; d(y, K)] \cap B[z; d(z, K)]$  and  $B[y; d(y, K)] \cup B[z; d(z, K)] \subseteq C$ ,  $x^*$  supports both  $B[y; d(y, K)]$  and  $B[z; d(z, K)]$  at  $x$ . It then follows, after translation and dilation, that  $x^*$  supports  $B_X$  at both

$$\frac{x - y}{\|x - y\|} \quad \text{and} \quad \frac{x - z}{\|x - z\|}.$$

However, by Lemma 2.38,  $(x - y)/\|x - y\| = (x - z)/\|x - z\|$  and we are done.  $\square$

In order to obtain some further single-valuedness implications of the notion of strict convexity to the metric projection mapping we will need to consider some notions of differentiability.

**DEFINITION 2.40.** Let  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|')$  be normed linear spaces and  $U$  be a nonempty open subset of  $X$ . We say a function  $f : U \rightarrow Y$  is Gâteaux differentiable at  $x \in U$  if there exists a bounded linear operator  $T_x : X \rightarrow Y$  such that

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = T_x(h) \tag{*}$$

for all  $h \in S_X$ . The operator  $T_x$  is called the Gâteaux derivative of  $f$  at  $x$ . If  $f$  is Gâteaux differentiable at each point of  $U$ , then we simply say that  $f$  is

Gâteaux differentiable. If the limit in Equation (\*) is uniform with respect to all  $h \in S_X$ , then we say that  $f$  is Fréchet differentiable at  $x \in U$  and we call  $T_x$  the Fréchet derivative of  $f$  at  $x$ . Finally, if  $f$  is Fréchet differentiable at each point of  $U$ , then we simply say that  $f$  is Fréchet differentiable.

**LEMMA 2.41.** *Let  $K$  be a nonempty closed subset of a normed linear space  $(X, \|\cdot\|)$ . Suppose that  $x \in X \setminus K$  is a point of Gâteaux differentiability of the distance function for  $K$ , with Gâteaux derivative  $f \in X^*$  and  $y \in P_K(x)$ . Then  $\|f\| = 1$  and  $f(x - y) = \|x - y\| = d(x, K)$ .*

**PROOF.** Since  $x \notin K$ ,  $d(x, K) = \|x - y\| > 0$ . Let  $0 \leq t \leq 1$ . Since

$$x + t(y - x) \in [x, y],$$

Proposition 2.17 tells us that  $y \in P_K(x + t(y - x))$  and so

$$d(x + t(y - x), K) = \|x + t(y - x) - y\| = (1 - t)\|x - y\|.$$

Therefore,

$$\begin{aligned} f(y - x) &= \lim_{t \rightarrow 0^+} \frac{d(x + t(y - x), K) - d(x, K)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{(1 - t)\|x - y\| - \|x - y\|}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{-t\|x - y\|}{t} = -\|x - y\|. \end{aligned}$$

Thus,  $f(x - y) = \|x - y\| = d(x, K)$ . Proposition 2.2 then implies that, for any  $z \in X$ ,

$$\begin{aligned} |f(z)| &= \left| \lim_{t \rightarrow 0} \frac{d(x + tz, K) - d(x, M)}{t} \right| \\ &= \lim_{t \rightarrow 0} \left| \frac{d(x + tz, K) - d(x, K)}{t} \right| \\ &\leq \lim_{t \rightarrow 0} \frac{\|x + tz - x\|}{|t|} = \|z\|. \end{aligned}$$

Thus,  $\|f\| = 1$ . □

**THEOREM 2.42 [13, Theorem 5.1].** *Let  $K$  be a nonempty closed subset of a strictly convex normed linear space  $(X, \|\cdot\|)$ . Suppose that  $x \in X$  is a point of Gâteaux differentiability of the distance function for  $K$ . Then  $P_K(x)$  contains at most one element.*

**PROOF.** If  $x \in K$ , then  $P_K(x) = \{x\}$ . Therefore, we may assume that  $x \notin K$ . Suppose that  $y, z \in P_K(x)$  and let  $f \in X^*$  be the Gâteaux derivative of the distance function for  $K$  at  $x$ . Applying Lemma 2.41 gives

$$f\left(\frac{x - y}{\|x - y\|}\right) = \|f\| = f\left(\frac{x - z}{\|x - z\|}\right).$$

As  $(X, \|\cdot\|)$  is strictly convex,  $f$  can attain its norm at one point of  $S_X$  at most. Therefore,

$$\frac{x-y}{\|x-y\|} = \frac{x-z}{\|x-z\|}.$$

Since  $\|x-y\| = \|x-z\| = d(x, K)$ , we have  $y = z$ , which completes the proof.  $\square$

In order to exploit Theorem 2.42 we need to know something about the differentiability of distance functions. The next classical result provides some general information in this direction.

**THEOREM 2.43** [64]. *Every real-valued Lipschitz function defined on a nonempty open subset of a finite-dimensional normed linear space is Gâteaux differentiable almost everywhere (with respect to the Lebesgue measure).*

**COROLLARY 2.44.** *Let  $K$  be a nonempty closed subset of a strictly convex finite-dimensional normed linear space  $(X, \|\cdot\|)$ . Then  $P_K(x)$  is a singleton for almost all  $x \in X$  (with respect to the Lebesgue measure).*

In order to generalise this result beyond finite-dimensional spaces one needs to know about the differentiability of real-valued Lipschitz functions defined on infinite-dimensional normed linear spaces; see [13, 53, 63].

Next we present some sufficient conditions for a subset of a normed linear space to be a proximal set. We start by considering some general facts concerning lower semicontinuous functions.

**DEFINITION 2.45.** Let  $(X, \tau)$  be a topological space. We say that a function  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  is lower semicontinuous if for every  $\alpha \in \mathbb{R}$ ,  $\{x \in X : f(x) \leq \alpha\}$  is a closed set.

**REMARK 2.46.** It follows from the definition of lower semicontinuity that if  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  is lower semicontinuous and  $x = \lim_{n \rightarrow \infty} x_n$ , then  $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ .

**PROPOSITION 2.47.** *Let  $C$  be a closed convex subset of a normed linear space  $(X, \|\cdot\|)$ . Then  $C$  is weakly closed.*

**PROOF.** If  $C$  is empty or the whole space, then  $C$  is weakly closed, so let us suppose otherwise. Let  $x_0 \in X \setminus C$ . Since  $C$  is closed and convex, we have, by the Hahn–Banach theorem, the existence of an  $f_{x_0} \in X^*$  such that  $f_{x_0}(x_0) > \sup_{x \in C} f_{x_0}(x)$ . Thus,  $x_0 \in f_{x_0}^{-1}((\sup_{x \in C} f_{x_0}(x), \infty))$ , which, being the inverse image of an open set, is weakly open. It is then straightforward to check that  $X \setminus C = \bigcup_{x_0 \in X \setminus C} f_{x_0}^{-1}((\sup_{x \in C} f_{x_0}(x), \infty))$ . Hence,  $X \setminus C$ , being the union of weakly open sets, is weakly open. Thus,  $C$  is weakly closed.  $\square$

**LEMMA 2.48.** *Let  $(X, \|\cdot\|)$  be a normed linear space and let  $f : X \rightarrow \mathbb{R}$  be a continuous convex function. Then  $f$  is lower semicontinuous with respect to the weak topology on  $X$ .*

**PROOF.** Let  $\alpha \in \mathbb{R}$ . We need to show that the set  $A := \{x \in X : f(x) \leq \alpha\}$  is closed with respect to the weak topology. Since  $f$  is convex and continuous,  $A = f^{-1}((-\infty, \alpha])$  is convex and closed with respect to the norm topology. Hence, by Proposition 2.47, we conclude that  $A$  is weakly closed.  $\square$

Next we see that, as with continuous real-valued functions, lower semicontinuous functions defined on compact spaces attain their minimum value and are hence, bounded from below.

**PROPOSITION 2.49.** *Let  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper lower semicontinuous function defined on a compact space  $X$ . Then  $\operatorname{argmin}(f) \neq \emptyset$ .*

**PROOF.** Let  $(r_n)_{n=1}^\infty$  be a strictly decreasing sequence of real numbers such that converge to  $\inf_{x \in X} f(x)$ . Since  $X$  is compact,  $\bigcap_{n \in \mathbb{N}} f^{-1}((-\infty, r_n]) \neq \emptyset$  because  $\{f^{-1}((-\infty, r_n]) : n \in \mathbb{N}\}$  is a decreasing sequence of nonempty closed subsets of  $X$ . Furthermore,  $\bigcap_{n \in \mathbb{N}} f^{-1}((-\infty, r_n])$  is contained in  $\operatorname{argmin}(f)$ . Indeed, if  $x \in \bigcap_{n \in \mathbb{N}} f^{-1}((-\infty, r_n])$ , then  $f(x) \leq r_n$  for all  $n \in \mathbb{N}$ , and so  $\inf_{y \in X} f(y) \leq f(x) \leq \lim_{n \rightarrow \infty} r_n = \inf_{y \in X} f(y)$ , which completes the proof.  $\square$

**THEOREM 2.50.** *Every nonempty weakly closed subset of a reflexive Banach space is proximal. In particular, every nonempty closed and convex subset of a reflexive Banach space is proximal.*

**PROOF.** Let  $K$  be a nonempty weakly closed subset of a reflexive Banach space  $(X, \|\cdot\|)$ . Fix  $x \in X \setminus K$  and let  $f : K \rightarrow [0, \infty)$  be defined by  $f(k) := \|x - k\|$ . Then

$$\emptyset \neq \operatorname{argmin}(f|_{K \cap B[x; d(x, K) + 1]}) = \operatorname{argmin}(f)$$

as  $f|_{K \cap B[x; d(x, K) + 1]}$  is weakly lower semicontinuous and  $K \cap B[x; d(x, K) + 1]$  is weakly compact.  $\square$

**COROLLARY 2.51.** *Every nonempty closed and convex subset of a strictly convex reflexive Banach space is a Chebyshev set.*

**PROOF.** The proof follows directly from Theorem 2.50 and Corollary 2.34.  $\square$

It is interesting to see that the result from Theorem 2.50 is optimal in the sense that if every nonempty closed and convex set is proximal, then the underlying space must be reflexive.

**THEOREM 2.52** [78, Corollary 2.4], [23, Corollary 2.12]. *Let  $(X, \|\cdot\|)$  be a Banach space. The following are equivalent:*

- (i)  $X$  is reflexive;
- (ii) every nonempty weakly closed subset of  $X$  is proximal;
- (iii) every nonempty closed convex subset of  $X$  is proximal;
- (iv) every closed subspace of  $X$  is proximal;
- (v) every closed hyperplane in  $X$  is proximal.



**PROOF.** Theorem 2.50 says that (i)  $\Rightarrow$  (ii). It is obvious that (ii)  $\Rightarrow$  (iii), (iii)  $\Rightarrow$  (iv) and (iv)  $\Rightarrow$  (v). Finally, (v)  $\Rightarrow$  (i) follows from Example 2.8 and James' theorem [43, 55].  $\square$

As an application of some of the results that we have obtained so far, let us show how Schauder's fixed point theorem can be deduced from Brouwer's fixed point theorem.

**THEOREM 2.53 (Schauder's fixed point theorem [77]).** *Let  $K$  be a nonempty compact convex subset of a normed linear space  $(X, \|\cdot\|)$ . If  $f : K \rightarrow K$  is a continuous function, then  $f$  has a fixed point, that is, there exists a point  $k \in K$  such that  $f(k) = k$ .*

**PROOF.** As the closed span of  $K$  is a separable normed linear space, we may assume, without loss of generality, that  $X$  is separable. Hence, by Appendix D, we can assume, after possibly renorming, that the norm on  $X$  is strictly convex. For each  $n \in \mathbb{N}$ , let  $F_n$  be a finite subset of  $K$  such that  $K \subseteq F_n + (1/n)B_X$ ; note that this is possible since  $K$  is totally bounded. Let  $K_n$  be the convex hull of  $F_n$ . Then  $K_n$  is a compact convex subset of  $K$ . Thus, by Corollaries 2.34 and 2.27,  $p_{K_n}$  is continuous. Define  $f_n : K_n \rightarrow K_n$  by  $f_n(x) := p_{K_n}(f(x))$ . By Brouwer's fixed point theorem [17],  $f_n$  has a fixed point  $x_n \in K_n$ . Now

$$\|f(x) - f_n(x)\| \leq 1/n \quad \text{for all } x \in K.$$

Therefore,  $\|f(x_n) - x_n\| \leq 1/n$ . It is now routine to check that every cluster point of the sequence  $(x_n)_{n=1}^\infty$  is a fixed point of  $f$ .  $\square$

**2.3. Continuity of the metric projection.** In Section 3 it is shown that continuity of the metric projection mapping plays a key role in deriving the convexity of a Chebyshev set. Therefore, it is natural to ask whether every Chebyshev set has a continuous metric projection mapping. Unfortunately, finding an example of a Chebyshev set with a discontinuous metric projection mapping is not as straightforward as one might first imagine. So we begin this section by presenting an example, due to B. Kripke, of such a set.

**EXAMPLE 2.54 [40, page 246].** There exists a reflexive space containing a convex Chebyshev set with a discontinuous metric projection mapping.

**PROOF.** Consider the Hilbert space  $\ell_2(\mathbb{Z})$ . We define a linear map  $T : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$  componentwise by

$$T(x)_n := \begin{cases} x_n & \text{if } n \leq 0, \\ x_{-n} + \frac{1}{n}x_n & \text{if } 0 < n, \end{cases}$$

where  $(\dots, x_{-1}, x_0, x_1, \dots) := x \in \ell_2(\mathbb{Z})$  and  $n \in \mathbb{Z}$ . It is straightforward to check that  $T$  is injective. Furthermore,  $T$  is continuous. To see this, suppose that  $x \in B_{\ell_2(\mathbb{Z})}$ , that is,

suppose that  $\|x\|_2^2 = \sum_{n \in \mathbb{Z}} x_n^2 \leq 1$ . Making use of the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|T(x)\|_2^2 &= \sum_{n \leq 0} x_n^2 + \sum_{n > 0} \left(x_{-n} + \frac{x_n}{n}\right)^2 \\ &= \sum_{n \leq 0} x_n^2 + \sum_{n > 0} x_{-n}^2 + \sum_{n > 0} \left(\frac{x_n}{n}\right)^2 + 2 \sum_{n > 0} \frac{x_{-n}x_n}{n} \\ &\leq \sum_{n \leq 0} x_n^2 + \sum_{n > 0} x_{-n}^2 + \sum_{n > 0} \left(\frac{x_n}{n}\right)^2 + 2 \sqrt{\left(\sum_{n > 0} x_{-n}^2\right)\left(\sum_{n > 0} \left(\frac{x_n}{n}\right)^2\right)} \\ &\leq 5, \end{aligned}$$

and so  $\|T\| \leq \sqrt{5}$ .

Next we define a new norm  $\|\cdot\|'$  on  $\ell_2(\mathbb{Z})$  by  $\|x\|' := \max\{\|x\|_2, 2|x_0|\} + \|T(x)\|_2$ . It is straightforward to check that this is an equivalent norm, since for any  $x \in \ell_2(\mathbb{Z})$ ,

$$\|x\|_2 \leq \|x\|' \leq (2 + \|T\|)\|x\|_2.$$

Therefore,  $(\ell_2(\mathbb{Z}), \|\cdot\|')$  is reflexive and complete. Since  $\|\cdot\|_2$  is strictly convex (it is induced by an inner product) and  $T$  is injective, it follows that  $\|\cdot\|'$  is strictly convex. Also, define  $K := \overline{\text{span}}\{e_n : 1 \leq n\}$ , where  $e_n$  is the element of  $\ell_2(\mathbb{Z})$  with 1 as its  $n$ th coordinate and zeros everywhere else. By Corollary 2.51,  $K$ , viewed as a subset of  $(\ell_2(\mathbb{Z}), \|\cdot\|')$ , is a Chebyshev set.

We now show that  $K$  has a discontinuous metric projection mapping. Let  $y := \sum_{n=1}^k \alpha_n e_n \in \text{span}\{e_n : 1 \leq n\}$  for some  $1 \leq k$  and  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ . Then

$$\begin{aligned} \|e_0 - y\|' &= \left\| e_0 - \sum_{n=1}^k \alpha_n e_n \right\|' \\ &= \max\left\{ \left\| e_0 - \sum_{n=1}^k \alpha_n e_n \right\|_2, 2 \right\} + \left\| T\left( e_0 - \sum_{n=1}^k \alpha_n e_n \right) \right\|_2 \\ &= \max\left\{ 2, \sqrt{1 + \sum_{n=1}^k \alpha_n^2} \right\} + \sqrt{1 + \sum_{n=1}^k \left(\frac{\alpha_n}{n}\right)^2}. \end{aligned}$$

This quantity can be minimised by setting  $\alpha_n = 0$  for all  $n \in \{1, \dots, k\}$ . Thus,

$$d(e_0, \text{span}\{e_n : 1 \leq n\}) = \|e_0 - 0\|' = 3.$$

Since  $d(e_0, \text{span}\{e_n : 1 \leq n\}) = d(e_0, \overline{\text{span}}\{e_n : 1 \leq n\})$  and  $K$  is a Chebyshev set, it follows that  $p_K(e_0) = 0$ . Fix  $j > 0$ . We may assume, without loss of generality,

that  $j \leq k$ . Then

$$\begin{aligned} \left\| \left( e_0 + \frac{e_{-j}}{j} \right) - y \right\|' &= \left\| e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n \right\|' \\ &= \max \left\{ \left\| e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n \right\|_2, 2 \right\} + \left\| T \left( e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n \right) \right\|_2 \\ &= \max \left\{ 2, \sqrt{1 + \frac{1}{j^2} + \sum_{n=1}^k \alpha_n^2} \right\} + \sqrt{1 + \frac{1}{j^2} + \left( \frac{1 - \alpha_j}{j} \right)^2 + \sum_{\substack{n=1 \\ n \neq j}}^k \left( \frac{\alpha_n}{n} \right)^2}. \end{aligned}$$

This quantity can be minimised by setting  $\alpha_j = 1$  and  $\alpha_n = 0$  for all  $n \neq j$ . Thus,

$$d \left( e_0 + \frac{e_{-j}}{j}, \text{span}\{e_n : 1 \leq n\} \right) = \left\| e_0 + \frac{e_{-j}}{j} - e_j \right\|' = 2 + \sqrt{1 + \frac{1}{j^2}}.$$

Since  $d(e_0 + e_{-j}/j, \text{span}\{e_n : 1 \leq n\}) = d(e_0 + e_{-j}/j, \overline{\text{span}\{e_n : 1 \leq n\}})$  and  $K$  is a Chebyshev set, it follows that  $p_K(e_0 + e_{-j}/j) = e_j$ . Finally, observe that

$$\lim_{j \rightarrow \infty} \left( e_0 + \frac{e_{-j}}{j} \right) = e_0 \quad \text{whilst} \quad \lim_{j \rightarrow \infty} p_K \left( e_0 + \frac{e_{-j}}{j} \right) = \lim_{j \rightarrow \infty} e_j \neq 0 = p_K(e_0),$$

showing that the metric projection mapping  $x \mapsto p_K(x)$  is not continuous. □

Another example of a Chebyshev set in a strictly convex reflexive space with a discontinuous metric projection mapping is given in [18].

We now give some hypotheses that guarantee that the metric projection mapping is continuous. To do this we consider the following property of a norm which was first considered in [65, 69, 70].

**DEFINITION 2.55** [47, 49]. Let  $(X, \|\cdot\|)$  be a normed linear space. We say  $\|\cdot\|$  is a Kadec–Klee norm if for every sequence  $(x_n)_{n=1}^\infty$  in  $X$ , if  $\lim_{n \rightarrow \infty} x_n = x$  with respect to the weak topology and  $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$ , then  $\lim_{n \rightarrow \infty} x_n = x$  with respect to the norm topology.

Many norms that we are familiar with are Kadec–Klee norms.

**EXAMPLE 2.56.** Every finite-dimensional space has a Kadec–Klee norm, as does every Hilbert space.

**PROOF.** In every finite-dimensional normed linear space the weak and norm topologies coincide and so it directly follows that every norm on a finite-dimensional normed linear space is a Kadec–Klee norm. In any Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ , the norm satisfies the identity

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle.$$

Hence, if  $(x_n)_{n=1}^\infty$  is a sequence in  $H$  and  $\lim_{n \rightarrow \infty} x_n = x$  with respect to the weak topology and  $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$ , then  $\lim_{n \rightarrow \infty} x_n = x$  with respect to the norm topology since the functional  $y \mapsto \langle y, x \rangle$  is weakly continuous and  $\langle x, x \rangle = \|x\|^2$ . □

*Note.* It follows from Clarkson’s inequalities [22, 39] that for any  $1 < p < \infty$ ,  $L_p$  has a Kadec–Klee norm.

**THEOREM 2.57** [23, Theorem 2.7 and Corollary 2.16]. *Let  $K$  be a weakly closed Chebyshev set in a reflexive normed linear space  $(X, \|\cdot\|)$  with a Kadec–Klee norm. Then  $K$  has a continuous metric projection mapping.*

**PROOF.** Let  $x \in X$ . To show that  $p_K$  is continuous at  $x$  it suffices to show that for each sequence  $(x_n)_{n=1}^\infty$  with  $x = \lim_{n \rightarrow \infty} x_n$  there exists a subsequence  $(x_{n_k})_{k=1}^\infty$  of  $(x_n)_{n=1}^\infty$  such that  $\lim_{k \rightarrow \infty} p_K(x_{n_k}) = p_K(x)$ . To this end, let  $(x_n)_{n=1}^\infty$  be a sequence in  $X$  with  $x = \lim_{n \rightarrow \infty} x_n$ . Since  $p_K$  is locally bounded (see Lemma 2.22) and  $X$  is reflexive, there exist a subsequence  $(x_{n_k})_{k=1}^\infty$  of  $(x_n)_{n=1}^\infty$  and an element  $y \in X$  such that  $(p_K(x_{n_k}))_{k=1}^\infty$  converges to  $y$  with respect to the weak topology on  $X$ . Since  $K$  is weakly closed,  $y \in K$ . Therefore,

$$\begin{aligned} d(x, K) &\leq \|x - y\| \leq \liminf_{k \rightarrow \infty} \|x_{n_k} - p_K(x_{n_k})\| \\ &\leq \limsup_{k \rightarrow \infty} \|x_{n_k} - p_K(x_{n_k})\| = \lim_{k \rightarrow \infty} d(x_{n_k}, K) = d(x, K) \end{aligned}$$

since  $(x_{n_k} - p_K(x_{n_k}))_{k=1}^\infty$  converges to  $(x - y)$  with respect to the weak topology on  $X$  and the norm is lower semicontinuous with respect to the weak topology on  $X$ .

Now because  $K$  is a Chebyshev set it follows that  $y = p_K(x)$ . Furthermore, since  $\|\cdot\|$  is a Kadec–Klee norm,  $(x_{n_k} - p_K(x_{n_k}))_{k=1}^\infty$  converges in norm to  $(x - y) = (x - p_K(x))$ . Therefore,  $(p_K(x_{n_k}))_{k=1}^\infty$  converges in norm to  $p_K(x)$ , which completes the proof.  $\square$

We finish this subsection by presenting a classical result of Robert R. Phelps, which shows that the convexity of a Chebyshev set in an inner product space is characterised by the metric projection mapping being nonexpansive. To prove this fact we first need a result that characterises best approximations for convex sets in inner product spaces.

**PROPOSITION 2.58** [2]. *Let  $K$  be a convex subset of an inner product space  $(X, \langle \cdot, \cdot \rangle)$  and let  $x \in X, y \in K$ . Then  $y \in P_K(x)$  if and only if  $\langle x - y, z - y \rangle \leq 0$  for all  $z \in K$ .*

**PROOF.** Let  $x \in X, y \in K$  and suppose that the above inequality holds. If  $x = y$ , then there is nothing to prove, so assume otherwise. For any  $z \in K$ ,

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x - y, x - z \rangle + \langle x - y, z - y \rangle \\ &\leq \langle x - y, x - z \rangle \\ &\leq \|x - y\| \|x - z\| \quad \text{by the Cauchy–Schwarz inequality.} \end{aligned}$$

Thus,  $\|x - y\| \leq \|x - z\|$  for all  $z \in K$ , and so  $y \in P_K(x)$ .

Conversely, suppose that for some  $z \in K, \langle x - y, z - y \rangle > 0$ . Clearly,  $z \neq y$ . Choose

$$0 < \lambda < \min\left\{1, \frac{2\langle x - y, z - y \rangle}{\|z - y\|^2}\right\},$$

which guarantees that  $2\langle x - y, z - y \rangle - \lambda\|z - y\|^2 > 0$ . Since  $K$  is convex,  $y_\lambda := \lambda z + (1 - \lambda)y$  is a member of  $K$ . Therefore,

$$\begin{aligned} \|x - y_\lambda\|^2 &= \langle x - y_\lambda, x - y_\lambda \rangle \\ &= \langle x - y - \lambda(z - y), x - y - \lambda(z - y) \rangle \\ &= \|x - y\|^2 - \lambda(2\langle x - y, z - y \rangle - \lambda\|z - y\|^2) \\ &< \|x - y\|^2. \end{aligned}$$

Thus,  $\|x - y_\lambda\| < \|x - y\|$ , and so  $y \notin P_K(x)$ . □

**THEOREM 2.59 [61].** *Let  $K$  be a Chebyshev set in an inner product space  $(X, \langle \cdot, \cdot \rangle)$ . Then  $K$  is convex if and only if  $\|p_K(x) - p_K(y)\| \leq \|x - y\|$  for all  $x, y \in X$ , that is, the metric projection mapping is nonexpansive.*

**PROOF.** Suppose that  $K$  is convex and  $x, y \in X$ . Clearly, the required inequality holds if  $p_K(x) = p_K(y)$ , so we will assume otherwise. By Proposition 2.58,

$$\langle x - p_K(x), p_K(y) - p_K(x) \rangle \leq 0 \quad \text{and} \quad \langle y - p_K(y), p_K(x) - p_K(y) \rangle \leq 0$$

or equivalently,

$$\langle x - p_K(x), p_K(y) - p_K(x) \rangle \leq 0 \quad \text{and} \quad \langle p_K(y) - y, p_K(y) - p_K(x) \rangle \leq 0.$$

Adding these two inequalities gives  $\langle x - y + p_K(y) - p_K(x), p_K(y) - p_K(x) \rangle \leq 0$ , and so

$$\langle p_K(y) - p_K(x), p_K(y) - p_K(x) \rangle \leq \langle y - x, p_K(y) - p_K(x) \rangle.$$

Therefore,

$$\begin{aligned} \|p_K(y) - p_K(x)\|^2 &= \langle p_K(y) - p_K(x), p_K(y) - p_K(x) \rangle \\ &\leq \langle y - x, p_K(y) - p_K(x) \rangle \\ &\leq \|y - x\| \|p_K(y) - p_K(x)\|. \end{aligned}$$

Thus,  $\|p_K(y) - p_K(x)\| \leq \|x - y\|$  as required.

For the converse, suppose that the metric projection mapping is nonexpansive but  $K$  is not convex. Since  $K$  is closed, it is not midpoint convex, and so we can find  $x, y \in K$  such that  $z := (x + y)/2 \notin K$ . Clearly,  $x \neq y$ . Set  $r := \|x - y\|/2 > 0$ . We claim that  $p_K(z) \in B[x; r] \cap B[y; r]$ . This follows since

$$\|p_K(z) - x\| = \|p_K(z) - p_K(x)\| \leq \|z - x\| = r$$

and

$$\|p_K(z) - y\| = \|p_K(z) - p_K(y)\| \leq \|z - y\| = r.$$

Since  $(X, \langle \cdot, \cdot \rangle)$  is strictly convex (as it is an inner product space) and  $\|x - y\| = 2r$ , Lemma 2.31 tells us that  $B[x; r] \cap B[y; r] = \{(x + y)/2\} = \{z\}$ . This forces  $z = p_K(z) \in K$ , which is impossible. Hence,  $K$  is convex. □

Let us end this section with a few remarks concerning the single-valuedness of the metric projection mapping. In 1963, Stečkin [81] proved that in a strictly convex normed linear space  $(X, \|\cdot\|)$ , for every nonempty subset  $K$  of  $X$ ,  $\{x \in X : P_K(x) \text{ is at most a singleton}\}$  is dense in  $X$ . This result can be deduced from the following geometric fact. In a strictly convex normed linear space  $(X, \|\cdot\|)$ , if  $x \notin K$  and  $z \in P_K(x)$ , then  $P_K(y) = \{z\}$  for all  $y \in (x, z)$ . To see why this is true we simply note that by Lemma 2.32,

$$B[y; \|z - y\|] \setminus \{z\} \subseteq B(x; \|z - x\|) \subseteq X \setminus K \quad \text{and so } B[y; \|z - y\|] \cap K = \{z\}$$

for all  $y \in (z, x)$ . Stečkin also proved that if  $(X, \|\cdot\|)$  is complete (that is, a Banach space) and the norm is ‘locally uniformly rotund’ (see [82] for the definition), then for each nonempty subset  $K$  of  $X$ , the set

$$\{x \in X : P_K(x) \text{ is at most a singleton}\}$$

is ‘residual’ (that is, contains the intersection of a countable family of dense open subsets of  $X$ ). Stečkin also conjectured that in any strictly convex Banach space  $(X, \|\cdot\|)$ ,

$$\{x \in X : P_K(x) \text{ is at most a singleton}\}$$

is residual in  $X$ , for every nonempty subset  $K$  of  $X$ . Stečkin’s conjecture is still open. For the latest state of play, see [71].

### 3. The convexity of Chebyshev sets

As shown by Example 2.11, a Chebyshev set need not be convex. However, in this section we present four distinct proofs which show that, under various additional hypotheses (all of which include the assumption that the metric projection mapping is continuous), a Chebyshev set is convex. In this way, this section follows the pattern of the paper [11].

**3.1. Proof by fixed point theorem.** In this subsection we make use of Schauder’s fixed point theorem to prove that a boundedly compact Chebyshev set in a smooth Banach space is necessarily convex.

Let us begin with the following simple observation.

**LEMMA 3.1.** *Let  $J$  be an open half space of a normed linear space  $(X, \|\cdot\|)$  and let  $x, y \in X$ . If  $\frac{1}{2}(x + y) \in J$ , then either  $x \in J$  or  $y \in J$ .*

We shall also require the following geometric consequence of smoothness.

**LEMMA 3.2 [1].** *Let  $(X, \|\cdot\|)$  be a normed linear space and let  $x_0 \in X \setminus \{0\}$ . If the norm is smooth at  $x_0$ , then  $\{y \in X : 0 < x^*(y)\} = \bigcup_{n \in \mathbb{N}} B(nx_0; n\|x_0\|)$ , where  $x^*$  is the Gâteaux derivative of the norm at  $x_0$ . Furthermore, if  $z \in X$ , then  $\{y \in X : x^*(z) < x^*(y)\}$  equals  $\bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|)$ .*

**PROOF.** We first show that  $\bigcup_{n \in \mathbb{N}} B(nx_0; n\|x_0\|) \subseteq \{y \in X : 0 < x^*(y)\}$ . To this end, let  $x \in \bigcup_{n \in \mathbb{N}} B(nx_0; n\|x_0\|)$ . Then there exists an  $n \in \mathbb{N}$  such that  $x \in B(nx_0; n\|x_0\|)$ , that is,  $\|x - nx_0\| < n\|x_0\|$ . Since  $x^*$  is the Gâteaux derivative of the norm at  $x_0$ , it follows that  $x^*(x_0) = \|x_0\|$  and  $\|x^*\| \leq 1$ . Therefore,

$$\begin{aligned} x^*(x) &= x^*(nx_0 + (x - nx_0)) \\ &= n\|x_0\| + x^*(x - nx_0) \\ &\geq n\|x_0\| - |x^*(x - nx_0)| \\ &\geq n\|x_0\| - \|x - nx_0\| \quad \text{since } \|x^*\| \leq 1 \\ &> 0 \quad \text{since } x \in B(nx_0; n\|x_0\|). \end{aligned}$$

Thus,  $\bigcup_{n \in \mathbb{N}} B(nx_0; n\|x_0\|) \subseteq \{y \in X : 0 < x^*(y)\}$ .

Suppose that  $x \in \{y \in X : 0 < x^*(y)\}$ . Then

$$\lim_{n \rightarrow \infty} \frac{\|x_0 + (1/n)(-x)\| - \|x_0\|}{1/n} = x^*(-x) = -x^*(x) < 0.$$

Therefore, there exists an  $n \in \mathbb{N}$  such that

$$\frac{\|x_0 + (1/n)(-x)\| - \|x_0\|}{1/n} < 0.$$

Thus,  $\|nx_0 - x\| < n\|x_0\|$ , and so  $x \in B(nx_0; n\|x_0\|) \subseteq \bigcup_{k \in \mathbb{N}} B(kx_0; k\|x_0\|)$ .

Finally, if  $z \in X$ , then

$$\begin{aligned} \{y \in X : x^*(z) < x^*(y)\} &= \{y \in X : 0 < x^*(y - z)\} \\ &= \{w + z \in X : 0 < x^*(w)\} \\ &= z + \{w \in X : 0 < x^*(w)\} \\ &= z + \bigcup_{n \in \mathbb{N}} B(nx_0; n\|x_0\|) = \bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|). \end{aligned}$$

This completes the proof. □

**DEFINITION 3.3 [32].** We say that a Chebyshev set  $K$  in a normed linear space  $(X, \|\cdot\|)$  is a sun if for every  $x \in X \setminus K$  and  $0 < \lambda$ ,  $p_K(x_\lambda) = p_K(x)$ , where  $x_\lambda := p_K(x) + \lambda(x - p_K(x))$ .

*Note.* In light of Proposition 2.20 and Proposition 2.17 we know that

$$I := \{\lambda \geq 0 : p_K(x_\lambda) = p_K(x)\}$$

is a closed interval containing  $[0, 1]$ . Thus, for  $K$  to be a sun we only require that  $(1, \infty) \subseteq I$ .

The following two theorems (the first of which is a folklore result) demonstrate the relationship between a Chebyshev set being a sun and being convex.

**THEOREM 3.4.** *In any normed linear space, every convex Chebyshev set is a sun.*

**PROOF.** Let  $K$  be a convex Chebyshev subset of a normed linear space  $(X, \|\cdot\|)$  and suppose that  $x \in X \setminus K$ . Then  $p_K(x) \in K \cap B[x; \|x - p_K(x)\|]$  and so  $0 \in B[x; \|x - p_K(x)\|] - K$ . However,

$$0 \notin B(x; \|x - p_K(x)\|) - K \quad \text{since } B(x; \|x - p_K(x)\|) \cap K = \emptyset.$$

Therefore,

$$0 \in B[x; \|x - p_K(x)\|] - K \subseteq \overline{B(x; \|x - p_K(x)\|) - K}$$

but

$$0 \notin B(x; \|x - p_K(x)\|) - K = \text{int}(\overline{B(x; \|x - p_K(x)\|) - K}), \quad \text{by Corollary 3.41.}$$

Thus,  $0 \in \text{Bd}(\overline{B(x; \|x - p_K(x)\|) - K})$ . Therefore, by the support theorem (see Theorem 2.36), there exists an  $f \in S_{X^*}$  such that  $\sup\{f(y) : y \in B[x; \|x - p_K(x)\|] - K\} \leq f(0) = 0$ . Thus,

$$f(y) \leq f(k) \quad \text{for all } y \in B[x; \|x - p_K(x)\|] \text{ and all } k \in K.$$

In particular,

$$\max_{y \in B[x; \|x - p_K(x)\|]} f(y) = f(p_K(x)) \quad \text{since } p_K(x) \in B[x; \|x - p_K(x)\|] \cap K$$

and

$$\min_{k \in K} f(k) = f(p_K(x)) \quad \text{since } p_K(x) \in B[x; \|x - p_K(x)\|] \cap K.$$

Let  $\alpha := f(p_K(x))$ . Then  $B[x; \|x - p_K(x)\|] \subseteq f^{-1}((-\infty, \alpha])$  and  $K \subseteq f^{-1}([\alpha, \infty))$ . It also follows that

$$f(p_K(x) - x) = \|x - p_K(x)\| \quad \text{or, equivalently, } f(x - p_K(x)) = -\|x - p_K(x)\|. \quad (*)$$

Let  $\lambda > 0$  and let  $x_\lambda := p_K(x) + \lambda(x - p_K(x))$ . Then  $p_K(x) \in K \cap B[x_\lambda; \|x_\lambda - p_K(x)\|]$  and so  $d(x_\lambda, K) \leq \|x_\lambda - p_K(x)\|$ . On the other hand,  $B(x_\lambda; \|x_\lambda - p_K(x)\|) \subseteq f^{-1}((-\infty, \alpha]) \subseteq X \setminus K$  since if  $y \in B(x_\lambda; \|x_\lambda - p_K(x)\|)$  then

$$\begin{aligned} f(y) &= f(y - x_\lambda) + f(x_\lambda) \\ &\leq \|y - x_\lambda\| + f(x_\lambda) \quad \text{since } \|f\| \leq 1. \\ &< \|p_K(x) - x_\lambda\| + f(x_\lambda) \quad \text{since } y \in B(x_\lambda; \|x_\lambda - p_K(x)\|). \\ &= \lambda\|x - p_K(x)\| + \lambda f(x - p_K(x)) + f(p_K(x)) \\ &= f(p_K(x)) = \alpha \quad \text{by } (*). \end{aligned}$$

Thus,  $\|x_\lambda - p_K(x)\| \leq d(x_\lambda, K)$  and so  $\|x_\lambda - p_K(x)\| = d(x_\lambda, K)$ . This shows that  $p_K(x_\lambda)$  equals  $p_K(x)$ . □

**REMARK 3.5.** We should note that there is also a short indirect proof of the previous theorem. Indeed, suppose to the contrary that  $K$  is a convex Chebyshev set that is not a sun. Then there exists an  $x \in X \setminus K$  and  $\lambda > 1$  such that  $p_K(x_\lambda) \neq p_K(x)$ , where



$x_\lambda := p_K(x) + \lambda(x - p_K(x))$ . Let  $\mu := 1/\lambda$ . Then  $0 < \mu < 1$  and  $x = \mu x_\lambda + (1 - \mu)p_K(x)$ . Now, since  $K$  is convex,  $\mu p_K(x_\lambda) + (1 - \mu)p_K(x) \in K$ . However,

$$\begin{aligned} \|x - [\mu p_K(x_\lambda) + (1 - \mu)p_K(x)]\| &= \|[\mu x_\lambda + (1 - \mu)p_K(x)] - [\mu p_K(x_\lambda) + (1 - \mu)p_K(x)]\| \\ &= \mu \|x_\lambda - p_K(x_\lambda)\| < \mu \|x_\lambda - p_K(x)\| = \|x - p_K(x)\|, \end{aligned}$$

which contradicts the fact that  $p_K(x)$  is the nearest point in  $K$  to  $x$ . This contradiction completes the proof.

**THEOREM 3.6** [1, 84]. *Let  $K$  be a Chebyshev set in a smooth normed linear space  $(X, \|\cdot\|)$ . If  $K$  is a sun, then  $K$  is convex.*

**PROOF.** Looking for a contradiction, suppose that  $K$  is not convex. Since  $K$  is closed, Proposition 2.12 tells us that there exist  $x, y \in K$  such that  $\frac{1}{2}(x + y) \notin K$ . Let

$$z := p_K\left(\frac{x + y}{2}\right) \quad \text{and} \quad x_0 := \frac{x + y}{2} - z.$$

By Lemma 3.2,

$$z + x_0 \in B(z + x_0; \|x_0\|) \subseteq J := \{w \in X : x^*(z) < x^*(w)\} \subseteq \bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|),$$

where  $x^*$  is the Gâteaux derivative of the norm at  $x_0$ . We claim that  $\bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|)$  is contained in  $X \setminus K$ . For  $1 < n \in \mathbb{N}$ ,  $z + x_0 = [1 - (1/n)]z + [1/n](z + nx_0) \in (z, z + nx_0)$ . Since  $K$  is a sun,

$$z = p_K(z + x_0) = p_K(z + nx_0) \quad \text{for all } n \in \mathbb{N}.$$

Therefore,  $d(z + nx_0, K) = \|(z + nx_0) - z\| = n\|x_0\|$  for all  $n \in \mathbb{N}$ . Thus,  $B(z + nx_0; n\|x_0\|)$  is contained in  $X \setminus K$  for all  $n \in \mathbb{N}$ . From Lemma 3.1 it follows that either  $x$  or  $y$  is in  $J \subseteq \bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|) \subseteq X \setminus K$ . However, this is impossible, since both  $x$  and  $y$  are members of  $K$ . □

To see that smoothness of the normed linear space is required, consider again Example 2.11. The key point is that since the unit ball in  $(\mathbb{R}^2, \|\cdot\|_1)$  has sharp corners, that is, the norm is not smooth, the union of the balls discussed in Lemma 3.2 does not have to fill out a whole half space.

In order to present our first proof concerning the convexity of Chebyshev sets we need to apply the following modification of Schauder’s fixed point theorem. To obtain this modification we use the fact that the closed convex hull of a compact subset of a Banach space is compact; see [54] or [28, page 416].

**THEOREM 3.7** [36, Theorem 7.6]. *Let  $K$  be a nonempty closed convex subset of a Banach space  $(X, \|\cdot\|)$ . If  $f : K \rightarrow K$  is a continuous function and  $\overline{f(K)}$  is compact, then  $f$  has a fixed point.*

**THEOREM 3.8** [84]. *Let  $(X, \|\cdot\|)$  be a smooth Banach space. Then every boundedly compact Chebyshev set in  $X$  is convex. In particular, every Chebyshev subset of a smooth finite-dimensional normed linear space is convex.*

**PROOF.** Let  $K$  be a Chebyshev set in  $X$ . In light of Theorem 3.6, it suffices to show that  $K$  is a sun. To this end, let  $x \in X \setminus K$  and let  $I_x := \{\lambda \geq 0 : p_K(x) = p_K(p_K(x) + \lambda(x - p_K(x)))\}$ . By Proposition 2.20,  $I_x$  is a nonempty closed interval. Looking for a contradiction, suppose that  $K$  is not a sun, that is,  $I_x \neq [0, \infty)$ . Then there exists  $\lambda_0 \geq 1$  such that  $I_x = [0, \lambda_0]$ . Let  $x_0 := p_K(x) + \lambda_0(x - p_K(x))$ . Note that  $x_0 \notin K$  since  $p_K(x_0) = p_K(x) \neq p_K(x) + \lambda_0(x - p_K(x)) = x_0$ , where we make use of the fact that  $K$  is closed and  $x \notin K$ . Choose  $\varepsilon \in (0, d(x_0, K))$  so that  $p_K(B[x_0; \varepsilon])$  is bounded, and consider the function  $f : B[x_0; \varepsilon] \rightarrow B[x_0; \varepsilon]$  defined by

$$f(x) := x_0 + \varepsilon \frac{x_0 - p_K(x)}{\|x_0 - p_K(x)\|}.$$

Since  $K$  is boundedly compact, we have by Theorem 2.26 that  $p_K$  is continuous. It then follows that (i)  $f$  is continuous and (ii)  $\overline{f(B[x_0; \varepsilon])} \subseteq B[x_0; \varepsilon]$  is compact, since  $f(B[x_0; \varepsilon])$  is contained in  $\overline{g(p_K(B[x_0; \varepsilon]))}$ , where  $g : X \setminus \{x_0\} \rightarrow X$  is the continuous function defined by

$$g(y) := x_0 + \varepsilon \frac{x_0 - y}{\|x_0 - y\|},$$

and  $\overline{p_K(B[x_0; \varepsilon])}$  is compact. Therefore, by Theorem 3.7, there exists an element  $x_\infty \in B[x_0; \varepsilon]$  such that  $f(x_\infty) = x_\infty$ . Rearranging the equation  $f(x_\infty) = x_\infty$ , we see that

$$x_0 = \left( \frac{\|x_0 - p_K(x_\infty)\|}{\|x_0 - p_K(x_\infty)\| + \varepsilon} \right) x_\infty + \left( \frac{\varepsilon}{\|x_0 - p_K(x_\infty)\| + \varepsilon} \right) p_K(x_\infty).$$

Hence,  $x_0 \in (x_\infty, p_K(x_\infty))$ , and so by Proposition 2.17,  $p_K(x_\infty) = p_K(x_0) = p_K(x)$ . On the other hand,

$$\begin{aligned} x_\infty &= x_0 + \varepsilon \frac{x_0 - p_K(x_\infty)}{\|x_0 - p_K(x_\infty)\|} \\ &= x_0 + \varepsilon \frac{x_0 - p_K(x)}{\|x_0 - p_K(x)\|} \quad \text{since } p_K(x_\infty) = p_K(x) \\ &= p_K(x) + \lambda_0(x - p_K(x)) + \varepsilon \frac{x - p_K(x)}{\|x - p_K(x)\|} \\ &= p_K(x) + \left( \lambda_0 + \frac{\varepsilon}{\|x - p_K(x)\|} \right) (x - p_K(x)). \end{aligned}$$

Thus, we have that  $\lambda_0 + \varepsilon/\|x - p_K(x)\| \in I_x$ . However, this is impossible since  $\lambda_0$  is strictly smaller than  $\lambda_0 + \varepsilon/\|x - p_K(x)\|$ . Therefore,  $K$  is a sun. □

**COROLLARY 3.9 [33].** *Let  $(X, \|\cdot\|)$  be a strictly convex smooth finite-dimensional normed linear space. Then a subset of  $X$  is a Chebyshev set if and only if it is nonempty, closed and convex.*

**PROOF.** This result follows directly from Corollary 2.51 and Theorem 3.8. □

**3.2. Proof using inversion in the unit sphere.** In this subsection we limit our discussion to Hilbert spaces. In particular, we use ‘inversion in the unit sphere’ to relate nearest points to farthest points. In this way, we prove that a Chebyshev set in a Hilbert space with a continuous metric projection mapping is convex.

**DEFINITION 3.10.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $K$  be a nonempty bounded subset of  $X$ . For any point  $x \in X$ , we define  $r(x, K) := \sup_{y \in K} \|x - y\|$ . We refer to the map  $x \mapsto r(x, K)$  as the radial function for  $K$ .

**PROPOSITION 3.11.** Let  $K$  be a nonempty bounded subset of a normed linear space  $(X, \|\cdot\|)$ . Then the radial function for  $K$  is convex and nonexpansive (and hence continuous).

**PROOF.** For each  $k \in K$ , let  $g_k : X \rightarrow [0, \infty)$  be defined by  $g_k(x) := \|x - k\|$ . Then each  $g_k$  is convex and nonexpansive. Now, for each  $x \in X$ ,  $r(x, K) = \sup\{g_k(x) : k \in K\}$ . Thus, as the pointwise supremum of a family of convex nonexpansive mappings, the radial function is itself convex and nonexpansive.  $\square$

**DEFINITION 3.12.** Let  $(X, \|\cdot\|)$  be a normed linear space and let  $K$  be a subset of  $X$ . We define a set-valued mapping  $F_K : X \rightarrow \mathcal{P}(K)$  by  $F_K(x) := \{y \in K : \|x - y\| = r(x, K)\}$  if  $K$  is nonempty and bounded, and by  $F_K(x) = \emptyset$  otherwise. We refer to the elements of  $F_K(x)$  as the farthest points from  $x$  in  $K$ .

We say that  $K$  is a remotal set if  $F_K(x)$  is a nonempty for each  $x \in X$ . Furthermore, we say that  $K$  is a uniquely remotal set if  $F_K(x)$  is a singleton for each  $x \in X$ .

For a uniquely remotal set we define a mapping  $f_K : X \rightarrow K$  as the mapping that assigns to each  $x \in X$  the unique element of  $F_K(x)$ .

We will refer to both  $F_K$  and  $f_K$  as the farthest point map (for  $K$ ).

**REMARK 3.13.** It follows from the definition that every remotal set is nonempty and bounded.

To begin, we need the following technical result concerning the farthest point mapping.

**LEMMA 3.14.** Let  $K$  be a remotal set in a normed linear space  $(X, \|\cdot\|)$ . Let  $x \in X$ ,  $z \in F_K(x)$  and suppose that  $x \notin F_K(x)$ . If for each  $\lambda \in (0, 1)$ ,  $x_\lambda := x + \lambda(z - x)$ , then

$$r(x_\lambda, K) \leq r(x, K) - \|x_\lambda - x\| \left(1 - \frac{\|z - z_\lambda\|}{\|x - z\|}\right),$$

where  $z_\lambda \in F_K(x_\lambda)$  for each  $\lambda \in (0, 1)$ .

**PROOF.** Let  $w_\lambda := \lambda z_\lambda + (1 - \lambda)x \in (x, z_\lambda)$ . Then

$$\begin{aligned} r(x_\lambda, K) &= \|x_\lambda - z_\lambda\| \\ &\leq \|x_\lambda - w_\lambda\| + \|w_\lambda - z_\lambda\| \quad \text{by the triangle inequality} \\ &= \lambda\|z - z_\lambda\| + (1 - \lambda)\|x - z_\lambda\| \\ &\leq \lambda\|z - z_\lambda\| + (1 - \lambda)\|x - z\| \quad \text{since } z \in F_K(x) \text{ and } z_\lambda \in K. \end{aligned}$$

Therefore,  $r(x_\lambda, K) \leq r(x, K) - \lambda(\|x - z\| - \|z - z_\lambda\|)$ . Now, from the definition of  $x_\lambda$ , we have that  $\lambda = \|x_\lambda - x\|/\|x - z\|$ . Thus,  $r(x_\lambda, K) \leq r(x, K) - \|x_\lambda - x\|(1 - \|z - z_\lambda\|/\|x - z\|)$ .  $\square$

**COROLLARY 3.15.** *Let  $K$  be a remotal set in a normed linear space  $(X, \|\cdot\|)$ . Suppose that  $x \in X$  and  $x \notin F_K(x)$ . If the farthest point map for  $K$  is continuous at  $x \in X$ , then*

$$\lim_{\lambda \rightarrow 0^+} \frac{\|x_\lambda - z_\lambda\| - \|x - z\|}{\|x_\lambda - x\|} = \lim_{\lambda \rightarrow 0^+} \frac{r(x_\lambda, K) - r(x, K)}{\|x_\lambda - x\|} = -1,$$

where  $\{z\} = F_K(x)$ ,  $x_\lambda := x + \lambda(z - x)$  and  $z_\lambda \in F_K(x_\lambda)$  for each  $\lambda \in (0, 1)$ .

**PROOF.** We see that for any  $\lambda \in (0, 1)$ ,

$$\begin{aligned} -1 &= -\frac{\|x_\lambda - x\|}{\|x_\lambda - x\|} \leq \frac{r(x_\lambda, K) - r(x, K)}{\|x_\lambda - x\|} \quad \text{since the radial function for } K \text{ is nonexpansive} \\ &\leq -\left(1 - \frac{\|z - z_\lambda\|}{\|x - z\|}\right) \quad \text{by Lemma 3.14.} \end{aligned}$$

Since  $\lim_{\lambda \rightarrow 0^+} x_\lambda = x$  and the farthest point map for  $K$  is continuous at  $x$ , the right-hand side of the above inequality converges to  $-1$  as  $\lambda \rightarrow 0^+$ . Hence,  $\lim_{\lambda \rightarrow 0^+} (r(x_\lambda, M) - r(x, M))/\|x_\lambda - x\| = -1$ .  $\square$

The next result that we require is well known in optimisation theory.

**PROPOSITION 3.16.** *Let  $(X, \|\cdot\|)$  be a reflexive Banach space and let  $f : X \rightarrow [0, \infty)$  be a continuous convex function such that  $\liminf_{\|x\| \rightarrow \infty} f(x) = \infty$ . Then  $\operatorname{argmin}(f) \neq \emptyset$ , that is,  $f$  attains its global minimum.*

**PROOF.** Since  $f$  is continuous and convex,  $f$  is lower semicontinuous with respect to the weak topology on  $X$  (see Lemma 2.48). Let  $K := \{x \in X : f(x) \leq f(0)\}$ . Then:

- (i)  $K \neq \emptyset$  since  $0 \in K$ ;
- (ii)  $K$  is bounded since  $\liminf_{\|x\| \rightarrow \infty} f(x) = \infty$ ;
- (iii)  $K$  is weakly compact, since it is closed and convex (hence weakly closed) and bounded.

Therefore, by Proposition 2.49,  $\emptyset \neq \operatorname{argmin}(f|_K)$ . Thus,  $\emptyset \neq \operatorname{argmin}(f|_K) = \operatorname{argmin}(f)$ .  $\square$

Our next result shows that within the context of reflexive Banach spaces there are very few uniquely remotal sets that possess a continuous farthest point mapping.

**PROPOSITION 3.17 [10].** *Let  $(X, \|\cdot\|)$  be a reflexive Banach space and let  $K$  be a uniquely remotal set in  $X$ . If the farthest point mapping,  $x \mapsto f_K(x)$ , is continuous on  $X$ , then  $K$  is a singleton.*

**PROOF.** It is easy to see that  $\liminf_{\|x\| \rightarrow \infty} r(x, K) = \infty$ . Thus, by Proposition 3.16, there exists  $x_0 \in X$  such that  $r(x_0, K) \leq r(x, K)$  for all  $x \in X$ . However, it follows from the continuity of  $f_K$  and Corollary 3.15, that unless  $x_0 = f_K(x_0)$  we have a contradiction. Thus,  $x_0 = f_K(x_0)$ , which implies that  $K$  is a singleton.  $\square$

One can show, though we will not do so here, that as with the metric projection map, the farthest point map has a closed graph. Hence, in finite-dimensional spaces the farthest point mapping of every uniquely remotal set is continuous. For more information on remotal and uniquely remotal sets, see [9, 60].

**COROLLARY 3.18 [50].** *Let  $(X, \|\cdot\|)$  be a finite-dimensional normed linear space. Then every uniquely remotal subset of  $X$  is a singleton.*

**DEFINITION 3.19.** Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space. We define  $i : X \setminus \{0\} \rightarrow X \setminus \{0\}$  by  $i(x) := x/\|x\|^2$  and we refer to this map as inversion in the unit sphere.

*Note.* It is clear that  $i$  is continuous. Furthermore, since  $i$  is invertible, with  $i^{-1} = i$ , we see that  $i$  is a homeomorphism.

**LEMMA 3.20.** *For any inner product space  $(X, \langle \cdot, \cdot \rangle)$ , and any  $x \in X \setminus \{0\}$ ,*

$$i(B[x; \|x\|] \setminus \{0\}) \subseteq \{y \in X : \frac{1}{2} \leq \langle y, x \rangle\}.$$

**PROOF.** Let  $w \in i(B[x; \|x\|] \setminus \{0\})$ . Then  $w = i(z)$  for some  $z \in B[x; \|x\|] \setminus \{0\}$ . Now,

$$\begin{aligned} z \in B[x; \|x\|] \setminus \{0\} &\Leftrightarrow z \neq 0 \text{ and } \|z - x\| \leq \|x\| \\ &\Leftrightarrow z \neq 0 \text{ and } \|z - x\|^2 \leq \|x\|^2 \\ &\Leftrightarrow z \neq 0 \text{ and } \|z\|^2 - 2\langle z, x \rangle + \|x\|^2 \leq \|x\|^2 \\ &\Leftrightarrow z \neq 0 \text{ and } \|z\|^2 \leq 2\langle z, x \rangle \\ &\Leftrightarrow \frac{1}{2} \leq \left\langle \frac{z}{\|z\|^2}, x \right\rangle = \langle w, x \rangle \\ &\Rightarrow w \in \left\{ y \in X : \frac{1}{2} \leq \langle y, x \rangle \right\}. \quad \square \end{aligned}$$

**PROPOSITION 3.21.** *Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space. Let  $x \in X$  and let  $\delta > \|x\|$ . Then*

$$i(B[x; \delta] \setminus \{0\}) = X \setminus B(-\delta'x; \delta'\delta) \quad \text{where } \delta' := \frac{1}{\delta^2 - \|x\|^2}.$$

*In particular, if  $y \notin B[x; \delta]$ , then  $i(y) \in B(-\delta'x; \delta'\delta)$ . Furthermore,  $i(S[x; \delta]) = S[-\delta'x; \delta\delta']$ .*

**PROOF.** Let  $w \in X \setminus B(-\delta'x; \delta'\delta)$ . Then  $w = i(y)$  for some  $y \in X \setminus \{0\}$ . Therefore,

$$\begin{aligned} w \in X \setminus B(-\delta'x; \delta'\delta) &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \left\| \frac{y}{\|y\|^2} - \frac{-x}{\delta^2 - \|x\|^2} \right\| \geq \frac{\delta}{\delta^2 - \|x\|^2} \\ &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \|y(\delta^2 - \|x\|^2) + x\|y\|^2 \geq \delta\|y\|^2 \\ &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \|y(\delta^2 - \|x\|^2) + x\|y\|^2 \geq \delta^2\|y\|^4 \\ &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \|y\|^2(\delta^2 - \|x\|^2)^2 + 2(\delta^2 - \|x\|^2)\|y\|^2\langle y, x \rangle + \|x\|^2\|y\|^4 \geq \delta^2\|y\|^4 \end{aligned}$$

$$\begin{aligned}
 &\Leftrightarrow y \in X \setminus \{0\} \text{ and } (\delta^2 - \|x\|^2)^2 + 2(\delta^2 - \|x\|^2)\langle y, x \rangle + \|x\|^2\|y\|^2 \geq \delta^2\|y\|^2 \\
 &\Leftrightarrow y \in X \setminus \{0\} \text{ and } (\delta^2 - \|x\|^2) + 2\langle y, x \rangle \geq \|y\|^2 \\
 &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \|y\|^2 - 2\langle y, x \rangle + \|x\|^2 \leq \delta^2 \\
 &\Leftrightarrow y \in X \setminus \{0\} \text{ and } \|y - x\|^2 \leq \delta^2 \\
 &\Leftrightarrow y \in B[x; \delta] \setminus \{0\} \\
 &\Leftrightarrow w \in i(B[x; \delta] \setminus \{0\}).
 \end{aligned}$$

Thus,  $i(B[x; \delta] \setminus \{0\}) = X \setminus B(-\delta'x; \delta'\delta)$ . Finally, to see that  $i(S[x; \delta]) = S[-\delta'x; \delta\delta']$ , observe that all the implications (apart from the first one and the last two) in the above working still hold if we replace each inequality by an equality.  $\square$

**THEOREM 3.22** [4, page 238]. *Let  $(X, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $A$  be a Chebyshev set in  $X$ . If the metric projection mapping  $p_A$  is continuous on  $X$ , then  $A$  is convex.*

**PROOF.** Suppose, looking for a contradiction, that  $A$  is not convex. Let  $x_0 \in \text{co}(A) \setminus A$  and define  $K := A - x_0$ . It is clear that  $0 \in \text{co}(K) \setminus K$ . Furthermore,  $K$  is a Chebyshev set with a continuous metric projection map. Indeed, for each  $x \in X$ ,  $p_K(x) = p_A(x + x_0) - x_0$ . To see this, first note that  $p_K(x) \in K$  and for any  $k \in K \setminus \{p_K(x)\}$ ,

$$\begin{aligned}
 \|p_K(x) - x\| &= \|p_A(x + x_0) - (x + x_0)\| \\
 &< \|(k + x_0) - (x + x_0)\| \quad \text{since } k + x_0 \in A \text{ and } k + x_0 \neq p_A(x + x_0) \\
 &= \|x - k\|.
 \end{aligned}$$

Since  $p_A$  is continuous, we see that  $p_K$  is continuous.

Let  $G := i(K)$ . Since  $K$  is a Chebyshev set (and so nonempty and closed) and  $0 \notin K$ , it follows that  $G$  is a nonempty closed and bounded subset of  $X \setminus \{0\}$ . We claim that  $G$  is a uniquely remotal set in  $X$  whose farthest point mapping is continuous. Let  $x \in X$ . It follows from the definition of  $r(x, G)$  that  $G \subseteq B[x; r(x, G)] \setminus \{0\}$ , and moreover,  $G \not\subseteq B[x; \delta]$  for any  $\delta \in (0, r(x, G))$ . We now show that  $\|x\| < r(x, G)$ . Looking for a contradiction, suppose that  $\|x\| \geq r(x, G)$ . By Lemma 3.20,

$$K = i(i(K)) = i(G) \subseteq i(B[x; r(x, G)] \setminus \{0\}) \subseteq i(B[x; \|x\|] \setminus \{0\}) \subseteq \{z \in X : \frac{1}{2} \leq \langle z, x \rangle\}.$$

Thus,  $0 \in \text{co}(K) \subseteq \{z \in X : \frac{1}{2} \leq \langle z, x \rangle\}$ , which is impossible. Therefore, we must have that  $\|x\| < r(x, G)$ . Next we claim that

$$d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right) = \frac{r(x, G)}{r^2(x, G) - \|x\|^2}.$$

From Proposition 3.21,

$$K = i(G) \subseteq i(B[x; r(x, G)] \setminus \{0\}) \subseteq X \setminus B\left(\frac{-x}{r^2(x, G) - \|x\|^2}; \frac{r(x, G)}{r^2(x, G) - \|x\|^2}\right).$$

Therefore,

$$\frac{r(x, G)}{r^2(x, G) - \|x\|^2} \leq d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right).$$

Next, let  $\delta \in (\|x\|, r(x, G))$ . By our earlier remark, we can find  $g_\delta \in G \setminus B[x; \delta]$ . Applying Proposition 3.21 again, we have that

$$i(g_\delta) \in K \cap B\left(\frac{-x}{\delta^2 - \|x\|^2}; \frac{\delta}{\delta^2 - \|x\|^2}\right).$$

Hence,

$$\begin{aligned} \left\| i(g_\delta) - \frac{-x}{r^2(x, G) - \|x\|^2} \right\| &\leq \left\| i(g_\delta) - \frac{-x}{\delta^2 - \|x\|^2} \right\| + \left\| \frac{-x}{\delta^2 - \|x\|^2} - \frac{-x}{r^2(x, G) - \|x\|^2} \right\| \\ &< \frac{\delta}{\delta^2 - \|x\|^2} + \frac{\|x\|(r^2(x, G) - \delta^2)}{(\delta^2 - \|x\|^2)(r^2(x, G) - \|x\|^2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right) &\leq \left\| i(g_\delta) - \frac{-x}{r^2(x, G) - \|x\|^2} \right\| \\ &< \frac{\delta}{\delta^2 - \|x\|^2} + \frac{\|x\|(r^2(x, G) - \delta^2)}{(\delta^2 - \|x\|^2)(r^2(x, G) - \|x\|^2)}. \end{aligned}$$

Taking the limit as  $\delta \rightarrow r(x, G)$ , we obtain

$$d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right) \leq \frac{r(x, G)}{r^2(x, G) - \|x\|^2}.$$

Thus,

$$d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right) = \frac{r(x, G)}{r^2(x, G) - \|x\|^2}$$

as claimed. Since

$$\begin{aligned} p_K\left(\frac{-x}{r^2(x, G) - \|x\|^2}\right) &\in K \cap S\left[\frac{-x}{r^2(x, G) - \|x\|^2}; \frac{r(x, G)}{r^2(x, G) - \|x\|^2}\right] \\ &= K \cap i(S[x; r(x, G)]) = i(G) \cap i(S[x; r(x, G)]), \end{aligned}$$

we have that

$$i\left(p_K\left(\frac{-x}{r^2(x, G) - \|x\|^2}\right)\right) \in G \cap S[x; r(x, G)].$$

Thus,

$$i\left(p_K\left(\frac{-x}{r^2(x, G) - \|x\|^2}\right)\right) \in F_G(x).$$

Moreover, since  $i$  is a bijection and  $K$  is a Chebyshev set, it follows that

$$F_G(x) = \left\{ i\left(p_K\left(\frac{-x}{r^2(x, G) - \|x\|^2}\right)\right) \right\},$$

and so  $G$  is a uniquely remotal set. Since  $p_K$  is continuous on  $X$ , it follows that  $f_G$  is continuous on  $X$ . Finally, it follows from Proposition 3.17 that  $G$  is a singleton, and so  $K$  is a singleton. However, this contradicts the fact that, by construction,  $K$  is nonconvex. □

**3.3. A proof using convex analysis.** Again we limit ourselves to Hilbert spaces, but this time we make use of tools from convex analysis.

**DEFINITION 3.23.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ . We say that a sequence  $(x_n)_{n=1}^\infty$  in  $X$  is minimising if  $\lim_{n \rightarrow \infty} f(x_n) = \inf_{x \in X} f(x)$ .

**DEFINITION 3.24.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ . We say that  $f$  has a strong minimum at  $x \in X$  if  $f$  has a global minimum at  $x$  and each minimising sequence  $(x_n)_{n=1}^\infty$  in  $X$  converges to  $x$ .

**DEFINITION 3.25.** Let  $(X, \|\cdot\|)$  be a normed linear space and let  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper function that is bounded below by a continuous linear functional. We define a function  $\overline{\text{co}}(f) : X \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$\overline{\text{co}}(f)(x) := \sup\{\psi(x) : \psi : X \rightarrow \mathbb{R} \text{ is continuous, convex and } \psi(y) \leq f(y) \text{ for all } y \in X\}.$$

It is immediately clear from this definition that  $\overline{\text{co}}(f)$  is convex, lower semicontinuous, and  $\overline{\text{co}}(f)(x) \leq f(x)$  for all  $x \in X$ . Moreover, if  $x^* \in X^*$ , then  $\overline{\text{co}}(f - x^*) = \overline{\text{co}}(f) - x^*$ .

**THEOREM 3.26 [76].** Let  $(X, \|\cdot\|)$  be a normed linear space. Suppose that  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  and  $\liminf_{\|x\| \rightarrow \infty} f(x)/\|x\| > 0$ . If  $f$  has a strong minimum at  $x_0 \in X$ , then  $\text{argmin}(\overline{\text{co}}(f)) = \text{argmin}(f)$ .

**PROOF.** Clearly,  $\{x_0\} = \text{argmin}(f) \subseteq \text{argmin}(\overline{\text{co}}(f))$  and  $f(x_0) = \overline{\text{co}}(f)(x_0)$ . So it suffices to show that

$$\text{argmin}(\overline{\text{co}}(f)) \subseteq \text{argmin}(f) = \{x_0\}.$$

To do this, we show that  $\text{argmin}(\overline{\text{co}}(f)) \subseteq B[x_0; \varepsilon]$  for each  $\varepsilon > 0$ . To this end, let  $\varepsilon > 0$ . We claim that for some  $n \in \mathbb{N}$ , the continuous convex function  $c_n : X \rightarrow \mathbb{R}$ , defined by

$$c_n(x) := f(x_0) + \frac{1}{n}d(x, B[x_0; \varepsilon]),$$

satisfies the property that  $c_n(x) \leq f(x)$  for all  $x \in X$ . Suppose, looking for a contradiction, that this is not the case. Then, for each  $n \in \mathbb{N}$ , there exists  $x_n \in X$  such that  $f(x_n) < c_n(x_n)$ .

Since  $\liminf_{\|x\| \rightarrow \infty} f(x)/\|x\| > 0$ , the sequence  $(x_n)_{n=1}^\infty$  is bounded. Therefore,  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ . However, since  $x_0$  is a strong minimum of  $f$ , we must have that  $\lim_{n \rightarrow \infty} x_n = x_0$ . Hence, for  $n$  sufficiently large,  $x_n \in B[x_0; \varepsilon]$ , and so  $f(x_n) < c_n(x_n) = f(x_0) + (1/n)d(x_n, B[x_0; \varepsilon]) = f(x_0)$ , which is clearly impossible since  $f(x_0) = \inf_{x \in X} f(x)$ . This proves the claim. Finally, since  $c_n$  is continuous and convex,  $c_n(x) \leq \overline{\text{co}}(f)(x)$  for all  $x \in X$ . This implies that  $\text{argmin}(\overline{\text{co}}(f))$  is contained in  $B[x_0; \varepsilon]$ , since  $\overline{\text{co}}(f)(x_0) = f(x_0) < c_n(x) \leq \overline{\text{co}}(f)(x)$  for all  $x \notin B[x_0; \varepsilon]$ .  $\square$

**THEOREM 3.27 [4, 76].** Let  $(X, \langle \cdot, \cdot \rangle)$  be a Hilbert space and let  $K$  be a Chebyshev set in  $X$ . If for each  $x \in X \setminus K$ , the function  $k \mapsto \|x - k\|$ , defined on  $K$ , has a strong minimum at  $p_K(x)$ , then  $K$  is convex.



**PROOF.** Define  $f : X \rightarrow [0, \infty]$  by

$$f(x) = \begin{cases} \|x\|^2 & \text{if } x \in K, \\ \infty & \text{if } x \notin K. \end{cases}$$

Looking for a contradiction, suppose that  $K$  is not convex. Then there exists

$$z \in \text{co}(K) \setminus K \subseteq \text{Dom}(\overline{\text{co}}(f)) \setminus K,$$

as  $\text{Dom}(\overline{\text{co}}(f))$  is convex and contains  $\text{Dom}(f) = K$ . Now, by the Brøndsted–Rockafellar theorem (see [16] or [62, Theorem 3.17]), there exists a point  $z_0 \in \text{Dom}(\partial(\overline{\text{co}}(f))) \setminus K$ . Let  $x^* \in \partial(\overline{\text{co}}(f))(z_0)$ . Then  $z_0 \in \text{argmin}(\overline{\text{co}}(f) - x^*)$  and

$$\liminf_{\|x\| \rightarrow \infty} \frac{(f - x^*)(x)}{\|x\|} = \infty.$$

By Riesz’s representation theorem (see [67] or, more recently, [74, page 248]), there exists  $x \in X$  such that  $x^*(y) = \langle y, x \rangle$  for all  $y \in X$ . Therefore,

$$(f - x^*)(k) = f(k) - \langle k, x \rangle = \left\| k - \frac{x}{2} \right\|^2 - \frac{\|x\|^2}{4}$$

for all  $k \in K = \text{Dom}(f - x^*)$ . Hence, it follows that  $f - x^*$  has a strong minimum at  $p_K(x/2)$ . Putting all of this together, we obtain the following:

$$\begin{aligned} z_0 \in \text{argmin}(\overline{\text{co}}(f) - x^*) &= \text{argmin}(\overline{\text{co}}(f - x^*)) \\ &= \text{argmin}(f - x^*) \quad \text{by Theorem 3.26} \\ &= \left\{ p_K\left(\frac{x}{2}\right) \right\} \subseteq K. \end{aligned}$$

However, this is impossible, since  $z_0 \notin K$ . Hence,  $K$  is convex. □

The drawback of the previous result is that it is difficult to determine whether or not the function  $k \mapsto \|x - k\|$  has a strong minimum at  $p_K(x)$ . So next we try to alleviate this problem.

**COROLLARY 3.28.** *Let  $(X, \langle \cdot, \cdot \rangle)$  be a Hilbert space and let  $K$  be a Chebyshev set in  $X$ . If  $x \mapsto d(x, K)$  is Fréchet differentiable on  $X \setminus K$ , then  $K$  is convex.*

**PROOF.** Let  $x \in X \setminus K$ . We will show that the function  $k \mapsto \|x - k\|$ , defined on  $K$ , has a strong minimum at  $p_K(x)$ . Since every subsequence of a minimising sequence is again a minimising sequence, it is sufficient to show that for every minimising sequence  $(z_n)_{n=1}^\infty$  in  $K$  there exists a subsequence  $(z_{n_k})_{k=1}^\infty$  of  $(z_n)_{n=1}^\infty$  such that  $\lim_{k \rightarrow \infty} z_{n_k} = p_K(x)$ . To this end, let  $(z_n)_{n=1}^\infty$  be a sequence in  $K$  with  $\lim_{n \rightarrow \infty} \|x - z_n\| = d(x, K)$ . Since  $(z_n)_{n=1}^\infty$  is bounded and  $X$  is reflexive, there exist a subsequence  $(z_{n_k})_{k=1}^\infty$  of  $(z_n)_{n=1}^\infty$  and an element  $z \in X$  such that  $(z_{n_k})_{k=1}^\infty$  converges to  $z$  with respect to the weak topology on  $X$ . Let  $x^*$  be the Fréchet derivative of the distance function for  $K$  at  $x$ . By

**Lemma 2.41.**  $\|x^*\| \leq 1$ . For each  $n \in \mathbb{N}$ , let  $\lambda_n := \sqrt{\|x - z_n\| - d(x, K) + 1/n}$ . Then  $\lim_{k \rightarrow \infty} (\|x - z_{n_k}\| - d(x, K))/\lambda_{n_k} = 0$  and so

$$\begin{aligned} x^*(z - x) &= \lim_{k \rightarrow \infty} x^*(z_{n_k} - x) \\ &= \lim_{k \rightarrow \infty} \frac{d(x + \lambda_{n_k}(z_{n_k} - x), K) - d(x, K)}{\lambda_{n_k}} \\ &\leq \lim_{k \rightarrow \infty} \frac{\|x + \lambda_{n_k}(z_{n_k} - x) - z_{n_k}\| - d(x, K)}{\lambda_{n_k}} \\ &= \lim_{k \rightarrow \infty} \frac{(1 - \lambda_{n_k})\|x - z_{n_k}\| - d(x, K)}{\lambda_{n_k}} \\ &= \lim_{k \rightarrow \infty} \left[ -\|x - z_{n_k}\| + \frac{\|x - z_{n_k}\| - d(x, K)}{\lambda_{n_k}} \right] = -d(x, K). \end{aligned}$$

Therefore,  $d(x, K) \leq x^*(x - z)$ , which implies that

$$d(x, K) \leq x^*(x - z) \leq \|x^*\| \|x - z\| \leq \|x - z\| \leq \liminf_{k \rightarrow \infty} \|x - z_{n_k}\| = d(x, K)$$

since  $(x - z_{n_k})_{k=1}^\infty$  converges to  $(x - z)$  with respect to the weak topology on  $X$  and the norm is lower semicontinuous with respect to the weak topology on  $X$ . Thus,  $(x - z_{n_k})_{k=1}^\infty$  converges to  $(x - z)$  with respect to the weak topology on  $X$  and  $\lim_{k \rightarrow \infty} \|x - z_{n_k}\| = \|x - z\| = d(x, K)$ . Hence, since the norm is a Kadec–Klee norm,  $(x - z_{n_k})_{k=1}^\infty$  converges in norm to  $(x - z)$ , and so  $z = \lim_{k \rightarrow \infty} z_{n_k}$ . Therefore,  $z \in K$ . However, since  $K$  is a Chebyshev set and  $\|x - z\|$  equals  $d(x, K)$ , it must be the case that  $z = p_K(x)$ . □

Of course we are now left wondering when the distance function  $x \mapsto d(x, K)$  is Fréchet differentiable on  $X \setminus K$ . To solve this we consider the following corollary.

**COROLLARY 3.29.** *Let  $(X, \langle \cdot, \cdot \rangle)$  be a Hilbert space and let  $K$  be a Chebyshev set in  $X$ . If the metric projection map,  $x \mapsto p_K(x)$ , is continuous on  $X \setminus K$ , then the distance function,  $x \mapsto d(x, K)$ , is Fréchet differentiable on  $X \setminus K$ , and so  $K$  is convex.*

**PROOF.** This follows directly from [4, Proposition]. □

**REMARK 3.30.** Several authors have approached the question of the convexity of Chebyshev sets by considering the differentiability of the associated distance function; see [37, 41, 89].

**3.4. Vlasov’s theorem.** In this subsection we prove Vlasov’s theorem, which has some of the weakest assumptions for a Chebyshev set to be convex, requiring only completeness, continuity of the metric projection map and strict convexity of the dual space. First, we require a number of preliminary results.

**LEMMA 3.31** [40, page 238]. *Let  $K$  be a proximal set in a normed linear space  $(X, \|\cdot\|)$ . Let  $x \in X \setminus K$  and  $z \in P_K(x)$ . If for each  $\lambda > 0$ ,  $x_\lambda := x + \lambda(x - z)$ , then*

$$d(x, K) + \|x_\lambda - x\| \left( 1 - \frac{\|z - z_\lambda\|}{\|x - z\|} \right) \leq d(x_\lambda, K),$$

where  $z_\lambda \in P_K(x_\lambda)$  for each  $\lambda > 0$ .

**PROOF.** Let  $w_\lambda := (1/(1 + \lambda))x_\lambda + (\lambda/(1 + \lambda))z_\lambda \in (x_\lambda, z_\lambda)$ . Then

$$\begin{aligned} d(x, K) &= \|x - z\| \\ &\leq \|x - z_\lambda\| \quad \text{since } z \in P_K(x) \text{ and } z_\lambda \in K \\ &\leq \|x - w_\lambda\| + \|w_\lambda - z_\lambda\| \quad \text{by the triangle inequality} \\ &= \left(\frac{\lambda}{1 + \lambda}\right)\|z_\lambda - z\| + \left(\frac{1}{1 + \lambda}\right)\|x_\lambda - z_\lambda\|. \end{aligned}$$

Therefore,  $d(x, K) + \lambda(d(x, K) - \|z_\lambda - z\|) \leq d(x_\lambda, K)$ . Now, from the definition of  $x_\lambda$ , we have that  $\lambda = \|x_\lambda - x\|/\|x - z\|$ . Thus,  $d(x, K) + \|x_\lambda - x\|(1 - \|z - z_\lambda\|/\|x - z\|) \leq d(x_\lambda, K)$ . □

**COROLLARY 3.32** [40, page 238]. *Let  $K$  be a proximal set in a normed linear space  $(X, \|\cdot\|)$ . If the metric projection map is continuous at  $x \in X \setminus K$ , then*

$$\lim_{\lambda \rightarrow 0^+} \frac{\|x_\lambda - z_\lambda\| - \|x - z\|}{\|x_\lambda - x\|} = \lim_{\lambda \rightarrow 0^+} \frac{d(x_\lambda, K) - d(x, K)}{\|x_\lambda - x\|} = 1,$$

where  $\{z\} = P_K(x)$ ,  $x_\lambda := x + \lambda(x - z)$  and  $z_\lambda \in P_K(x_\lambda)$  for each  $\lambda > 0$ .

**PROOF.** We see that for any  $\lambda > 0$ ,

$$\begin{aligned} 1 &= \frac{\|x_\lambda - x\|}{\|x_\lambda - x\|} \\ &\geq \frac{d(x_\lambda, K) - d(x, K)}{\|x_\lambda - x\|} \quad \text{since the distance function for } K \text{ is nonexpansive} \\ &\geq 1 - \frac{\|z - z_\lambda\|}{\|x - z\|} \quad \text{by Lemma 3.31.} \end{aligned}$$

Since  $\lim_{\lambda \rightarrow 0^+} x_\lambda = x$  and the metric projection map is continuous at  $x$ , the right-hand side of the above inequality converges to 1 as  $\lambda \rightarrow 0^+$ . Hence,  $\lim_{\lambda \rightarrow 0^+} (d(x_\lambda, M) - d(x, M))/\|x_\lambda - x\| = 1$ . □

We require the following variational result from optimisation theory.

**THEOREM 3.33 (Primitive Ekeland theorem [35]).** *Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function on  $X$ , bounded below. If  $\varepsilon > 0$  and  $x_0 \in X$ , then there exists  $x_\infty \in X$  such that:*

- (i)  $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$ ; and
- (ii)  $f(x_\infty) - \varepsilon d(x_\infty, x) < f(x)$  for all  $x \in X \setminus \{x_\infty\}$ .

**PROOF.** See Appendix A. □

**COROLLARY 3.34** [85]. *Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow \mathbb{R}$  be a continuous function on  $X$ , bounded above. If  $\varepsilon > 0$  and  $x_0 \in X$ , then there exists*

$x_\infty \in X$  such that:

- (i)  $f(x_\infty) \geq f(x_0) + \varepsilon d(x_\infty, x_0)$ ; and
- (ii)  $f(x) < f(x_\infty) + \varepsilon d(x_\infty, x)$  for all  $x \in X \setminus \{x_\infty\}$ .

In particular,

$$\limsup_{x \rightarrow x_\infty} \frac{f(x) - f(x_\infty)}{d(x_\infty, x)} \leq \varepsilon.$$

**PROOF.** The result follows from applying Theorem 3.33 to  $-f$ . □

Rather than looking directly at convexity, we shall investigate the weaker property of almost convexity.

**DEFINITION 3.35 [86].** We will say that a closed subset  $A$  of a normed linear space  $(X, \|\cdot\|)$  is almost convex if, for any closed ball  $B[x; r] \subseteq X \setminus A$  and  $N > 0$ , there exist  $x' \in X$  and  $r' > N$  such that

$$B[x; r] \subseteq B[x'; r'] \subseteq X \setminus A.$$

**THEOREM 3.36 [86].** Let  $K$  be a Chebyshev set in a Banach space  $(X, \|\cdot\|)$ . If the metric projection map for  $K$  is continuous, then  $K$  is almost convex.

**PROOF.** Suppose that  $B[x_0; \beta] \subseteq X \setminus K$  for some  $x_0 \in X$  and  $0 < \beta$ . Let  $N > 0$  be arbitrary. Choose  $\varepsilon \in (0, 1)$  such that  $d(x_0, K) + \varepsilon N > \beta + N$  (notethat this is possible since  $d(x_0, K) > \beta$ ). Consider the function  $f : B[x_0, N] \rightarrow \mathbb{R}$  defined by  $f(x) := d(x, K)$ . Since  $B[x_0; N]$  is a complete metric space and the distance function for  $K$  is continuous on  $B[x_0; N]$ , Corollary 3.34 gives the existence of an  $x_\infty \in B[x_0; N]$  such that:

- (i)  $f(x_\infty) \geq f(x_0) + \varepsilon d(x_\infty, x_0)$ ; and
- (ii)  $\limsup_{x \rightarrow x_\infty} ((f(x) - f(x_\infty))/d(x_\infty, x)) \leq \varepsilon < 1$ .

Now, if  $\|x_\infty - x\| < N$ , then it follows from Corollary 3.32 that  $\limsup_{x \rightarrow x_\infty} ((f(x) - f(x_\infty))/d(x_\infty, x)) = 1$ , which is impossible by (ii). Thus,  $\|x_\infty - x\| = N$ . Hence, from (i) we obtain that

$$d(x_\infty, K) \geq d(x_0, K) + \varepsilon N > \beta + N.$$

Therefore,  $B[x_0; \beta] \subseteq B[x_\infty; \beta + N] \subseteq X \setminus K$ . Since  $N > 0$  was arbitrary, we conclude that  $K$  is almost convex. □

We now establish a condition that guarantees that an almost convex set is convex. To achieve this we need to prove some preliminary results.

**LEMMA 3.37 [40, page 241].** Let  $(X, \|\cdot\|)$  be a normed linear space with a strictly convex dual space and let  $f \in S_{X^*}$ . If  $(B[z_n; r_n])_{n=1}^\infty$  is an increasing sequence of closed balls in  $X$  such that:

- (i)  $B[z_n; r_n] \subseteq \{x \in X : f(x) \leq 1\}$  for all  $n \in \mathbb{N}$ ;
- (ii)  $\lim_{n \rightarrow \infty} r_n = \infty$ ,

then there exists some  $r \leq 1$  such that  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]} = \{z \in X : f(z) \leq r\}$ , that is,  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}$  is a closed half space.

**PROOF.** Let  $r := \sup\{f(x) : x \in \overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}\} \leq 1$ . Suppose, looking for a contradiction, that

$$\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]} \neq \{x \in X : f(x) \leq r\}.$$

Then there exists a  $z \in \{x \in X : f(x) \leq r\} \setminus \overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}$ . Since  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}$  is nonempty, closed and convex, there exists some  $g \in S_{X^*}$  such that

$$\sup\left\{g(x) : x \in \overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}\right\} < g(z).$$

Clearly,  $f \neq g$ . For each  $n \in \mathbb{N}$ , let  $x_n := (z_1 - z_n)/r_n$ . Since  $z_1 \in B[z_1; r_1] \subseteq B[z_n; r_n]$  for all  $n \in \mathbb{N}$ , it follows that  $\|x_n\| = \|z_1 - z_n\|/r_n \leq 1$ . Let  $\varepsilon := r - f(z_1) > 0$ . Then, for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} f(z_1) &= r - \varepsilon \\ &\geq \sup\{f(x) : x \in B[z_n; r_n]\} - \varepsilon \quad \text{since } B[z_n; r_n] \subseteq \overline{\bigcup_{k \in \mathbb{N}} B[z_k; r_k]} \\ &= (f(z_n) + r_n) - \varepsilon \quad \text{since } B[z_n; r_n] = z_n + r_n B_X \text{ and } \|f\| = 1. \end{aligned}$$

Therefore,  $1 \geq f(x_n) = (f(z_1) - f(z_n))/r_n \geq 1 - (\varepsilon/r_n)$  for all  $n \in \mathbb{N}$ . Hence,  $\lim_{n \rightarrow \infty} f(x_n) = 1$ . By a similar argument,  $\lim_{n \rightarrow \infty} g(x_n) = 1$ . Thus,  $\lim_{n \rightarrow \infty} ((f + g)/2)(x_n) = 1$ , and so  $\|(f + g)/2\| = 1$ . However, this contradicts the strict convexity of  $X^*$ . □

**LEMMA 3.38.** *Let  $C$  be a closed convex subset of a normed linear space  $(X, \|\cdot\|)$ . Then either  $C = X$ , or else  $C$  is contained in a half space.*

**PROOF.** Suppose that  $C \neq X$ . Let  $x \in X \setminus C$ . By the Hahn–Banach theorem, there exists a hyperplane separating  $x$  and  $C$ . Thus,  $C$  is contained in one of the half spaces determined by this hyperplane. □

**LEMMA 3.39.** *Let  $J$  be a closed half space of a normed linear space  $(X, \|\cdot\|)$ . If for some  $x, y \in X$ ,  $(x + y)/2 \in \text{int}(J)$ , then either  $x \in \text{int}(J)$  or  $y \in \text{int}(J)$ .*

**PROOF.** This follows directly from Lemma 3.1. □

**LEMMA 3.40.** *Let  $(C_n)_{n=1}^\infty$  be an increasing sequence of convex sets in a normed linear space  $(X, \|\cdot\|)$ . Suppose that  $\text{int}(C_k) \neq \emptyset$  for some  $k \in \mathbb{N}$ . Then  $\text{int}(\overline{\bigcup_{n=1}^\infty C_n}) = \bigcup_{n=1}^\infty \text{int}(C_n)$ .*

**PROOF.** Clearly,  $\bigcup_{n=1}^\infty \text{int}(C_n) \subseteq \text{int}(\overline{\bigcup_{n=1}^\infty C_n})$ , so it suffices to show the reverse set inclusion. Let  $x \in \text{int}(\overline{\bigcup_{n=1}^\infty C_n})$ . Then there exists  $r > 0$  such that  $B(x; r) \subseteq$

$\text{int}(\overline{\bigcup_{n=1}^{\infty} C_n})$ . Moreover, there exist  $k \in \mathbb{N}$ ,  $s > 0$ , and  $y \in C_k$  such that  $B(y; s) \subseteq C_k$ . Now choose  $\lambda > 0$  sufficiently small so that

$$B(x + \lambda(x - y); \lambda s) \subseteq B(x; r) \subseteq \overline{\bigcup_{n=1}^{\infty} C_n}$$

Hence, there exist  $z \in B(x + \lambda(x - y); \lambda s)$  and  $m \geq k$  such that  $z \in C_m$ . It is straightforward to verify that

$$x \in B\left(\frac{z + \lambda y}{1 + \lambda}; \frac{\lambda s}{1 + \lambda}\right) = \left(\frac{1}{1 + \lambda}\right)z + \left(\frac{\lambda}{1 + \lambda}\right)B(y; s).$$

Since  $z \in C_m$  and  $B(y; s) \subseteq C_k \subseteq C_m$ , we have from the convexity of  $C_m$  that  $x \in \text{int}(C_m)$ . Therefore,  $x \in \bigcup_{n=1}^{\infty} \text{int}(C_n)$ . □

**COROLLARY 3.41.** *Let  $C$  be a convex subset of a normed linear space  $(X, \|\cdot\|)$ . If  $\text{int}(C) \neq \emptyset$ , then  $\text{int}(\overline{C}) = \text{int}(C)$ .*

**THEOREM 3.42 [86].** *Let  $(X, \|\cdot\|)$  be a normed linear space with strictly convex dual space. Then any closed almost convex subset of  $X$  is convex.*

**PROOF.** Looking for a contradiction, suppose that  $C \subseteq X$  is a closed almost convex set that is not convex. By Proposition 2.12,  $C$  is not midpoint convex. Therefore, there exist  $x, y \in C$  such that  $c := (x + y)/2 \notin C$ . As  $C$  is closed,  $d(c, M) > 0$ . Since

$$B\left[c; \frac{d(c, M)}{2}\right] \subseteq X \setminus C$$

and  $C$  is almost convex, there exist sequences  $(z_n)_{n=1}^{\infty}$  and  $(r_n)_{n=1}^{\infty}$ , in  $X$  and  $[0, \infty)$  respectively, such that

$$B\left[c; \frac{d(c, M)}{2}\right] \subseteq B[z_n; r_n] \subseteq B[z_{n+1}; r_{n+1}] \quad \text{and} \quad B[z_n; r_n] \subseteq X \setminus C$$

for all  $n \in \mathbb{N}$ , and  $\lim_{n \rightarrow \infty} r_n = \infty$ . If  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]} = X$ , Lemma 3.40 tells us that

$$\bigcup_{n \in \mathbb{N}} B(z_n; r_n) = \text{int}\left(\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}\right) = \text{int}(X) = X.$$

Thus,  $x \in B(z_k; r_k)$  for some  $k \in \mathbb{N}$ , which is impossible since  $x \in C$ . Therefore,  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}$  is not the whole space, and so, by Lemma 3.38, is contained in a half space. Hence, by Lemma 3.37,  $\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}$  is a closed half space. Since  $c \in \text{int}(B[c; d(c, M)/2]) \subseteq \text{int}(\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]})$ , it follows by Lemmas 3.39 and 3.40 that either  $x$  or  $y$  is in

$$\text{int}\left(\overline{\bigcup_{n \in \mathbb{N}} B[z_n; r_n]}\right) = \bigcup_{n \in \mathbb{N}} B(z_n; r_n),$$

which is impossible since  $x, y \in C$ . Hence,  $C$  is convex. □

**THEOREM 3.43 [86].** *In any Banach space with a strictly convex dual space, every Chebyshev set with a continuous metric projection mapping is convex.*

**PROOF.** Since  $K$  has a continuous metric projection mapping, it is almost convex, by Theorem 3.36. Since any Chebyshev set is necessarily closed and  $X^*$  is strictly convex,  $K$  is convex by Theorem 3.42.  $\square$

The strict convexity of the dual is essential here. Consider Example 2.11. By Corollary 2.27, the metric projection mapping for  $K$  is continuous and it is straightforward to check that  $K$  is almost convex. However,  $K$  is clearly not convex.

**COROLLARY 3.44.** *For any  $1 < p < \infty$ , every Chebyshev set in  $L_p$  with a continuous metric projection mapping is convex.*

**PROOF.** Again we need only check that  $L_p$  has a strictly convex dual for  $1 < p < \infty$ . However, by [38, 68] or, more recently, [74, page 284], the dual of  $L_p$  is  $L_q$  (where  $q := p/(p - 1)$ ), which is strictly convex [22, 39].  $\square$

**COROLLARY 3.45** [4]. *In any Hilbert space, every Chebyshev set with a continuous metric projection mapping is convex.*

**PROOF.** By the previous theorem, all we need to check is that a Hilbert space has a strictly convex dual. However, this follows since the dual of a Hilbert space is again a Hilbert space and so is strictly convex [22, 39].  $\square$

**3.5. Historical review and future directions.** The problem of the convexity of Chebyshev sets in infinite-dimensional spaces was considered by Efimov and Stečkin in a series of papers published between 1958 and 1961 in *Doklady* [31–34]. In these papers the term ‘Chebyshev set’ was first coined. However, earlier and independently, the ‘Chebyshev set problem’ had been considered by Klee [48]. Moreover, Klee also considered the problem of the existence, in a Hilbert space, of a Chebyshev set with a convex complement (nowadays called a *Klee cavern*). Note, however, that later in [4], Asplund showed that the existence of a Klee cavern is equivalent to the existence of a nonconvex Chebyshev set. Going back further in time, we note that according to [50], Frederick A. Ficken had earlier, in 1951, shown that in a Hilbert space, every compact Chebyshev set is convex, by using the method of inversion in the unit sphere. So this appears to be the first result concerning the ‘Chebyshev set problem’ in infinite-dimensional normed linear spaces. Not surprisingly, though, convexity of sets in finite-dimensional normed linear spaces that admit unique best approximations had been considered earlier. As far as the authors of this paper know, the first results concerning the convexity of sets that admit unique best approximations are due to Bunt [19] in 1934, Motzkin [57, 58] in 1935 and Kritikos [52] in 1938. All of these authors independently showed that in Euclidean space (of various dimensions) every Chebyshev set is convex. In 1940 Jessen [44], aware of Kritikos’s proof, gave yet another proof in  $\mathbb{R}^n$ . Busemann noted in 1947 [20] that Jessen’s proof could be extended to ‘straight-line spaces’, and in 1955 [21] showed how this could be done. Since a finite-dimensional normed linear space is a ‘straight-line space’ if and only if it is strictly convex, Busemann’s result implies that in a smooth, strictly convex

finite-dimensional normed linear space, every Chebyshev set is convex. Valentine [83] independently in 1964 gave essentially the same proof as Busemann.

In 1953, Klee [48] stated that in a finite-dimensional normed linear space, every Chebyshev set is a ‘sun’ and gave a characterisation of Chebyshev sets in smooth, strictly convex finite-dimensional normed linear spaces. However, as he noted in 1961 [50], the argument in [48] was garbled and he proceeded to prove a stronger result, which in finite dimensions shows that the requirement of strict convexity could be dropped. Thus, Klee was the first to show that in a smooth finite-dimensional normed linear space every Chebyshev set is convex.

As can be seen from this section of the paper, the main impediment to proving that every Chebyshev subset of a Hilbert space is convex, is removing the continuity assumptions on the metric projection mapping. So we point out here that some progress has already been made in this direction; see [6, 7, 27, 88].

We should also point out that there are many other approaches to showing the convexity of Chebyshev sets in Hilbert spaces that are not pursued here. For example, by exploiting the theory of maximal monotone operators, one can obtain results that are very similar to those presented in this section. Indeed, it follows from [4] that if  $K$  is a proximal subset of a Hilbert space  $H$ , then  $P_K(x) \subseteq \partial f(x)$  for all  $x \in H$ , where  $f(x) := \sup\{\langle x, y \rangle - \|y\|^2 : y \in K\}$  is a continuous convex function and  $\partial f(x)$  is the set of all subgradients of  $f$  at  $x$ ; see [62, page 6]. Now, by [73], the mapping  $x \mapsto \partial f(x)$  is a maximal monotone operator. On the other hand, by [72, Theorem 2], the norm closure of the range of a maximal monotone operator on a reflexive Banach space is convex, and so the norm closure of the range of the subdifferential mapping,  $x \mapsto \partial f(x)$ , is convex. Let us now recall that the subdifferential mapping,  $x \mapsto \partial f(x)$ , is a minimal weak cusco; see [62, Theorem 7.9]. Hence, if  $x \mapsto P_K(x)$  is a weak cusco, then  $\partial f(x) = P_K(x)$  for all  $x \in H$  (here we are identifying  $H$  with its dual  $H^*$ ). This then tells us that  $K$ , which is the range of  $P_K$ , is convex. So the only question that remains is: ‘when is  $x \mapsto P_K(x)$  a weak cusco?’. One answer to this is when  $K$  is weakly closed and  $P_K(x)$  is nonempty and convex for each  $x \in H$ .

## 4. A nonconvex Chebyshev set in an inner product space

**4.1. Background and motivation.** In this section we will construct a nonconvex Chebyshev set in the inner product space consisting of all real sequences with only finitely many nonzero terms, equipped with the Euclidean inner product. This construction was first proposed by Johnson [46]. Whilst the motivation behind this construction is certainly geometric, Johnson’s proof was predominantly algebraic and analytic and contained at least two errors, two of which were later corrected by Jiang [45]. The proof presented in this paper is more geometric in nature, based upon a proof of Balaganskii and Vlasov [7]. We hope that the more detailed proof presented here assists the reader in understanding the numerous technical details of the construction, as well as appreciating the general results that contribute to the construction.



Throughout the remainder of this section we will be working in Euclidean space, which we denote by  $E$  (or  $E_n$  if we wish to make the dimension explicit). That is,  $\mathbb{R}^n$  for some  $n$  in  $\mathbb{N} \cup \{0\}$ , equipped with the usual Euclidean inner product. We will also make use of the following notation: for any  $x \in E$  we will write  $x'$  for  $(x, 0) \in E \times \mathbb{R}$ .

The idea behind the construction is as follows. We start with  $M_1 := E_1 \setminus (-2, 1)$ , which is clearly not a Chebyshev set in  $E_1$  since  $P_{M_1}(\frac{-1}{2}) = \{-2, 1\}$ . On the other hand,  $P_{M_1}(0) = \{1\}$  is a singleton. We now construct a nonconvex set  $M_2 \subseteq E_2$  such that  $M_1 \times \{0\} = M_2 \cap (E_1 \times \{0\})$ , every point in  $E_1 \times \{0\}$  has a unique nearest point in  $M_2$ , and  $P_{M_1}(0) \times \{0\} = P_{M_2}(0)$ . Since  $M_2$  is nonconvex, by Theorem 3.8, it is not a Chebyshev set in  $E_2$ .

This gives us an idea of how to proceed. Construct a nonconvex set  $M_3 \subseteq E_3$  so that  $M_2 \times \{0\} = M_3 \cap (E_2 \times \{0\})$ , every point in  $E_2 \times \{0\}$  has a unique nearest point in  $M_3$ , and  $P_{M_2}(x) \times \{0\} = P_{M_3}(x')$  for all  $x \in E_1 \times \{0\}$ . Repeating this process indefinitely (formally we will use induction) produces a sequence of nonconvex sets  $(M_n)_{n=1}^\infty$  such that  $M_n \subseteq E_n$ ,  $M_n \times \{0\} = M_{n+1} \cap (E_n \times \{0\})$ , each point in  $E_n \times \{0\}$  has a unique nearest point in  $M_{n+1}$ , and  $P_{M_n}(x) \times \{0\} = P_{M_{n+1}}(x')$  for all  $x \in E_{n-1} \times \{0\}$ , for all  $n \in \mathbb{N}$ .

The problem is how to actually construct  $M_{n+1}$  given  $M_n$ , for some  $n \in \mathbb{N}$ , such that it has the aforementioned properties. Whilst the arguments are to a great extent quite technical, the key idea is given in Theorem 2.42, which states that if  $K$  is a nonempty subset of a strictly convex normed linear space  $(X, \|\cdot\|)$  and  $x \in X$  is a point of Gâteaux differentiability of the distance function for  $K$ , then  $P_K(x)$  has at most one element. Of course, this begs the question how to prove that a given set has a differentiable distance function. To get around this problem, rather than starting with a set and trying to show that its distance function is differentiable, we will start with a differentiable function and, using a clever trick [7, pages 1175–1180], construct a set that has this function as its distance function.

## 4.2. The construction.

4.2.1. *Preliminaries.* We now define various objects that will be used throughout the rest of this section.

**DEFINITION 4.1.** Let  $M \subseteq E$  and let  $b > 0$ . We say  $M$  has the  $b$ -boundary property if, for any  $y \in \text{Bd}(M)$ , there exists  $x \in E$  such that  $\|x - y\| = d(x, M) = b$ .

**DEFINITION 4.2.** Given a Euclidean space  $E$  (with dimension  $n$ ), we define the subspace  $E^{(0)}$  by

$$E^{(0)} := \{(x_1, \dots, x_n) \in E : x_n = 0\}.$$

**DEFINITION 4.3.** Let  $M$  be a nonempty, closed subset of  $E$ . Suppose that  $M$  has the  $b$ -boundary property for some  $b > 0$ ,  $E \setminus M$  is convex and bounded,  $0 \notin M$ , and every point in  $E^{(0)}$  has a unique nearest point in  $M$ . We define the following objects:

$$C := \{(x, r) \in E \times \mathbb{R} : x \in \overline{E \setminus M}, |r| \leq d(x, M)\}, \quad A := \{(x, r) \in C : r \geq 0\},$$

$$R : C \rightarrow \mathbb{R} \quad \text{by } R(w) := \frac{d(x, M) - r}{\sqrt{2}} \quad \text{for all } w := (x, r) \in C,$$

$$\xi := \frac{d\sqrt{8}}{b} \quad \text{where } d := \text{diam}(E \setminus M) = \sup\{\|x - y\| : x, y \in E \setminus M\},$$

$$Q := \{w \in C : d(w, E^{(0)} \times \mathbb{R}) \leq \xi R(w)\},$$

$$D := (E \times [0, \infty)) \cap \left( \bigcup_{w \in Q} B(w; R(w)) \right).$$

**REMARK 4.4.** It is clear that  $b \leq d/2$ .

Before establishing some further properties of these objects, we briefly describe the motivation behind their definitions. The set  $C$  is simply the volume contained between the graph of the distance function for  $M$ , restricted to  $\overline{E \setminus M}$ , and its reflection in  $E \times 0$ . The function  $R$  measures the distance from a point  $w \in C$  to the graph of the distance function for  $M$ , restricted to  $\overline{E \setminus M}$ . Rewriting the definition of  $Q$  gives

$$Q = \left\{ (x, r) \in C : r \leq d(x, M) - \frac{\sqrt{2}}{\xi} d(x, E^{(0)}) \right\},$$

which shows that  $Q$  is just  $C$  with points removed in a linear manner as we move away from  $E^{(0)}$ . The inclusion of the  $d(\cdot, E^{(0)})$  term in the definition of  $Q$  will later be used to show that our metric projections coincide on  $E^{(0)}$ . Now that we have a proper subset of  $C$ , we are able to construct the ‘smooth’ set  $D$  inside  $A$  (it looks like  $A$  without a sharp point at the top) by taking the union of balls of an appropriate radius (given by the function  $R$ ). This ‘smooth’ set is then used to construct a Gâteaux differentiable function,  $\rho : \overline{E \setminus M} \rightarrow \mathbb{R}$ , defined by

$$\rho(x) := \max\{r : (x, r) \in \overline{D}\}$$

(we will prove later that this is well defined). From this we construct a set  $M''$  in  $E \times \mathbb{R}$  that has  $\rho$  as its distance function (for at least some of its points). It is essential that our smoothed set  $D$  lies inside  $A$ , since we require  $M \times \{0\} = M'' \cap (E \times \{0\})$ . The differentiability of this distance function then enables us to prove the uniqueness of nearest points.

**PROPOSITION 4.5.**  $C$  and  $A$  are closed, convex, and bounded (and hence compact).

**PROOF.** Since the distance function for  $M$  is continuous, it is straightforward to check that  $C$  and  $A$  are closed. To see the convexity of  $C$ , suppose that  $(x, r), (y, s) \in C$  and  $0 \leq \lambda \leq 1$ . Since  $\overline{E \setminus M}$  is convex by assumption, we have  $\lambda x + (1 - \lambda)y \in \overline{E \setminus M}$ . Furthermore,

$$\begin{aligned} |\lambda r + (1 - \lambda)s| &\leq \lambda|r| + (1 - \lambda)|s| \\ &\leq \lambda d(x, M) + (1 - \lambda)d(y, M) \\ &\leq d(\lambda x + (1 - \lambda)y, M), \end{aligned}$$

since the distance function for  $M$  restricted to  $\overline{E \setminus M}$  is concave by Proposition 2.16. Thus,  $\lambda(x, r) + (1 - \lambda)(y, s) \in C$ , and so  $C$  is convex. An almost identical proof

shows that  $A$  is convex. Finally, to see that  $C$  (and hence  $A \subseteq C$ ) is bounded, observe that since  $E \setminus M$  is bounded and the distance function for  $M$  is continuous, we have  $r := \max\{d(x, M) : x \in \overline{E \setminus M}\} < \infty$ , and so  $\overline{E \setminus M} \subseteq B(y; s)$  for some  $y \in E, s > 0$ . Therefore,  $C \subseteq B(y'; \sqrt{r^2 + s^2})$ , and so  $C$  is bounded.  $\square$

**PROPOSITION 4.6.** *The map  $R : C \rightarrow \mathbb{R}$  is continuous and concave.*

**PROOF.** Suppose that  $w := (x, r), v := (y, s) \in C$  and  $0 \leq \lambda \leq 1$ . Since  $x, y \in \overline{E \setminus M}$ , Proposition 2.16 gives

$$\begin{aligned} \lambda R(w) + (1 - \lambda)R(v) &= \lambda \left( \frac{d(x, M) - r}{\sqrt{2}} \right) + (1 - \lambda) \left( \frac{d(y, M) - s}{\sqrt{2}} \right) \\ &= \frac{\lambda d(x, M) + (1 - \lambda)d(y, M) - (\lambda r + (1 - \lambda)s)}{\sqrt{2}} \\ &\leq \frac{d(\lambda x + (1 - \lambda)y, M) - (\lambda r + (1 - \lambda)s)}{\sqrt{2}} \\ &= R(\lambda w + (1 - \lambda)v). \end{aligned}$$

Thus,  $R$  is concave. Continuity of  $R$  follows by the continuity of the distance function for  $M$ .  $\square$

**PROPOSITION 4.7.**  *$Q$  is closed and bounded (and hence compact).*

**PROOF.** Let  $(w_n)_{n=1}^\infty$  be a sequence in  $Q$  converging to some  $w \in E \times \mathbb{R}$ . Since  $C$  is closed and  $Q \subseteq C, w \in C$ . By the continuity of both  $R$  and the distance function for  $E^{(0)} \times \mathbb{R}$ , we have

$$d(w, E^{(0)} \times \mathbb{R}) = \lim_{n \rightarrow \infty} d(w_n, E^{(0)} \times \mathbb{R}) \leq \lim_{n \rightarrow \infty} \xi R(w_n) = \xi R(w).$$

Thus,  $w \in Q$ , and so  $Q$  is closed. Moreover, since  $C$  is bounded, so is  $Q$ .  $\square$

**PROPOSITION 4.8.** *The closure of  $D$  is given by  $\overline{D} = (E \times [0, \infty)) \cap (\bigcup_{w \in Q} B[w; R(w)])$ .*

**PROOF.** It is clear that set inclusion holds in one direction, so let  $x \in \overline{D}$  and  $(x_n)_{n=1}^\infty$  be a sequence in  $D$  converging to  $x$ . Clearly,  $x \in E \times [0, \infty)$ . By the definition of  $D$ , there exists a sequence  $(w_n)_{n=1}^\infty$  in  $Q$  such that  $\|x_n - w_n\| < R(w_n)$  for all  $n \in \mathbb{N}$ . Since  $Q$  is compact, we may assume, without loss of generality, that  $\lim_{n \rightarrow \infty} w_n = w$ , for some  $w \in Q$ . As  $R$  is continuous, it follows that

$$\|x - w\| = \left\| \lim_{n \rightarrow \infty} x_n - \lim_{n \rightarrow \infty} w_n \right\| = \lim_{n \rightarrow \infty} \|x_n - w_n\| \leq \lim_{n \rightarrow \infty} R(w_n) = R(w).$$

Hence,  $x \in B[w; R(w)]$  and we are done.  $\square$

**LEMMA 4.9.** *For any  $x \in E \setminus M$ , we have  $d(x', E^{(0)} \times \mathbb{R}) \leq d$ .*

**PROOF.** Let  $x \in E \setminus M$ . Since  $0 \in E^{(0)} \times \mathbb{R}$  and  $0 \in E \setminus M$ , we have

$$d(x', E^{(0)} \times \mathbb{R}) \leq \|x' - 0\| = \|x - 0\| \leq \text{diam}(E \setminus M) =: d \quad \square$$

**PROPOSITION 4.10.**  $(E \setminus M) \times \{0\} \subseteq D$ .

**PROOF.** Let  $x' \in (E \setminus M) \times \{0\}$ . Clearly,  $x' \in E \times [0, \infty)$  and  $x' \in C$ . Firstly, suppose that  $d(x, M) \geq b/2$ , so that  $R(x') = d(x, M)/\sqrt{2} \geq b/\sqrt{8} > 0$ . Using Lemma 4.9, we have

$$d(x', E^{(0)} \times \mathbb{R}) \leq d = \frac{d\sqrt{8}}{b} \cdot \frac{b}{\sqrt{8}} = \xi \frac{b}{\sqrt{8}} \leq \xi R(x').$$

Thus,  $x' \in Q$ . Since  $x' \in B(x'; R(x'))$ , it follows that  $x' \in D$ .

Alternatively, suppose that  $d(x, M) < b/2$ . Let  $y \in P_M(x)$ . Since  $M$  has the  $b$ -boundary property, there exists  $z \in E$  such that  $d(z, M) = \|z - y\| = b$ . Thus,  $z \in P_M(y)$ , and so by Lemma 2.39,  $x, y, z$  are collinear. Since  $0 < \|x - y\| < b/2 < b = \|y - z\|$ , we have  $x \in (y, z)$ , and so  $\|x - z\| < \|y - z\| = b$ . Let  $w := (z, -b)$ . We will now show that  $w \in Q$ . Firstly, it is clear that  $w \in C$ . Then

$$R(w) = \frac{d(z, M) - (-b)}{\sqrt{2}} = b\sqrt{2},$$

and so  $d(w, E^{(0)} \times \mathbb{R}) \leq d \leq d\sqrt{8}/b \cdot b\sqrt{2} = \xi R(w)$ , as required. Finally, we have

$$\|x' - w\|^2 = \|(x, 0) - (z, -b)\|^2 = \|x - z\|^2 + b^2 < 2b^2 = R(w)^2.$$

Therefore,  $x' \in B(w; R(w))$ , and so  $x' \in D$ . □

**PROPOSITION 4.11.**  $\bar{D} \subseteq A$ .

**PROOF.** Let  $z := (y, t) \in \bar{D} = (E \times [0, \infty)) \cap (\bigcup_{w \in Q} B[w; R(w)])$ . Therefore,  $t \geq 0$  and there exists some  $w := (x, r) \in Q \subseteq C$  such that  $\|z - w\| \leq R(w)$ . Suppose, looking for a contradiction, that  $d(x, M) < t$ . As  $w \in C$ , it follows that  $r \leq |r| \leq d(x, M) < t$ . Then since

$$\|z - w\|^2 = \|y - x\|^2 + (t - r)^2 \leq R(w)^2,$$

we have

$$t \leq r + R(w) = r + \frac{d(x, M) - r}{\sqrt{2}} \leq r + d(x, M) - r = d(x, M),$$

which is impossible. Thus,  $d(x, M) \geq t$ . Observe that

$$\begin{aligned} & (d(x, M) - t)^2 - \left( \left( \frac{d(x, M) - r}{\sqrt{2}} \right)^2 - (t - r)^2 \right) \\ &= (d(x, M) - r - (t - r))^2 - \left( \left( \frac{d(x, M) - r}{\sqrt{2}} \right)^2 - (t - r)^2 \right) \\ &= \left( \frac{d(x, M) - r}{\sqrt{2}} \right)^2 + 2(t - r)^2 - 2(d(x, M) - r)(t - r) \\ &= \left( \frac{d(x, M) - r}{\sqrt{2}} - \sqrt{2}(t - r) \right)^2 \geq 0. \end{aligned}$$

Rearranging and taking the square root of both sides gives

$$d(x, M) - t \geq \sqrt{\left(\frac{d(x, M) - r}{\sqrt{2}}\right)^2 - (t - r)^2}.$$

Using this and the nonexpansivity of the distance function for  $M$  gives

$$\begin{aligned} d(y, M) &\geq d(x, M) - \|y - x\| \\ &\geq d(x, M) - \sqrt{R(w)^2 - (r - t)^2} \\ &= d(x, M) - \sqrt{\left(\frac{d(x, M) - r}{\sqrt{2}}\right)^2 - (r - t)^2} \\ &\geq t \geq 0. \end{aligned}$$

To complete the proof we need to show that  $y \in \overline{E \setminus M}$ . By the above working, if  $t > 0$ , then  $d(y, M) > 0$ , so we may as well suppose that  $t = 0$ . Since

$$d(x, M)^2 - R(w)^2 + r^2 = \frac{d(x, M)^2}{2} + d(x, M)r + \frac{r^2}{2} = \left(\frac{d(x, M) + r}{\sqrt{2}}\right)^2 \geq 0,$$

we have that  $\|x - y\|^2 \leq R(w)^2 - r^2 \leq d(x, M)^2$ . As  $w \in C$ ,  $x \in \overline{E \setminus M}$ , and so we conclude that  $y \in \overline{E \setminus M}$ . Hence,  $z := (y, t) \in A$ , and so  $\overline{D} \subseteq A$ . □

**PROPOSITION 4.12.**  *$Q$  and  $D$  are convex.*

**PROOF.** Let  $\lambda \in [0, 1]$  and suppose that  $w := (x, r), v := (y, s) \in Q$ . Since  $E^{(0)} \times \mathbb{R}$  is convex, by Propositions 2.15 and 4.6, we have

$$\begin{aligned} d(\lambda w + (1 - \lambda)v, E^{(0)} \times \mathbb{R}) &\leq \lambda d(w, E^{(0)} \times \mathbb{R}) + (1 - \lambda)d(v, E^{(0)} \times \mathbb{R}) \\ &\leq \lambda \xi R(w) + (1 - \lambda) \xi R(v) \\ &\leq \xi R(\lambda w + (1 - \lambda)v). \end{aligned}$$

Since  $C$  is convex,  $\lambda w + (1 - \lambda)v \in Q$ , and so  $Q$  is convex. Now let  $x, y \in D$ . Therefore,  $x, y \in E \times [0, \infty)$  and there exist  $w, v \in Q$  such that  $\|x - w\| < R(w)$  and  $\|y - v\| < R(v)$ . Making use of the concavity of  $R$  again, we see that

$$\begin{aligned} \|\lambda x + (1 - \lambda)y - (\lambda w + (1 - \lambda)v)\| &= \|\lambda(x - w) + (1 - \lambda)(y - v)\| \\ &\leq \lambda \|x - w\| + (1 - \lambda) \|y - v\| \\ &< \lambda R(w) + (1 - \lambda) R(v) \\ &\leq R(\lambda w + (1 - \lambda)v). \end{aligned}$$

Since  $Q$  and  $E \times [0, \infty)$  are convex, we conclude that  $D$  is convex. □

4.2.2. *Smoothness and unique nearest points.* In Section 2 we introduced the related concepts of Gâteaux differentiability and smoothness of normed linear spaces. We now define what it means for a subset of a normed linear space to be smooth.

**DEFINITION 4.13.** Let  $K$  be a subset of a normed linear space  $(X, \|\cdot\|)$  and let  $x \in K$ . We say that  $K$  is smooth at  $x$  if there exists a unique  $f \in S_{X^*}$  such that  $f$  supports  $K$  at  $x$ .

**REMARK 4.14.** It is well known (see [80], [36, Lemma 8.4] or [40, page 124]) that a normed linear space is smooth, in the sense that the norm is Gâteaux differentiable everywhere except at 0 if and only if the closed unit ball is smooth at every point of  $S_X$ .

We now present a number of results that will later be used to prove a smoothness condition for  $\bar{D}$ . First, we introduce the concept of a cone in Euclidean space and derive some of its properties.

**DEFINITION 4.15.** Let  $x \in E$  and  $r \geq 0$ . Define the cone of height and base radius  $r$  as

$$K[x; r] := \{(y, t) \in E \times [0, \infty) : \|y - x\| + t \leq r\} \subseteq E \times \mathbb{R}.$$

That is,  $K[x; r]$  is the right-circular cone with vertex  $(x, r)$  and base  $B[x; r] \times \{0\}$ .

The following result gives a different way of describing a cone, and will be used to prove a smoothness condition for cones.

**PROPOSITION 4.16.** For any  $x \in E$  and  $r > 0$ ,

$$K[x; r] = E \times [0, \infty) \cap \bigcup_{\lambda \in [0, 1]} B[(x, (1 - 2\lambda)r); \lambda\sqrt{2}r].$$

**PROOF.** See Appendix B. □

**COROLLARY 4.17.** Let  $x \in E$  and  $r > 0$ . Then  $K[x; r]$  is smooth at any  $(y, t) \in E \times (0, r)$ , such that  $\|y - x\| + t = r$ .

**PROOF.** See Appendix B. □

**PROPOSITION 4.18.** Let  $x \in E \setminus M$  and suppose that  $(x, d(x, M)) \in \bar{D}$ . Then  $K[x; d(x, M)] \subseteq \bar{D}$ .

**PROOF.** See Appendix B. □

We now recall some results regarding the Gâteaux differentiability of the distance function.

**THEOREM 4.19** [7, Theorem 1.4]. Let  $K$  be a nonempty closed subset of a Fréchet smooth normed linear space  $(X, \|\cdot\|)$  and let  $x_0 \in X \setminus K$ . If  $P_K(x_0)$  is nonempty and

$$\lim_{t \rightarrow 0^+} \frac{d(x_0 + tv, K) - d(x_0, K)}{t} = 1$$

for some  $v \in S_X$ , then the distance function for  $K$  is Gâteaux differentiable at  $x_0$ .

**PROOF.** See Appendix C. □

**COROLLARY 4.20.** *Let  $K$  be a nonempty, proximal subset of a smooth normed linear space  $(X, \|\cdot\|)$  and  $x \in X \setminus K$  be such that the metric projection mapping for  $K$  is continuous at  $x$ . Then the distance function for  $K$  is Gâteaux differentiable at  $x$ .*

**PROOF.** Let  $v := (x - z)/\|x - z\| \in S_X$ , where  $\{z\} = P_K(x)$ . By Corollary 3.32, setting  $\lambda := t/\|x - z\|$  gives

$$\lim_{t \rightarrow 0^+} \frac{d(x + tv, K) - d(x, M)}{t} = \lim_{\lambda \rightarrow 0^+} \frac{d(x_\lambda, K) - d(x, K)}{\|x_\lambda - x\|} = 1,$$

where  $x_\lambda = x + \lambda(x - z)$ . Theorem 4.19 then shows that the distance function for  $K$  is Gâteaux differentiable at  $x$ . □

Finally, we are ready to prove the smoothness condition on  $\overline{D}$ . This result will be required to prove the uniqueness of nearest points in our construction.

**PROPOSITION 4.21.**  *$\overline{D}$  is smooth at any  $(x, r) \in \text{Bd}(D)$  such that  $r > 0$ .*

**PROOF.** Let  $w := (x, r) \in \text{Bd}(D)$  with  $r > 0$ . By Proposition 4.8, there exists a  $w_0 := (x_0, r_0)$  in  $Q$  such that  $\|w - w_0\| = R(w_0)$ . If  $R(w_0) > 0$ , then any hyperplane supporting  $\overline{D}$  at  $w$  will also support the closed ball  $B[w_0; R(w_0)] \subseteq \overline{D}$  at  $w$ . Since  $(E, \langle \cdot, \cdot \rangle)$  is an inner product space, it is smooth, and so this hyperplane must be unique. Alternatively,  $R(w_0) = 0$ , which implies that  $w = w_0 \in Q$  and  $d(x, M) = r > 0$ . Hence,  $x \notin M$  and  $w \in \text{Bd}(A)$ . By the definition of  $Q$ ,  $d(w, E^{(0)} \times \mathbb{R}) \leq \xi R(w) = 0$ , and since  $E^{(0)} \times \mathbb{R}$  is closed, we must have  $w \in E^{(0)} \times \mathbb{R}$ . Hence,  $x \in E^{(0)}$ , and so  $P_M(x) = \{\hat{x}\}$  for some  $\hat{x} \in M$ . Theorem 2.26 tells us that the metric projection mapping for  $M$  is continuous at  $x$ , and so, by Corollary 4.20, the distance function for  $M$  is Gâteaux differentiable at  $x$ . Since

$$A = \{(y, s) : y \in \overline{E \setminus M}, 0 \leq s \leq d(y, M)\},$$

it follows that  $w \in \text{Bd}(A)$  is a smooth point for  $A$ . Let  $H$  be the unique supporting hyperplane for  $A$  at  $w$ . Since  $\overline{D} \subseteq A$ ,  $H$  also supports  $\overline{D}$  at  $w$ . Suppose, looking for a contradiction, that  $\overline{D}$  is not smooth at  $w$ . Hence, there exists a hyperplane  $H_1 \neq H$ , that also supports  $\overline{D}$  at  $w$ . Next we implement a neat trick, which enables us to view the problem in just two dimensions. Consider the two-dimensional plane  $Z$  in  $E \times \mathbb{R}$  containing the points  $x', \hat{x}'$ , and  $w$ , see Figure 1 (it is clear that these points are not collinear so we do indeed have a two-dimensional plane). Suppose, looking for a contradiction, that  $H \cap Z = H_1 \cap Z$ . Using Corollary 4.20 again, we see that  $H$  has slope 1 in the direction  $x - \hat{x}$ , and so  $(w + \hat{x}')/2 \in H$ . Since  $(w + \hat{x}')/2 \in Z$  also,  $(w + \hat{x}')/2 \in H_1$ . As  $H_1$  supports  $\overline{D}$  at  $w$  and, by Proposition 4.18,  $K[x; d(x, M)] \subseteq \overline{D}$ , it follows that  $H_1$  supports  $K[x; d(x, M)]$  at  $(w + \hat{x}')/2$ . Similarly, since  $H$  supports  $A$  at  $w$  and

$$K[x; d(x, M)] \subseteq \overline{D} \subseteq A,$$

$H$  supports  $K[x; d(x, M)]$  at  $(w + \hat{x}')/2$  as well. However,  $(w + \hat{x}')/2 = ((x + \hat{x})/2, d(x, M)/2)$ , and since  $0 < d(x, M)/2 < r$ , Corollary 4.17 implies that the cone  $K[x; d(x, M)]$  is smooth at  $(w + \hat{x}')/2$ . Thus,  $H = H_1$ , which is impossible. Therefore,  $H \cap Z \neq H_1 \cap Z$ .

Next consider the sequences  $(x_n)_{n=1}^\infty$  and  $(r_n)_{n=1}^\infty$  defined by

$$x_n := x + \frac{1}{n}(x - \hat{x}) \quad \text{and} \quad r_n := \left(1 - \frac{1}{n}\right)d(x, M) \quad \text{for all } n \in \mathbb{N},$$

respectively. Define the sequence  $(w_n)_{n=1}^\infty$  by  $w_n := (x_n, r_n)$  for all  $n \in \mathbb{N}$ . Clearly, we have  $\lim_{n \rightarrow \infty} w_n = w$ . Let  $0 < \alpha \leq \pi/2$  be the angle between  $H \cap Z$  and  $H_1 \cap Z$ . For each  $n \in \mathbb{N}$ , let  $u_n$  be the nearest point on  $H_1 \cap Z$  to  $w_n$ . Thus,  $\langle w - u_n, w_n - u_n \rangle = 0$  for all  $n \in \mathbb{N}$ , and taking the limit of this expression shows that  $\lim_{n \rightarrow \infty} u_n = w$ . For each  $n \in \mathbb{N}$ , define  $w_n^\dagger := (x, r_n)$ . Clearly,  $\lim_{n \rightarrow \infty} w_n^\dagger = w$ .

Therefore, for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \angle w w_n^\dagger w_n &= \frac{\pi}{2}, \\ \|w_n^\dagger - w_n\| &= \|x - x_n\| = d(x, M) - r_n = \|w - x'\| - r_n = \|w - w_n^\dagger\|, \\ \angle w_n^\dagger w w_n &= \frac{\pi}{4}, \\ \angle x' w x' &= \frac{\pi}{4}, \\ \angle \hat{x}' w w_n &= \frac{\pi}{2}. \end{aligned}$$

Thus, since  $0 < \alpha \leq \pi/2$ ,  $1 > \cos(\alpha) = \|u_n - w_n\|/\|w - w_n\|$  for all  $n \in \mathbb{N}$ . Furthermore,  $\sqrt{2}\|x - x_n\| = \|w - w_n\|$  and  $r_n = d(x, M) - \|x - x_n\|$  for all  $n \in \mathbb{N}$ . Hence, for all  $n \in \mathbb{N}$ ,

$$R(w_n) = \frac{d(x_n, M) - r_n}{\sqrt{2}} = \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{\sqrt{2}}.$$

By Theorem 2.26 and Corollary 3.32,  $\lim_{n \rightarrow \infty} (d(x_n, M) - d(x, M))/\|x_n - x\| = 1$ . Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{R(w_n)}{\|w - w_n\|} &= \lim_{n \rightarrow \infty} \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{\sqrt{2}\|w - w_n\|} \\ &= \lim_{n \rightarrow \infty} \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{2\|x - x_n\|} \\ &= \lim_{n \rightarrow \infty} \frac{d(x_n, M) - d(x, M)}{2\|x - x_n\|} + \frac{1}{2} = 1. \end{aligned}$$

As  $x \in E^{(0)}$ ,  $w_n^\dagger \in E^{(0)} \times \mathbb{R}$  for all  $n \in \mathbb{N}$ , and so

$$\frac{d(w_n, E^{(0)} \times \mathbb{R})}{R(w_n)} \leq \frac{\|w_n - w_n^\dagger\|}{R(w_n)} \leq \frac{\|w - w_n\|}{\sqrt{2}R(w_n)}$$

for all  $n \in \mathbb{N}$ . Since  $b \leq d/2$ , our earlier working gives

$$\lim_{n \rightarrow \infty} \frac{\|w - w_n\|}{\sqrt{2}R(w_n)} = \frac{1}{\sqrt{2}} < \frac{d\sqrt{8}}{b} = \xi.$$



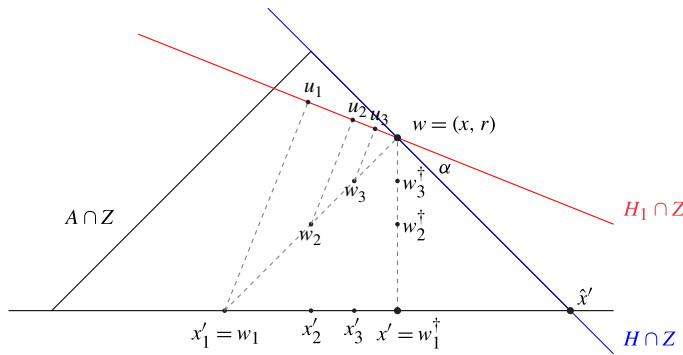


FIGURE 1. Diagram showing the first few terms of each sequence.

Hence, there exists some  $N \in \mathbb{N}$  such that, for all  $n > N$ ,  $d(w_n, E^{(0)} \times \mathbb{R}) \leq \xi R(w_n)$ . Since

$$d(x_n, M) \geq d(x, M) - \|x - x_n\| = \left(1 - \frac{1}{n}\right)d(x, M) = r_n = |r_n|,$$

$w_n \in C$  for all  $n \in \mathbb{N}$ , and so  $w_n \in Q$  for all  $n > N$ . Hence, by Proposition 4.8,  $B[w_n; R(w_n)]$  is contained in  $\bar{D}$  for all  $n > N$ . Since  $u_n$  is the nearest point on  $H_1 \cap Z$  to  $w_n$  for any  $n \in \mathbb{N}$ , and  $H_1$  supports  $\bar{D}$ , we have

$$\frac{R(w_n)}{\|w - w_n\|} \leq \frac{\|u_n - w_n\|}{\|w - w_n\|} = \cos(\alpha) < 1$$

for all  $n > N$ , which contradicts  $\lim_{n \rightarrow \infty} (R(w_n)/\|w - w_n\|) = 1$ . Hence,  $\bar{D}$  is smooth at  $w$ . □

We now work towards the key theorem for this section.

**LEMMA 4.22.** *Suppose that  $u := (u_1, \dots, u_{n+1}), v := (v_1, \dots, v_{n+1}) \in E_{n+1}, \|u\| = \|v\|$ , and  $(u_1, \dots, u_n) = (v_1, \dots, v_n)$ . If  $u_{n+1}, v_{n+1} \geq 0$  or  $u_{n+1}, v_{n+1} \leq 0$ , then  $u = v$ .*

**PROOF.** We have  $u_{n+1}^2 = \|u\|^2 - \sum_{k=1}^n u_k^2 = \|v\|^2 - \sum_{k=1}^n v_k^2 = v_{n+1}^2$ . Since  $u_{n+1}$  and  $v_{n+1}$  have the same sign, it follows that  $u_{n+1} = v_{n+1}$ , and so  $u = v$ . □

The following result is a slight modification of Theorem 2.42. We will make use of it later to prove the uniqueness of nearest points.

**PROPOSITION 4.23.** *Let  $U$  be a nonempty open subset of  $E$  and let  $K'$  be a nonempty subset of  $E \times [0, \infty)$ . If  $\phi : U \rightarrow \mathbb{R}$  is defined by  $\phi(y) := d(y', K')$  and  $x \in U$  is a point of Gâteaux differentiability of  $\phi$ , then  $P_{K'}(x')$  contains at most one element.*

**PROOF.** Clearly, if  $P_{K'}(x')$  is empty or  $\{x'\}$  we are done, so we might as well assume that  $y \in P_{K'}(x')$  and  $y \neq x'$ . For any  $z \in E$  and  $\lambda > 0$  we have

$$\|x' + \lambda z' - y\| \geq d(x' + \lambda z', K').$$

Since  $U$  is open, for sufficiently small  $\lambda > 0$ ,  $x + \lambda z \in E \setminus K$ . Thus, for sufficiently small  $\lambda > 0$ ,

$$\frac{\|x' + \lambda z' - y\| - \|x' - y\|}{\lambda} \geq \frac{d(x' + \lambda z', K') - d(x', K')}{\lambda} = \frac{\phi(x + \lambda z) - \phi(x)}{\lambda}.$$

Taking the limit of both sides as  $\lambda \rightarrow 0^+$  gives  $g(z') \geq f(z)$ , where  $g \in (E \times \mathbb{R})^*$  is the Gâteaux derivative of the norm at  $(x' - y) \neq 0$  (which exists since  $E \times \mathbb{R}$  is smooth) and  $f \in E^*$  is the Gâteaux derivative of  $\phi$  at  $x$ . It is straightforward to show that  $g(v) = (x' - y)/\|x' - y\| \cdot v$  for any  $v$  in  $E \times \mathbb{R}$ . Since all of the above inequalities still hold if we replace  $z$  by  $-z$  and  $z \in E$  was arbitrary, we have  $g(z') = f(z)$  for all  $z \in E$ . Thus, if  $w \in P_{K'}(x')$  is another best approximation to  $x'$  in  $K'$ , we have  $g(z') = f(z) = h(z')$  for all  $z \in E$ , where  $h \in (E \times \mathbb{R})^*$  is the Gâteaux derivative of the norm at  $x' - w \neq 0$ . As before,  $h(v) = (x' - w)/\|x' - w\| \cdot v$  for all  $v \in E \times \mathbb{R}$ . Therefore,  $((x' - y)/\|x' - y\|)|_{E \times \{0\}} = ((x' - w)/\|x' - w\|)|_{E \times \{0\}}$ , that is, these points agree apart from at possibly their last coordinate. Since  $\|((x' - y)/\|x' - y\|)\| = 1 = \|((x' - w)/\|x' - w\|)\|$  and  $x' - y, x' - w \in E \times [0, -\infty)$ , it follows by Lemma 4.22 that  $(x' - y)/\|x' - y\| = (x' - w)/\|x' - w\|$ . As  $\|x' - y\| = \|x' - w\| = d(x', K')$ , we have  $y = w$ , which completes the proof.  $\square$

4.2.3. *The inductive step.* Firstly, we define a function on  $\overline{E \setminus M}$  using  $\overline{D}$  and derive some of its properties.

**DEFINITION 4.24.** Define  $\rho : \overline{E \setminus M} \rightarrow \mathbb{R}$  by  $\rho(x) := \max\{r : (x, r) \in \overline{D}\}$  for all  $x$  in  $\overline{E \setminus M}$ .

**REMARK 4.25.** This function is well defined since  $\overline{D}$  is closed and bounded and  $(E \setminus M) \times \{0\} \subseteq D$ . Furthermore, it is straightforward to show that for any nonnegative concave function  $f : U \rightarrow \mathbb{R}$ , where  $U$  is a convex subset of  $E$ , with nonempty interior, the hypograph of  $f$ ,

$$\text{hyp}(f) := \{(x, r) : x \in U, 0 \leq r \leq f(x)\},$$

is smooth at  $(x, f(x))$  for some  $x \in \text{int}(U)$  if and only if  $f|_{\text{int}(U)}$  is Gâteaux differentiable at  $x$ .

**PROPOSITION 4.26.** For all  $x \in E \setminus M$ ,  $0 < \rho(x) \leq d(x, M)$ . Furthermore,  $\rho$  is concave,  $\rho|_{\text{Bd}(M)} = 0$ , and  $\rho|_{E \setminus M}$  is Gâteaux differentiable.

**PROOF.** Let  $x \in \overline{E \setminus M}$ . By the definition of  $\rho$  and Proposition 4.11,  $(x, \rho(x)) \in \overline{D} \subseteq A$ . Hence, by the definition of  $A$ ,  $0 \leq \rho(x) \leq d(x, M)$ . Thus, for any  $x \in \text{Bd}(M)$ ,

$$0 \leq \rho(x) \leq d(x, M) = 0,$$

and so  $\rho|_{\text{Bd}(M)} = 0$ . To see the concavity of  $\rho$ , suppose that  $x, y \in \overline{E \setminus M}$  and  $0 \leq \lambda \leq 1$ . Since  $(x, \rho(x)), (y, \rho(y)) \in \overline{D}$  and  $\overline{D}$  is convex, we have

$$(\lambda x + (1 - \lambda)y, \lambda \rho(x) + (1 - \lambda)\rho(y)) = \lambda(x, \rho(x)) + (1 - \lambda)(y, \rho(y)) \in \overline{D}.$$

Thus,  $\lambda \rho(x) + (1 - \lambda)\rho(y) \leq \rho(\lambda x + (1 - \lambda)y)$ . Suppose, looking for a contradiction, that  $\rho(x) = 0$  for some  $x \in E \setminus M$ . Since  $\rho$  is concave and  $\rho|_{\text{Bd}(M)} = 0$ , this implies that

$\rho = 0$ . We now show that  $\rho(0) > 0$ , which is of course a contradiction. As  $0 \in E \setminus M$ , we have  $(0, d(0, M)) \in C$  and  $d(0, M) > 0$ . Furthermore, since  $0 \in E^{(0)}$ ,  $(0, d(0, M)) \in Q$ . By Proposition 4.8, we have  $(0, d(0, M)) \in \overline{D}$ , and so  $\rho(0) \geq d(0, M) > 0$ . Thus,  $\rho(x) > 0$  for all  $x \in E \setminus M$ .

Finally, we check that  $\rho|_{E \setminus M}$  is Gâteaux differentiable. Let  $x \in E \setminus M$ . Since  $\rho(x) > 0$ ,  $(x, \rho(x)) \in \text{Bd}(D)$ , and  $\overline{D} = \text{hyp}(\rho)$ , Proposition 4.21 and the previous remark tell us that  $\rho$  is Gâteaux differentiable at  $x$ . □

**PROPOSITION 4.27.** *Let  $x \in \overline{E \setminus M}$ :*

- (i) *if  $d(x, M) \leq b/2$ , then  $\rho(x) = d(x, M)$ ;*
- (ii) *if  $d(x, M) \geq b/2$ , then  $\rho(x) \geq b/2$ .*

**PROOF.** (i) Let  $x \in \overline{E \setminus M}$  be such that  $d(x, M) \leq b/2$  and  $y \in P_M(x)$ . Since  $M$  has the  $b$ -boundary property, there exists some  $z \in E \setminus M$  such that  $\|y - z\| = d(z, M) = b$ . Since  $y \in P_M(z)$ , Lemma 2.39 tells us that  $x, y, z$  are collinear. Let  $w := (y + z)/2$ . Since  $w \in [y, z]$ ,  $y \in P_M(w)$  and  $d(w, M) = \|y - w\| = \|y - z\|/2 = b/2$ . We now show that  $\rho(w) = b/2$ . Since  $\rho(w) \leq d(w, M)$ , it is sufficient to show that  $(w, b/2) \in \overline{D}$ . Clearly,  $z' \in C$  and  $R(z') = d(z, M)/\sqrt{2} = b/\sqrt{2}$ . As  $E^{(0)} \cap (E \setminus M) \neq \emptyset$ , we have that

$$\begin{aligned} d(z', E^{(0)} \times \mathbb{R}) &= d(z, E^{(0)}) \\ &\leq d(z, E^{(0)} \cap (E \setminus M)) \\ &\leq d \leq \frac{d\sqrt{8}}{b} \cdot \frac{b}{\sqrt{2}} = \xi R(z'), \end{aligned}$$

and so  $z' \in Q$ . Also,  $\|(w, b/2) - z'\|^2 = \|w - z'\|^2 + (b/2)^2 = (b/2)^2 + (b/2)^2 = b^2/2$ . Hence,  $\|(w, b/2) - z'\| = b/\sqrt{2} = R(z')$ , and so  $(w, b/2) \in B[z; R(z')] \subseteq \overline{D}$  as required. Finally, since  $d(x, M) = \|x - y\| \leq b/2 = \|y - w\|$  and  $x, y, z$  are collinear, there exists some  $\lambda \in [0, 1]$  such that  $x = \lambda y + (1 - \lambda)w$ . As  $\rho$  is concave and  $\rho(y) = 0$  (as  $y \in \text{Bd}(M)$ ), we have

$$\rho(x) \geq \lambda\rho(y) + (1 - \lambda)\rho(w) = (1 - \lambda)\rho(w) = (1 - \lambda)\frac{b}{2} = \|y - x\| = d(x, M).$$

Since  $\rho(x) \leq d(x, M)$  by Proposition 4.26,  $\rho(x) = d(x, M)$  as required.

(ii) Let  $x \in \overline{E \setminus M}$  be such that  $d(x, M) \geq b/2$ . Take any straight line in  $E$  passing through  $x$ . Since  $E \setminus M$  is bounded and convex, this line must intercept  $\text{Bd}(M)$  at two distinct points  $y_1$  and  $y_2$ . As  $d(y_1, M) = d(y_2, M) = 0$  and the distance function for  $M$  is continuous, the intermediate value theorem tells us that there exist  $z_1 \in [x, y_1]$  and  $z_2 \in [x, y_2]$  such that  $d(z_1, M) = d(z_2, M) = b/2$ . By (i),  $\rho(z_1) = \rho(z_2) = b/2$ . Since  $\rho$  is concave and  $x \in [z_1, z_2]$ , it follows that  $\rho(x) \geq b/2$ . □

We now use a clever trick to construct a set in  $E \times [0, \infty)$  such that the distance function for this set, restricted to  $E \setminus M$ , is given by the smooth function  $\rho$ . Making use of Proposition 4.23, we will be able to prove the uniqueness of nearest points.

**DEFINITION 4.28.** Define the set  $M' := (E \times [0, \infty)) \setminus \bigcup_{x \in E \setminus M} B(x'; \rho(x))$ .

**PROPOSITION 4.29.** *M' has the following properties.*

- (i)  $M \times \{0\} = M' \cap (E \times \{0\})$ ,  $M'$  is closed in  $E \times \mathbb{R}$ , and  $(E \times [0, \infty)) \setminus M'$  is nonempty, convex and bounded.
- (ii) For every  $x \in E$  there exists a unique  $y \in M'$  such that  $\|x' - y\| = d(x', M')$ . Furthermore, for  $x \in E \setminus M$ ,  $d(x', M') = \rho(x)$ .
- (iii)  $0 \notin M'$ .
- (v) For every  $y \in \text{Bd}(M') \setminus (\text{int}(M) \times \{0\})$  there exists  $x \in E \setminus M$  such that  $P_{M'}(x') = \{y\}$ .

**PROOF.** (i) Firstly, we show that  $M \times \{0\} \subseteq M'$ . Suppose, looking for a contradiction, that there exists  $x \in M$  such that  $x' \notin M'$ . Since  $x' \in E \times [0, \infty)$ , it follows that

$$x' \in \bigcup_{y \in E \setminus M} B(y'; \rho(y)).$$

Hence, for some  $y \in E \setminus M$  we have  $\|x - y\| = \|x' - y'\| < \rho(y) \leq d(y, M)$ , which is impossible since  $x \in M$ . Therefore,  $M \times \{0\} \subseteq M'$ . Secondly, we show that

$$(E \setminus M) \times \{0\} \subseteq (E \times [0, \infty)) \setminus M'.$$

It is straightforward to show that  $(E \times [0, \infty)) \setminus M' = (E \times [0, \infty)) \cap \bigcup_{x \in E \setminus M} B(x'; \rho(x))$ . Let  $x' \in (E \setminus M) \times \{0\}$ . Clearly,  $x' \in E \times [0, \infty)$ . By Proposition 4.26,  $\rho(x) > 0$ , and so

$$x' \in B(x'; \rho(x)) \subseteq \bigcup_{z \in E \setminus M} B(z'; \rho(z)).$$

Thus,  $x' \in E \times [0, \infty) \setminus M'$ . Combining these two set inclusions gives the equation  $M \times \{0\} = M' \cap (E \times \{0\})$  as required. Furthermore, since  $0 \in E \setminus M$ ,  $(E \times [0, \infty)) \setminus M'$  is nonempty.

To see that  $M'$  is closed in  $E \times \mathbb{R}$ , observe that  $(E \times \mathbb{R}) \setminus M'$  is a union of two open sets, that is,  $(E \times \mathbb{R}) \setminus M' = (E \times (-\infty, 0)) \cup \bigcup_{x \in E \setminus M} B(x'; \rho(x))$ .

The convexity of  $(E \times [0, \infty)) \setminus M' = (E \times [0, \infty)) \cap \bigcup_{x \in E \setminus M} B(x'; \rho(x))$  follows from the concavity of  $\rho$ . Let  $x, y \in (E \times [0, \infty)) \setminus M'$  and  $0 \leq \lambda \leq 1$ . Clearly, the convex combination  $\lambda x + (1 - \lambda)y \in E \times [0, \infty)$ . Moreover, there exist  $w, z \in E \setminus M$  such that  $\|x - w'\| < \rho(w)$  and  $\|y - z'\| < \rho(z)$ . Then

$$\begin{aligned} \|\lambda x + (1 - \lambda)y - (\lambda w + (1 - \lambda)z)'\| &= \|\lambda(x - w') + (1 - \lambda)(y - z')\| \\ &\leq \lambda\|x - w'\| + (1 - \lambda)\|y - z'\| \\ &< \lambda\rho(w) + (1 - \lambda)\rho(z) \\ &\leq \rho(\lambda w + (1 - \lambda)z). \end{aligned}$$

Therefore,  $\lambda x + (1 - \lambda)y \in B((\lambda w + (1 - \lambda)z)'; \rho(\lambda w + (1 - \lambda)z))$ . Since  $E \setminus M$  is convex, it follows that  $\lambda w + (1 - \lambda)z \in E \setminus M$ , and so  $\lambda x + (1 - \lambda)y \in (E \times [0, \infty)) \setminus M'$  as required.

Finally, since  $E \setminus M$  is bounded and  $\rho(x) \leq d(x, M)$  for all  $x \in \overline{E \setminus M}$ , it follows that  $\rho$  is bounded, and so  $(E \times [0, \infty)) \setminus M'$  is bounded.

(ii) Let  $x \in E$ . If  $x \in M$ , then  $x' \in M'$ , and so  $x'$  has itself as its unique nearest point in  $M'$ . Alternatively,  $x \in E \setminus M$ . We will now find a point  $y \in M'$  such that  $\|x' - y\| = \rho(x) = d(x', M')$ . Observe that  $d(x', M') \geq \rho(x)$  by the definition of  $M'$ , so it is sufficient to find  $y \in M'$  such that  $\|x' - y\| = \rho(x)$ . Let  $H$  be a hyperplane in  $E \times \mathbb{R}$ , supporting the convex set  $\overline{D}$  at  $(x, \rho(x)) \in \text{Bd}(\overline{D})$ . Firstly, suppose that  $H$  is parallel to  $E \times \{0\}$ . Let  $y := (x, \rho(x))$ . Since  $\rho$  is concave and smooth on  $E \setminus M$ ,  $\rho$  must have a global maximum at  $x$ . Clearly,  $\|x' - y\| = \rho(x)$ , so all we need show is that  $y \in M'$ . Suppose, looking for a contradiction, that  $y \notin M'$ . Since  $\rho(x) > 0$ ,  $y \in E \times [0, \infty)$ , and so  $y \in B(w'; \rho(w'))$  for some  $w \in E \setminus M$ . Then

$$\begin{aligned} \|y - w'\|^2 &= \|(x, \rho(x)) - (w, 0)\|^2 \\ &= \|w - x\|^2 + \rho(x)^2 \\ &\geq \rho(x)^2, \end{aligned}$$

and so  $\rho(x) \leq \|y - w'\| < \rho(w)$ , which contradicts the maximality of  $\rho(x)$ . Hence,  $y \in M'$ , and so  $y \in P_{M'}(x')$ .

Alternatively,  $H$  and  $E \times \{0\}$  are not parallel. The intersection of these two hyperplanes is a translate of a codimension-two subspace of  $E \times \mathbb{R}$ , which we call  $N$ . Suppose, looking for a contradiction, that there exists some  $y' \in N \cap ((E \setminus M) \times \{0\})$ . Since  $y' \in (E \setminus M) \times \{0\} \subseteq D$  and  $H$  supports  $\overline{D}$ , we have that  $\rho(y) = 0$ , which is impossible by Proposition 4.26. Thus,  $N \cap ((E \setminus M) \times \{0\}) = \emptyset$ . As  $x \in E \setminus M$ , this implies that  $H$  is not perpendicular to  $E \times \{0\}$ . Let  $H_1$  be a hyperplane in  $E \times \mathbb{R}$  passing through  $N$  and tangent to  $B[x'; \rho(x)]$  such that the point of contact  $y$  is in  $E \times [0, \infty)$ . This is possible since  $B[x'; \rho(x)] \cap (E \times \{0\}) \subseteq E \setminus M \times \{0\}$ . Clearly,  $\|x' - y\| = \rho(x) > 0$ , so again it is sufficient to show that  $y \in M'$ . Suppose that  $H$  and  $E \times \{0\}$  form an angle  $0 < \alpha \leq \pi/4$ , whilst  $H_1$  and  $E \times \{0\}$  form an angle  $\alpha_1$ . Since  $H_1$  and the vector  $y - x'$  must be orthogonal, it follows that  $\tan(\alpha) = \sin(\alpha_1)$ . Therefore, the distance from any point  $(w, r) \in E \times \mathbb{R}$  on  $H$  to  $E \times \{0\}$ , which is just  $r$ , is equal to the distance from  $w'$  to  $H_1$ . Looking for a contradiction, suppose that  $y \notin M'$ . By construction,  $y \in E \times [0, \infty)$ , so there exists  $z \in E \setminus M$  such that  $\|y - z'\| \leq \rho(z)$ . Let  $v$  be the point directly above  $z'$  lying on  $H$ . Since  $H$  supports  $\overline{D}$ ,  $(z, \rho(z)) \in \overline{D}$  and  $y \in H_1$ , it follows by the previous observation that  $\rho(z) \leq d(v, E \times \{0\}) = d(z', H_1) \leq \|y - z'\| < \rho(z)$ , which is clearly impossible. Therefore,  $y \in M'$  as required.

We have now established that  $d(z', M') = \rho(z)$  for all  $z \in E \setminus M$ . As  $\rho|_{E \setminus M}$  is smooth, Proposition 4.23 tells us that  $P_{M'}(x)$  is a singleton.

(iii) By assumption,  $0 \in E \setminus M$ . By (ii) and Proposition 4.26, we have  $d(0, M') = \rho(0) > 0$ . Hence,  $0 \notin M'$ .

(iv) We begin by showing that

$$\overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))} = \bigcup_{x \in E \setminus M} B[x'; \rho(x)].$$

Let  $y \in \overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))}$ . Hence, there exist sequences  $(y_n)_{n=1}^\infty$  and  $(x_n)_{n=1}^\infty$  in  $E \times \mathbb{R}$  and  $E \setminus M$  respectively, such that  $\lim_{n \rightarrow \infty} y_n = y$  and  $\|y_n - x'_n\| < \rho(x_n)$  for all  $n \in \mathbb{N}$ .

Since  $\overline{E \setminus M}$  is compact, we may assume, without loss of generality, that  $(x_n)_{n=1}^\infty$  converges to some  $x \in \overline{E \setminus M}$ . Firstly, suppose that  $x \in E \setminus M$ . Then

$$\|y - x'\| = \left\| \lim_{n \rightarrow \infty} y_n - \lim_{n \rightarrow \infty} x'_n \right\| = \lim_{n \rightarrow \infty} \|y_n - x'_n\| \leq \lim_{n \rightarrow \infty} \rho(x_n) = \rho(x),$$

since  $\rho$  is smooth (and hence continuous) on  $E \setminus M$ . Thus,  $y \in B[x'; \rho(x)]$ . Alternatively,  $x \in \text{Bd}(M)$ . Therefore,

$$\|y - x'\| \leq \lim_{n \rightarrow \infty} \rho(x_n) \leq \lim_{n \rightarrow \infty} d(x_n, M) = d(x, M) = 0,$$

and so  $y = x' \in \text{Bd}(M) \times \{0\}$ . Since  $M$  has the  $b$ -boundary property, there exists  $z \in E \setminus M$  such that  $\|y - z\| = d(z, M) = b$ . Consider the point  $w := (y + z)/2$ . Since  $y, z \in \overline{E \setminus M}$ , which is convex, we have  $w \in \overline{E \setminus M}$ . As  $w \in [y, z]$ ,

$$d(w, M) = \|y - w\| = \left\| \frac{y - z}{2} \right\| = \frac{b}{2} > 0$$

by Proposition 2.17, and so  $w \in E \setminus M$ . By Proposition 4.27,  $\rho(w) = d(w, M) = b/2 > 0$ . Therefore,  $y \in B[w; d(w, M)] \times \{0\} \subseteq B[w'; \rho(w)]$ . Thus,  $\overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))}$  is contained in  $\bigcup_{x \in E \setminus M} B[x'; \rho(x)]$ . Set inclusion obviously holds in the reverse direction, which establishes the required equality.

Now suppose that  $w \in \text{Bd}(M') \setminus (\text{int}(M) \times \{0\})$ . By what we have just shown, there exists some  $x \in E \setminus M$  such that  $w \in B[x'; \rho(x)]$ . Thus,  $\|x' - w\| = \rho(x)$ , since otherwise  $w \notin M'$ . By (ii), there exists a unique point in  $M'$  with distance  $\rho(x) = d(x', M')$  from  $x'$ . Hence,  $\{w\} = P_{M'}(x')$  as required.  $\square$

**PROPOSITION 4.30.** *For any  $x \in \text{Bd}(M') \setminus (\text{int}(M) \times \{0\})$  there exists  $y \in E \times [0, \infty)$  such that*

$$\|x - y\| = d(y, M') = \frac{b}{2}.$$

**PROOF.** Let  $x := (v, r) \in \text{Bd}(M') \setminus (\text{int}(M) \times \{0\})$ . By part (iv) of Proposition 4.29, there exists  $w \in E \setminus M$  such that  $P_{M'}(w') = \{x\}$ , and by part (ii) of the same proposition,  $\|w' - x\| = d(w', M') = \rho(w)$ . Firstly, suppose that  $d(w', M') \geq b/2$ . Then there exists  $y \in [w', x]$  such that  $\|y - x\| = d(y, M') = b/2$ . Clearly,  $y \in E \times [0, \infty)$ , so we are done. Alternatively, suppose that  $d(w', M') = \rho(w) < b/2$ . By part (ii) of Proposition 4.27,  $d(w, M) < b/2$ , and hence, by part (i) of the same result,  $d(w, M) = \rho(w) = d(w', M')$ . Since  $M \times \{0\} \subseteq M'$ ,  $M$  is proximal, and  $P_{M'}(w') = \{x\}$ , we must have that  $x := (v, 0) \in M \times \{0\}$ . Clearly,  $v \in \text{Bd}(M)$ , and so there exists  $z \in E \setminus M$  such that  $\|v - z\| = d(z, M) = b$ , since  $M$  has the  $b$ -boundary property. Let  $y := ((v + z)/2)' \in (E \setminus M) \times \{0\} \subseteq E \times [0, \infty)$ . By Proposition 4.27 and part (ii) of Proposition 4.29,  $b/2 = d((v + z)/2, M) = \rho((v + z)/2) = d(y', M')$  since  $d((v + z)/2, M) = \|(v + z)/2 - v\| = b/2$ .  $\square$

Whilst we have shown that  $M'$  has convex complement in  $E \times [0, \infty)$ , its complement in  $E \times \mathbb{R}$  is not convex. We use the following flip-stretch operator to extend  $M'$  to a set with convex complement in  $E \times \mathbb{R}$ .

**DEFINITION 4.31.** For  $\theta > 1$ , define  $\psi_\theta : E \times \mathbb{R} \rightarrow E \times \mathbb{R}$  by  $\psi_\theta(x, r) = (x, -\theta r)$  for all  $(x, r) \in E \times \mathbb{R}$ .

**LEMMA 4.32.** *The map  $\psi_\theta$  is a bijective linear map, with inverse given by*

$$\psi_\theta^{-1}(x, r) = \left(x, \frac{-r}{\theta}\right) \text{ for all } (x, r) \in E \times \mathbb{R}.$$

We also have the inequalities  $\|(x, r)\| \leq \|\psi_\theta(x, r)\| \leq \theta\|(x, r)\|$  for any  $(x, r)$  in  $E \times \mathbb{R}$ , with the first inequality being strict unless  $r = 0$  and the second strict unless  $x = 0$ . Thus, both  $\psi_\theta$  and its inverse are bounded and hence continuous.

**PROOF.** Linearity and bijectiveness are trivial. For the inequality, observe that for any  $(x, r)$  in  $E \times \mathbb{R}$ ,

$$\|(x, r)\|^2 = \|x\|^2 + r^2 \leq \|x\|^2 + (\theta r)^2 = \|\psi_\theta(x, r)\|^2 \leq (\theta\|x\|)^2 + (\theta r)^2 = \theta^2\|(x, r)\|^2.$$

Replacing  $(x, r)$  by  $\psi_\theta^{-1}(x, r)$  in these inequalities shows that

$$\frac{1}{\theta}\|(x, r)\| \leq \|\psi_\theta^{-1}(x, r)\| \leq \|(x, r)\|$$

for all  $(x, r) \in E \times \mathbb{R}$ . The first inequality is strict unless  $x = 0$  and the second strict unless  $r = 0$ . Hence,  $\psi_\theta$  and  $\psi_\theta^{-1}$  are bounded, and so continuous. □

**DEFINITION 4.33.** Let  $1 < \theta \leq 2$  and define  $M'' \subseteq E \times \mathbb{R}$  by  $M'' := M' \cup \psi_\theta(M')$ .

We would like to show that the properties of  $M'$  established in Proposition 4.29 are inherited by this extension. The next result is used to show that  $M''$  has the  $b/4$ -boundary property.

**LEMMA 4.34.** *Let  $S$  be the ellipsoidal surface in  $\mathbb{R}^n$  defined by the equation*

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \dots + \frac{x_n^2}{\beta^2} = 1,$$

where  $\alpha > \beta > 0$ . For any  $y \in S$  there exists  $w \in \text{co}(S)$  such that  $\|w - y\| = d(y, S) = \beta^2/\alpha$ .

**PROOF.** See Appendix E. □

**PROPOSITION 4.35.**  *$M''$  has the following properties.*

- (i)  $M \times \{0\} = M'' \cap (E \times \{0\})$ ,  $M''$  is closed in  $E \times \mathbb{R}$ ,  $0 \notin M''$ , and  $(E \times \mathbb{R}) \setminus M''$  is nonempty, convex and bounded.
- (ii) For every  $x \in E$  there exists a unique  $y \in M''$  such that

$$\|x' - y\| = d(x', M'').$$

Furthermore, when  $x \in E \setminus M$ ,  $d(x', M'') = \rho(x)$ .

- (iii)  $M''$  has the  $b/4$ -boundary property.

**PROOF.** (i) These statements follow from Proposition 4.29. Since  $\psi_\theta(E \times \{0\}) = E \times \{0\}$ , we have

$$M \times \{0\} = M' \cap (E \times \{0\}) = M'' \cap (E \times \{0\}).$$

Also,  $\emptyset \neq (E \times [0, \infty)) \setminus M' \subseteq (E \times \mathbb{R}) \setminus M'$ . It is straightforward to verify that

$$(E \times \mathbb{R}) \setminus M'' = (E \times [0, \infty)) \setminus M' \cup \psi_\theta((E \times [0, \infty)) \setminus M').$$

Since  $\psi_\theta^{-1}$  is continuous and  $M'$  is closed in  $E \times \mathbb{R}$ , the above set is the finite union of closed sets in  $E \times \mathbb{R}$ , and so is closed in  $E \times \mathbb{R}$ .

As  $0 \notin M'$  and  $\psi_\theta(0) = 0$ , it follows that  $0 \notin M''$ .

Since  $(E \times [0, \infty)) \setminus M'$  is convex and  $\psi_\theta$  is linear,  $\psi_\theta((E \times [0, \infty)) \setminus M')$  is also convex. Thus, to prove that  $(E \times \mathbb{R}) \setminus M''$  is convex we need only show that it contains  $[x, y]$  where  $x := (u, r) \in (E \times [0, \infty)) \setminus M'$  and  $y := (v, s) \in \psi_\theta((E \times [0, \infty)) \setminus M')$ . Since  $x \notin M'$ ,

$$\|u' - z\| \leq \|x - z\| < \rho(z) \leq d(z, M)$$

for some  $z \in E \setminus M$ . Therefore,  $u' \in E \setminus M$ . Similarly,  $v' \in E \setminus M$ . Since  $r \geq 0$  and  $s \leq 0$ , whilst  $E \setminus M$  is convex, there exists  $\lambda \in [0, 1]$  such that  $\lambda x + (1 - \lambda)y \in E \setminus M \times \{0\}$ . As

$$(E \setminus M) \times \{0\} \subseteq ((E \times [0, \infty)) \setminus M') \cap \psi_\theta((E \times [0, \infty)) \setminus M')$$

and the two sets in this intersection are convex, it follows that  $[x, y] \subseteq (E \times \mathbb{R}) \setminus M''$  as required.

Finally, since  $\psi_\theta$  is bounded, it follows that  $(E \times \mathbb{R}) \setminus M''$  is the finite union of bounded sets, so is bounded.

(ii) Let  $x \in E$ . If  $x \in M$ , then  $x' \in M''$  and the claim holds trivially, so we may as well assume that  $x \in E \setminus M$ . By part (ii) of Proposition 4.29, there exists a unique  $y \in M'$  such that  $\|x' - y\| = d(x', M') = \rho(x)$ . Looking for a contradiction, suppose that there exists  $z \in M'' \setminus M'$ , such that  $\|x' - z\| \leq d(x', M')$ . Since  $x' - z \notin E \times \{0\}$ , Lemma 4.32 gives

$$\begin{aligned} \|x' - \psi_\theta^{-1}(z)\| &= \|\psi_\theta^{-1}(x') - \psi_\theta^{-1}(z)\| \quad \text{since } x' \in E \times \{0\} \\ &= \|\psi_\theta^{-1}(x' - z)\| \\ &< \|x' - z\| \\ &\leq d(x, M'), \end{aligned}$$

which is impossible since  $\psi_\theta^{-1}(z) \in M'$ . Hence,  $y$  is the unique element of  $M''$  such that

$$\|x' - y\| = d(x', M'') = \rho(x).$$

(iii) Let  $x \in \text{Bd}(M'')$ . It is straightforward to verify that

$$\text{Bd}(M'') = \text{Bd}(M') \setminus (\text{int}(M) \times \{0\}) \cup \psi_\theta(\text{Bd}(M') \setminus (\text{int}(M) \times \{0\})).$$

If  $x \in \text{Bd}(M') \setminus (\text{int}(M) \times \{0\})$ , by Proposition 4.30, there exists  $y \in E \times [0, \infty)$  such that

$$\|x - y\| = d(y, M') = \frac{b}{2}.$$



Using the same argument as in part (ii), we conclude that  $\|x - y\| = d(y, M'') = b/2$ . Therefore,

$$\left\| x - \frac{x+y}{2} \right\| = d\left(\frac{x+y}{2}, M''\right) = \frac{b}{4}.$$

Alternatively,  $x \in \psi_\theta(\text{Bd}(M') \setminus (\text{int}(M) \times \{0\}))$ . As before, we can find  $y \in E \times [0, \infty)$  such that  $\|\psi_\theta^{-1}(x) - y\| = d(y, M'') = b/2$ . It follows that  $\psi_\theta(B[y; b/2]) \subseteq \overline{(E \times \mathbb{R}) \setminus M''}$  is an ellipsoid centred at  $\psi_\theta(y)$  with minor axes of length  $b/2$  and a major axis of length  $b\theta/2$  such that

$$x \in \text{Bd}\left(\psi_\theta\left(B\left[y; \frac{b}{2}\right]\right)\right).$$

By Lemma 4.34, there exists  $w \in \psi_\theta(B[y; b/2])$  such that  $B[w; b/2\theta] \subseteq \psi_\theta(B[y; b/2])$  and  $x \in B[w; b/2\theta]$ . Hence,

$$\|w - x\| = d(w, M'') = \frac{b}{2\theta}.$$

Since  $\theta \leq 2$ , we have that  $b/4 \leq b/2\theta$ . Therefore, there exists a  $z \in [w, x] \subseteq E \times \mathbb{R}$  such that  $\|z - x\| = d(z, M'') = b/4$ . □

The next result explains why we chose to define  $Q$  as in Definition 4.3.

**PROPOSITION 4.36.** *For any  $x \in E^{(0)}$ ,  $P_M(x) \times \{0\} = P_{M''}(x')$ .*

**PROOF.** Let  $x \in E^{(0)}$ . If  $x \in M$ , then there is nothing to prove, so assume that  $x \notin M$ . By assumption,  $P_M(x)$  is a singleton. Proposition 4.35 tells us that  $P_{M''}(x')$  is a singleton and  $d(x', M'') = \rho(x)$ .

We now show that  $d(x, M) = \rho(x)$ . Since  $\rho(x) \leq d(x, M)$ , it is sufficient to check that  $d(x, M) \leq \rho(x)$ , which is true provided  $w := (x, d(x, M)) \in \overline{D}$ . Clearly,  $w \in C$ , with  $R(w) = 0$ . Since  $x \in E^{(0)}$ , we have  $w \in E^{(0)} \times \mathbb{R}$ , and so  $d(w, E^{(0)} \times \mathbb{R}) = 0 = \xi R(w)$ . Thus,  $w \in Q$ , and so  $w \in \overline{D}$  by Proposition 4.8. Finally, since  $M \times \{0\} \subseteq M''$ , it follows that  $P_M(x) \times \{0\} = P_{M''}(x')$ . □

The following two results will be used to prove that our nonconvex Chebyshev set has bounded complement.

**PROPOSITION 4.37.**

$$\sup\{\|x\| : x \in (E \times \mathbb{R}) \setminus M''\} \leq 3\theta \sup\{\|x\| : x \in E \setminus M\}.$$

**PROOF.** To begin with, let

$$x \in (E \times [0, \infty)) \setminus M' = (E \times [0, \infty)) \cap \bigcup_{z \in E \setminus M} B(z'; \rho(z)).$$

Hence, there exists  $y \in E \setminus M$  such that  $\|y' - x\| < \rho(y)$ . Since the distance function for  $M$  is nonexpansive,

$$\begin{aligned} \|x\| &< \|y'\| + \rho(y) \\ &\leq \|y\| + d(y, M) \\ &\leq 2\|y\| + d(0, M) \\ &\leq 3 \sup\{\|z\| : z \in E \setminus M\}. \end{aligned}$$

Thus,

$$\sup\{\|x\| : x \in (E \times [0, \infty)) \setminus M'\} \leq 3 \sup\{\|x\| : x \in E \setminus M\}.$$

Since

$$(E \times \mathbb{R}) \setminus M'' = ((E \times [0, \infty)) \setminus M') \cup \psi_\theta((E \times [0, \infty)) \setminus M')$$

and  $\theta > 1$ , the result follows. □

**THEOREM 4.38.** *Let  $(a_n)_{n=1}^\infty$  be a sequence with nonnegative terms. Then the infinite product  $\prod_{n=1}^\infty (1 + a_n)$  converges if and only if the infinite series  $\sum_{n=1}^\infty a_n$  converges.*

**PROOF.** See [66]. □

4.2.4. *The nonconvex Chebyshev set.*

**DEFINITION 4.39.** For any  $K \subseteq E_n$ , let  $K \otimes 0 := \{(x_1, \dots, x_n, 0, 0, \dots) : (x_1, \dots, x_n) \in K\}$ . Similarly, for any  $x := (x_1, \dots, x_n) \in E_n$ , define  $x \otimes 0 := (x_1, \dots, x_n, 0, 0, \dots) \in E_n \otimes 0$ .

**DEFINITION 4.40.** Let  $E$  be the set of all real sequences with only finitely many nonzero terms, that is,

$$E := \bigcup_{n=1}^\infty (E_n \otimes 0).$$

Equip  $E$  with the inner product  $\langle \cdot, \cdot \rangle$  defined by  $\langle x, y \rangle := \sum_{k=1}^\infty x_k y_k$  for all  $x := (x_1, x_2, \dots)$  and  $y := (y_1, y_2, \dots) \in E$ . The induced norm  $\|\cdot\|$  on  $E$  is then given by  $\|x\| := \sqrt{\langle x, x \rangle}$  for all  $x \in E$ .

**THEOREM 4.41** [7, page 1175]. *In the (incomplete) inner product space  $(E, \langle \cdot, \cdot \rangle)$  there exists a nonconvex Chebyshev set with bounded convex complement.*

**PROOF.** Firstly, we inductively define a sequence of sets  $(M_n)_{n=1}^\infty$  such that, for all  $n \in \mathbb{N}$ ,  $M_n$  is a nonempty closed subset of  $E_n$ ,  $E_n \setminus M_n$  is convex,  $0 \notin M_n$ , every point in  $E_n^{(0)}$  has a unique nearest point in  $M_n$ , and  $M_n$  has the  $(\frac{1}{4})^{n-1}$ -boundary property.

To begin let  $M_1 =: E_1 \setminus (-2, 1)$ . It is clear that  $M_1$  is a closed, nonempty subset of  $E_1$ . Furthermore, it is straightforward to check that  $0 \notin M_1$ ,  $E_1 \setminus M_1$  is bounded and convex,  $M_1$  has the 1-boundary property, and every point in  $E_1^{(0)} = \{0\}$  has a unique nearest point in  $M_1$ .

Suppose then that  $M_k$  is defined for some  $k \in \mathbb{N}$ , such that  $M_k$  has the properties listed above. Construct  $M'_k$  as in Definition 4.28. Then let  $M_{k+1} := M''_k$  where  $M''_k := M'_k \cup \psi_{\theta_k}(M'_k)$  and  $\theta_k := 1 + (1/k^2)$ . By Proposition 4.35, it follows that  $M_{k+1}$

has the required properties, which completes the construction. Furthermore, by Propositions 4.35–4.37, we have

$$M_n \times \{0\} = M_{n+1} \cap (E_n \times \{0\}), \quad P_{M_n}(x) \times \{0\} = P_{M_{n+1}}(x') \quad \text{for all } x \in E_n^{(0)}$$

and

$$\sup\{\|x\| : x \in E_{n+1} \setminus M_{n+1}\} \leq 3\theta \sup\{\|x\| : x \in E_n \setminus M_n\} \quad \text{for all } n \in \mathbb{N}.$$

We now show that  $M := \bigcup_{n=1}^\infty (M_n \otimes 0)$  is a Chebyshev set in  $(E, \langle \cdot, \cdot \rangle)$ . Firstly, we show that

$$M \cap (E_k \otimes 0) = M_k \otimes 0$$

for all  $k \in \mathbb{N}$ . Let  $k \in \mathbb{N}$ . It is clear that  $M_k \otimes 0 \subseteq M \cap (E_k \otimes 0)$ , so suppose that  $x \in M \cap (E_k \otimes 0)$ . Thus, for some  $n \in \mathbb{N}$ ,  $x \in (M_n \otimes 0) \cap (E_k \otimes 0)$ . Repeatedly using the fact that  $M_m \times \{0\} = M_{m+1} \cap (E_m \times \{0\})$  for all  $m \in \mathbb{N}$ , we conclude that  $(M_n \otimes 0) \cap (E_k \times 0) \subseteq M_k \otimes 0$ . Hence,  $x \in M_k \otimes 0$ , and we are done.

Now let  $x \in E$ . Since  $x$  can have only finitely many nonzero components, there exists  $n \in \mathbb{N}$  such that  $x \in E_n^{(0)} \otimes 0$ . By construction,  $P_{M_n \otimes 0}(x) = \{y\}$  for some  $y \in M_n \otimes 0 \subseteq M$ . Furthermore, since  $P_{M_n \otimes 0}(x) = P_{M_{n+1} \otimes 0}(x)$  and  $E_n^{(0)} \otimes 0 \subseteq E_{n+1}^{(0)} \otimes 0$ , it follows by induction that  $P_{M_m \otimes 0}(x) = P_{M_n \otimes 0}(x) = \{y\}$  for all  $m \geq n$ . To show that  $y$  is the unique nearest point to  $x$  in  $M$ , suppose that there exists  $z \in M$  such that  $\|x - z\| \leq \|x - y\|$ . Let  $k \geq n$  be such that  $z \in E_k \otimes 0$ . Thus,  $P_{M_k \otimes 0}(x) = \{y\}$ . Since

$$z \in (E_k \otimes 0) \cap M = M_k \otimes 0 \quad \text{and} \quad \|x - z\| \leq \|x - y\| = d(x, M_k \otimes 0),$$

it follows that  $z = y$ . Therefore,  $P_M(x) = \{y\}$ , and so  $M$  is a Chebyshev set.

We now show that  $M$  is nonconvex. Clearly  $P_{M_1 \otimes 0}(0) = \{(1) \otimes 0\}$ . By the previous working,  $P_M(0) = \{(1) \otimes 0\}$ , and so  $0 \notin M$ . However,  $(-2) \otimes 0, (1) \otimes 0 \in M_1 \otimes 0 \subseteq M$ , so  $M$  is not convex.

Making use of the fact that  $(M_n \otimes 0) \cap (E_k \times 0) \subseteq M_k \otimes 0$  for all  $k, n \in \mathbb{N}$ , we have that

$$\begin{aligned} E \setminus M &= \left( \bigcup_{n=1}^\infty (E_n \otimes 0) \right) \setminus \left( \bigcup_{n=1}^\infty (M_n \otimes 0) \right) \\ &= \bigcup_{n=1}^\infty (E_n \otimes 0) \setminus (M_n \otimes 0) \\ &= \bigcup_{n=1}^\infty ((E_n \setminus M_n) \otimes 0). \end{aligned}$$

Hence,  $E \setminus M$  is the union of expanding convex sets, so is convex. Furthermore, since

$$\sup\{\|z\| : z \in (E_{n+1} \setminus M_{n+1}) \otimes 0\} \leq 3\theta_k \sup\{\|z\| : z \in (E_n \setminus M_n) \otimes 0\}$$

for all  $k \in \mathbb{N}$ , we have by induction that

$$\begin{aligned} \sup\{\|z\| : z \in E \setminus M\} &\leq 3 \left( \prod_{n=1}^{\infty} \theta_n \right) \sup\{\|z\| : z \in (E_1 \setminus M_1) \otimes 0\} \\ &= 6 \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right). \end{aligned}$$

By Theorem 4.38, the infinite product  $\prod_{n=1}^{\infty} (1 + (1/n^2))$  converges, since the series  $\sum_{n=1}^{\infty} (1/n^2)$  converges. Thus, the right-hand side of the previous expression is finite, and so  $E \setminus M$  is bounded.  $\square$

Theorem 4.41 seems to be the closest known answer to the conjecture of Klee [51], that a nonconvex Chebyshev set exists in some infinite-dimensional Hilbert space.

### Appendix A

**THEOREM (Primitive Ekeland theorem [35]).** *Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function on  $X$ , bounded below. If  $\varepsilon > 0$  and  $x_0 \in X$ , then there exists  $x_\infty \in X$  such that:*

- (i)  $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$ ; and
- (ii)  $f(x_\infty) - \varepsilon d(x_\infty, x) < f(x)$  for all  $x \in X \setminus \{x_\infty\}$ .

**PROOF.** We shall inductively define a sequence  $(x_n)_{n=1}^{\infty}$  in  $X$  and a sequence  $(D_n)_{n=1}^{\infty}$  of closed subsets of  $X$  such that:

- (i)  $D_n := \{x \in D_{n-1} : f(x) \leq f(x_{n-1}) - \varepsilon d(x, x_{n-1})\}$ ;
- (ii)  $x_n \in D_n$ ;
- (iii)  $f(x_n) \leq \inf_{x \in D_n} f(x) + \varepsilon^2/(n + 1)$ .

Set  $D_0 := X$ . In the base step we let

$$D_1 := \{x \in D_0 : f(x) \leq f(x_0) - \varepsilon d(x, x_0)\}$$

and choose  $x_1 \in D_1$  so that  $f(x_1) \leq \inf_{x \in D_1} f(x) + \varepsilon^2/2$ . Then at the  $(n + 1)$ th step we let

$$D_{n+1} := \{x \in D_n : f(x) \leq f(x_n) - \varepsilon d(x, x_n)\}$$

and we choose  $x_{n+1} \in D_{n+1}$  such that

$$f(x_{n+1}) \leq \inf_{x \in D_{n+1}} f(x) + \varepsilon^2/(n + 2).$$

This completes the induction.

Now, by construction, the sets  $(D_n)_{n=1}^{\infty}$  are closed and  $\emptyset \neq D_{n+1} \subseteq D_n$  for all  $n \in \mathbb{N}$ . It is also easy to see that  $\sup\{d(x, x_n) : x \in D_{n+1}\} \leq \varepsilon/(n + 1)$ . Indeed, if  $x \in D_{n+1}$  and

$\varepsilon/(n + 1) < d(x, x_n)$ , then

$$\begin{aligned} f(x) &< \left[ f(x_n) - \varepsilon \left( \frac{\varepsilon}{n + 1} \right) \right] = f(x_n) - \frac{\varepsilon^2}{n + 1} \\ &\leq \left[ \inf_{y \in D_n} f(y) + \frac{\varepsilon^2}{n + 1} \right] - \frac{\varepsilon^2}{n + 1} = \inf_{y \in D_n} f(y), \end{aligned}$$

which contradicts the fact that  $x \in D_{n+1} \subseteq D_n$ .

Let  $\{x_\infty\} := \bigcap_{n=1}^\infty D_n$ . Fix  $x \in X \setminus \{x_\infty\}$  and let  $n$  be the first natural number such that  $x \notin D_n$ , that is,  $x \in D_{n-1} \setminus D_n$ . Then

$$f(x_{n-1}) - \varepsilon d(x, x_{n-1}) < f(x). \tag{*}$$

On the other hand, since  $x_\infty \in D_n$ ,

$$f(x_\infty) \leq f(x_{n-1}) - \varepsilon d(x_\infty, x_{n-1}),$$

and so

$$\begin{aligned} f(x_\infty) - \varepsilon d(x, x_\infty) &\leq f(x_{n-1}) - \varepsilon [d(x, x_\infty) + d(x_\infty, x_{n-1})] \\ &\leq f(x_{n-1}) - \varepsilon d(x, x_{n-1}) \quad \text{by the triangle inequality} \\ &< f(x) \quad \text{by (*)}. \end{aligned}$$

Finally, note that  $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$ , since  $x_\infty \in D_1$ . □

### Appendix B

**PROPOSITION.** For any  $x \in E$  and  $r > 0$ ,

$$K[x; r] = E \times [0, \infty) \cap \bigcup_{\lambda \in [0,1]} B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r].$$

**PROOF.** Let  $(y, t) \in K[x; r]$ . Hence,  $t \geq 0$  and  $\|y - x\| + t \leq r$ . Define

$$\lambda := \frac{r + \|y - x\| - t}{2r}.$$

Since

$$\|y - x\| - t \leq \|y - x\| + t \leq r$$

and

$$0 \leq 2\|y - x\| = \|y - x\| + t + \|y - x\| - t \leq r + \|y - x\| - t,$$

we see that  $\lambda \in [0, 1]$ . It is straightforward to show that

$$\|(y, t) - (x, (1 - 2\lambda)r)\|^2 = \|y - x\|^2 + (t - (1 - 2\lambda)r)^2 = 2\|y - x\|^2$$

whilst

$$(\lambda \sqrt{2}r)^2 = \frac{1}{2}(r + \|y - x\| - t)^2 \geq 2\|y - x\|^2.$$

Therefore,  $(y, t) \in B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r]$  and we have set inclusion in one direction.

For the opposite direction, suppose that

$$(y, t) \in \mathbb{R}^n \times [0, \infty) \cap \bigcup_{\lambda \in [0,1]} B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r].$$

Therefore,  $t \geq 0$  and  $\|(y, t) - (x, (1 - 2\lambda)r)\|^2 = \|y - x\|^2 + (t - (1 - 2\lambda)r)^2 \leq (\sqrt{2}\lambda r)^2$  for some  $\lambda \in [0, 1]$ . Since

$$(r - t)^2 - ((\sqrt{2}\lambda r)^2 - (t - (1 - 2\lambda)r)^2) = (\sqrt{2}(\lambda - 1)r + \sqrt{2}t)^2 \geq 0,$$

it follows that  $\|y - x\|^2 \leq (r - t)^2$ . Finally, for  $\lambda \in [0, 1]$ ,

$$t \leq (1 - 2\lambda)r + \lambda \sqrt{2}r = r + \lambda(\sqrt{2} - 2)r \leq r,$$

and so we conclude that  $\|y - x\| + t \leq r$ . Therefore,  $(y, t) \in K[x; r]$  and we have set inclusion in the opposite direction. □

**COROLLARY.** *If  $x \in E$  and  $r > 0$ , then  $K[x; r]$  is smooth at any  $(y, t) \in E \times (0, r)$ , such that  $\|y - x\| + t = r$ .*

**PROOF.** Let  $(y, t) \in E \times (0, r)$  and suppose that  $\|y - x\| + t = r$ . Clearly,  $(y, t) \in \text{Bd}(K[x; r])$ . By Proposition 4.16, there exists  $\lambda \in [0, 1]$  such that

$$(y, t) \in E \times [0, \infty) \cap B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r] \subseteq K[x; r].$$

Since  $t \neq r$ , we have  $\lambda > 0$ , and so  $B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r]$  is smooth. Furthermore, since  $t \neq 0$ ,  $(y, t)$  is a smooth point of  $E \times [0, \infty) \cap B[(x, (1 - 2\lambda)r); \lambda \sqrt{2}r]$ . Therefore,  $K[x; r]$  is smooth at  $(y, t)$ . □

**PROPOSITION.** *Let  $x \in E \setminus M$  and suppose that  $(x, d(x, M)) \in \overline{D}$ . Then  $K[x; d(x, M)] \subseteq \overline{D}$ .*

**PROOF.** Since  $\overline{D}$  is convex and  $K[x; d(x, M)]$  is simply the convex hull of the set

$$\{(x, d(x, M))\} \cup B[x; d(x, M)] \times \{0\},$$

we need only show that  $B[x; d(x, M)] \times \{0\} \subseteq \overline{D}$ . This follows from Proposition 4.10 and the fact that  $B[x; d(x, M)] \subseteq E \setminus M$ . □

### Appendix C

One of the fundamental tools of elementary analysis is the ‘sandwich theorem’ (sometimes called the ‘squeeze theorem’). The theory of differentiation is no different. In particular, we have the following result.

If  $f : X \rightarrow \mathbb{R}$ ,  $g : X \rightarrow \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}$  are functions defined on a normed linear space  $(X, \|\cdot\|)$  and:

- (i)  $f(y) \leq g(y) \leq h(y)$  for all  $y \in X$ ;
- (ii)  $f(x) = g(x) = h(x)$  for some  $x \in X$ ; and
- (iii) both  $\nabla f(x)$  and  $\nabla h(x)$  exist;

then  $\nabla g(x)$  exists and equals  $\nabla f(x) = \nabla h(x)$ . This result holds for both Gâteaux and Fréchet derivatives. Furthermore, it has direct implications for the differentiability of distance functions.

Suppose that  $K$  is a Chebyshev subset of a Gâteaux (Fréchet) smooth normed linear space  $(X, \|\cdot\|)$ . Then for each  $x \in X \setminus K$  the distance function,  $y \mapsto d(y, K)$ , is Gâteaux (Fréchet) differentiable at each point of  $(x, p_K(x))$ .

To see this, let  $f : X \rightarrow \mathbb{R}$ ,  $g : X \rightarrow \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}$  be defined by

$$f(y) := d(x, K) - \|y - x\|, \quad g(y) := d(y, K) \quad \text{and} \quad h(y) := \|y - p_K(x)\|.$$

Since  $g$  is nonexpansive and  $f(x) = g(x)$ , we have that  $f(y) \leq g(y)$  for all  $y \in X$ . Similarly, since  $h(p_K(x)) = g(p_K(x)) = 0$ , we have that  $g(y) \leq h(y)$  for all  $y \in X$ . Therefore,  $f(y) \leq g(y) \leq h(y)$  for all  $y \in X$ . On the other hand,  $h - f$  is convex and  $(h - f)(x) = (h - f)(p_K(x)) = 0$ . Therefore,  $(h - f)(y) \leq 0$  for all  $y \in [x, p_K(x)]$ , and so  $h(y) \leq f(y)$  for all  $y \in [x, p_K(x)]$ . Thus, we have that  $f(y) = g(y) = h(y)$  for all  $y \in [x, p_K(x)]$ . Finally, both  $f$  and  $h$  are Gâteaux (Fréchet) differentiable on  $(x, p_K(x))$ . In fact,  $f$  is Gâteaux (Fréchet) differentiable on  $X \setminus \{x\}$  and  $h$  is Gâteaux (Fréchet) differentiable on  $X \setminus \{p_K(x)\}$ . The result now follows from the ‘sandwich theorem’ given above.

The following results, which are in essence due to Simon Fitzpatrick [37], are an attempt to extend the aforementioned result on the differentiability of the distance function on  $(x, p_K(x))$  to the differentiability of the distance function on  $[x, p_K(x)]$ .

**PROPOSITION.** *Let  $(X, \|\cdot\|)$  be a Banach space and suppose that  $g : X \rightarrow \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}$  are functions. Let  $\delta, \varepsilon > 0$  and  $x_0 \in X$ . Suppose also that (a)  $g(x) \leq h(x)$  for all  $x \in X$  and  $g(x_0) = h(x_0)$  and (b)  $h$  is Gâteaux differentiable at  $x_0$  with Gâteaux derivative  $\nabla h(x_0)$ . If there exist an  $x^* \in X^*$  and functions  $f_\lambda : X \rightarrow \mathbb{R}$  (for  $0 < \lambda < \delta$ ) such that:*

- (i)  $f_\lambda(x) \leq g(x)$  for all  $x \in X$ ;
- (ii)  $\lim_{\lambda \rightarrow 0^+} (f_\lambda(x_0) - g(x_0))/\lambda = 0$ ; and
- (iii)  $\liminf_{\lambda \rightarrow 0^+} (f_\lambda(x_0 + \lambda y) - f_\lambda(x_0))/\lambda \geq x^*(y) - \varepsilon$  for all  $y \in S_X$ ;

then

$$x^*(y) - \varepsilon \leq \liminf_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all  $y \in S_X$ .

**PROOF.** The straightforward proof is left as an exercise for the reader. □

**THEOREM [7, Theorem 1.4].** *Let  $K$  be a nonempty closed subset of a Fréchet smooth normed linear space  $(X, \|\cdot\|)$  and let  $x_0 \in X \setminus K$ . If  $P_K(x_0)$  is nonempty and*

$$\lim_{t \rightarrow 0^+} \frac{d(x_0 + tv, K) - d(x_0, K)}{t} = 1$$

for some  $v \in S_X$ , then the distance function for  $K$  is Gâteaux differentiable at  $x_0$ .

**PROOF.** We shall apply the previous proposition. Let  $\varepsilon$  be an arbitrary positive real number and let  $\delta := 1$ . Let  $g : X \rightarrow \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}$  be defined by  $g(x) := d(x, K)$  and  $h(x) := \|x - p_K(x_0)\|$ . Then  $g(x) \leq h(x)$  for all  $x \in X$ ,  $g(x_0) = h(x_0)$ , and  $\nabla h(x_0) = \nabla \|x_0 - p_K(x_0)\|$ . Let  $x^* = \nabla \|v\|$  and choose  $t > 0$  so that

$$\left| \frac{\|v + tv\| - \|v\|}{t} - \nabla \|v\|(y) \right| < \varepsilon \quad \text{for all } y \in S_X.$$

Note that this is possible since the norm  $\|\cdot\|$  is Fréchet differentiable at  $v \in S_X$ . For each  $0 < \lambda < 1$ , let  $f_\lambda : X \rightarrow \mathbb{R}$  be defined by  $f_\lambda(x) := d(x_0 + t^{-1}\lambda v, K) - \|x - (x_0 + t^{-1}\lambda v)\|$ . Then

- (i)  $f_\lambda(x) \leq g(x)$  for all  $x \in X$ ,
- (ii)

$$\begin{aligned} \lim_{\lambda \rightarrow 0^+} \frac{f_\lambda(x_0) - g(x_0)}{\lambda} &= \lim_{\lambda \rightarrow 0^+} \frac{d(x_0 + t^{-1}\lambda v, K) - d(x_0, K) - t^{-1}\lambda}{\lambda} \\ &= \lim_{\lambda \rightarrow 0^+} t^{-1} \left( \frac{d(x_0 + t^{-1}\lambda v, K) - d(x_0, K)}{t^{-1}\lambda} - 1 \right) \\ &= t^{-1} \lim_{s \rightarrow 0^+} \left( \frac{d(x_0 + sv, K) - d(x_0, K)}{s} - 1 \right) = 0. \end{aligned}$$

- (iii) Let  $0 < \lambda < 1$  and  $y \in S_X$ . Then

$$\begin{aligned} \frac{f_\lambda(x_0 + \lambda y) - f_\lambda(x_0)}{\lambda} &= - \left( \frac{\|\lambda y - t^{-1}\lambda v\| - \|t^{-1}\lambda v\|}{\lambda} \right) \\ &= -t^{-1} \lambda \left( \frac{\|v - ty\| - \|v\|}{\lambda} \right) \\ &= - \left( \frac{\|v + t(-y)\| - \|v\|}{t} \right) \\ &\geq -\nabla \|v\|(-y) - \varepsilon = \nabla \|v\|(y) - \varepsilon = x^*(y) - \varepsilon. \end{aligned}$$

Therefore,  $\liminf_{\lambda \rightarrow 0^+} (f_\lambda(x_0 + \lambda y) - f_\lambda(x_0))/\lambda \geq x^*(y) - \varepsilon$  for all  $y \in S_X$ .

Hence, by the previous proposition,

$$x^*(y) - \varepsilon \leq \liminf_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all  $y \in S_X$ . Since  $\varepsilon > 0$  was arbitrary, we have that

$$x^*(y) \leq \liminf_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \rightarrow 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all  $y \in S_X$ . Thus,  $x^* = \nabla h(x_0)$ , and so  $\nabla h(x_0)(y) = \lim_{\lambda \rightarrow 0^+} (g(x_0 + \lambda y) - g(x_0))/\lambda$ . It is now routine to show that  $\nabla h(x_0)(y) = \lim_{\lambda \rightarrow 0} (g(x_0 + \lambda y) - g(x_0))/\lambda = \nabla g(x_0)(y)$  for all  $y \in S_X$ . □

Note that by being a little more careful in the proof of the above theorem, one can show that the assumption that the norm is Fréchet smooth can be relaxed to the norm being Gâteaux smooth.



### Appendix D

Let us start by giving a quick proof that the norm on a Hilbert space is rotund. In any Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  the norm satisfies the identities  $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$  and  $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$ . Therefore,  $\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2)$ —the parallelogram law. Hence,

$$\left\| \frac{x + y}{2} \right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} - \left\| \frac{x - y}{2} \right\|^2 < \frac{\|x\|^2 + \|y\|^2}{2} = 1 \quad \text{for all } x \neq y \in S_H.$$

This shows that  $(H, \langle \cdot, \cdot \rangle)$  is rotund.

**THEOREM [22, Theorem 9].** *If  $(X, \|\cdot\|)$  is a separable normed linear space, then  $X$  admits an equivalent rotund norm.*

**PROOF.** If  $(Y, \|\cdot\|_Y)$  is a rotund normed linear space and  $T : X \rightarrow Y$  is a one-to-one bounded linear mapping, then  $\|x\|' := \|x\| + \|T(x)\|_Y$  is an equivalent rotund norm on  $X$  (apply Lemma 2.32 for the proof of rotundity). Since  $X$  is separable, there exists a dense subset  $\{x_n\}_{n=1}^\infty$  of  $S_X$ . Now, by applying the Hahn–Banach extension theorem, there exists a subset  $\{x_n^*\}_{n=1}^\infty$  of  $S_{X^*}$  such that  $x_n^*(x_n) = 1$  for all  $n \in \mathbb{N}$ . Define  $T : (X, \|\cdot\|) \rightarrow (\ell_2(\mathbb{N}), \|\cdot\|_2)$  by

$$T(x) := \left( \frac{x_1^*(x)}{2}, \frac{x_2^*(x)}{4}, \dots, \frac{x_n^*(x)}{2^n}, \dots \right).$$

Then  $T$  is one-to-one, bounded and linear. Thus,  $X$  has an equivalent rotund norm.  $\square$

Note that the argument above works whenever the dual space  $X^*$  is separable with respect to the weak\* topology on  $X^*$ . That is, the following statement is true. If  $(X, \|\cdot\|)$  is a normed linear space and  $(X^*, \text{weak}^*)$  is separable, then  $X$  admits an equivalent rotund norm. In particular,  $\ell_\infty(\mathbb{N})$  admits an equivalent rotund norm. On the other hand, one should also note that not every Banach space admits an equivalent strictly convex norm. For example,  $\ell^\infty/c_0$  does not possess an equivalent strictly convex norm; see [15, 42, 56].

### Appendix E

**LEMMA.** *Let  $S$  be the ellipsoidal surface in  $\mathbb{R}^n$  defined by the equation*

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \dots + \frac{x_n^2}{\beta^2} = 1,$$

where  $\alpha > \beta > 0$ . For any  $y \in S$  there exists  $w \in \text{co}(S)$  such that  $\|w - y\| = d(y, S) = \beta^2/\alpha$ .

**PROOF.** Let  $y := (y_1, \dots, y_n) \in S$ , and consider  $\hat{y} := (y_1(1 - (\beta^2/\alpha^2)), 0, \dots, 0) \in \text{co}(S)$ . We will show that  $y$  is a nearest point to  $\hat{y}$  in  $S$ . To do this we will minimise the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$f(x_1, x_2, \dots, x_n) := \left\| (x_1, x_2, \dots, x_n) - \left( y_1 \left( 1 - \frac{\beta^2}{\alpha^2} \right), 0, \dots, 0 \right) \right\|^2,$$

for all  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , subject to the constraint that

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \dots + \frac{x_n^2}{\beta^2} = 1.$$

Since  $f(x_1, x_2, \dots, x_n) = (x_1 - y_1(1 - (\beta^2/\alpha^2)))^2 + x_2^2 + \dots + x_n^2$ , substituting in the constraint shows that we need only minimise (without constraint) the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$g(x_1) := \left(x_1 - y_1\left(1 - \frac{\beta^2}{\alpha^2}\right)\right)^2 + \beta^2 - \frac{x_1^2\beta^2}{\alpha^2}.$$

As  $1 - (\beta^2/\alpha^2) > 0$ ,

$$g'(x_1) = 2\left(x_1 - y_1\left(1 - \frac{\beta^2}{\alpha^2}\right)\right) - \frac{2x_1\beta^2}{\alpha^2}$$

is zero only when  $x_1 = y_1$ . Furthermore, since  $g''(x_1) = 2(1 - (\beta^2/\alpha^2)) > 0$ , we see that  $g$  is minimised at  $y_1$ . Therefore,  $f$ , subject to the given constraint, is minimised at  $(y_1, x_2, \dots, x_n)$ , where  $(y_1^2/\alpha^2) + (x_2^2/\beta^2) + \dots + (x_n^2/\beta^2) = 1$ . Hence,  $f$  is minimised (amongst other points) at  $(y_1, y_2, \dots, y_n)$ , and so  $y$  is a nearest point to  $\hat{y}$  in  $S$ . Finally, since

$$\begin{aligned} \left\| (y_1, y_2, \dots, y_n) - \left(y_1\left(1 - \frac{\beta^2}{\alpha^2}\right), 0, \dots, 0\right) \right\|^2 &= \left(\frac{y_1\beta^2}{\alpha^2}\right)^2 + y_2^2 + \dots + y_n^2 \\ &\geq \left(\frac{y_1\beta^2}{\alpha^2}\right)^2 + \frac{y_2^2\beta^2}{\alpha^2} + \dots + \frac{y_n^2\beta^2}{\alpha^2} \\ &= \left(\frac{\beta^2}{\alpha}\right)^2 \left(\frac{y_1^2}{\alpha^2} + \frac{y_2^2}{\beta^2} + \dots + \frac{y_n^2}{\beta^2}\right) \\ &= \left(\frac{\beta^2}{\alpha}\right)^2, \end{aligned}$$

there exists  $w \in [\hat{y}, y] \subseteq \text{co}(S)$  such that  $\|w - y\| = d(y, S) = \beta^2/\alpha$ . □

### Index of notation and assumed knowledge

- The natural numbers,  $\mathbb{N} := \{1, 2, 3, \dots\}$ .
- The integers,  $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ .
- The rational numbers,  $\mathbb{Q} := \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$ .
- For any set  $X$ ,  $\mathcal{P}(X)$  is the set of all subsets of  $X$ .
- For any subset  $A$  of a topological space  $(X, \tau)$ , we define:
  - $\text{int}(A)$ , called the interior of  $A$ , as the union of all open sets contained in  $A$ ;
  - $\bar{A}$ , called the closure of  $A$ , as the intersection of all closed sets containing  $A$ ;
  - $\text{Bd}(A)$ , called the boundary of  $A$ , as  $\bar{A} \setminus \text{int}(A)$ .

- For any points  $x$  and  $y$  in a vector space  $X$ , we define the following intervals:
  - $[x, y] := \{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}$ ;
  - $(x, y) := \{x + \lambda(y - x) : 0 < \lambda < 1\}$ ;
  - $[x, y) := \{x + \lambda(y - x) : 0 \leq \lambda < 1\}$ ;
  - $(x, y] := \{x + \lambda(y - x) : 0 < \lambda \leq 1\}$ .
- For any normed linear space  $(X, \|\cdot\|)$ , we define:
  - $B[x; r] := \{y \in X : \|x - y\| \leq r\}$ , for any  $x \in X$  and  $r \geq 0$  (note that this implies that  $B[x; 0] = \{x\}$ , for any  $x \in X$ );
  - $B(x; r) := \{y \in X : \|x - y\| < r\}$ , for any  $x \in X$  and  $r > 0$ ;
  - $B_X := B[0; 1]$ ;
  - $S_X := \{x \in X : \|x\| = 1\}$ .

- For any inner product space  $(X, \langle \cdot, \cdot \rangle)$ ,  $\angle xyz$  denotes the angle between the vectors  $(y - x)$  and  $(y - z)$ , where  $x, y, z$  are distinct points in  $X$ . That is,

$$\angle xyz := \cos^{-1} \left( \frac{\langle y - x, y - z \rangle}{\|y - x\| \|y - z\|} \right),$$

where  $\|\cdot\|$  is the norm induced by the inner product.

- Given a compact Hausdorff space  $K$ , we write  $C(K)$  for the set of all real-valued continuous functions on  $K$ . This is a vector space under the operations of pointwise addition and pointwise scalar multiplication.  $C(K)$  becomes a Banach space when equipped with the uniform norm  $\|\cdot\|_\infty$ , defined by

$$\|f\|_\infty := \sup_{x \in K} |f(x)| \quad \text{for all } f \in C(K).$$

- Let  $A$  and  $B$  be sets. Given a function  $f : A \rightarrow B$ , we define  $f(A) := \bigcup_{a \in A} \{f(a)\}$ . Similarly, given a set-valued mapping  $\Phi : A \rightarrow \mathcal{P}(B)$ , we define  $\Phi(A) := \bigcup_{a \in A} \Phi(a)$ .
- For a normed linear space  $(X, \|\cdot\|)$ ,  $X^*$ , the set of bounded linear maps from  $X$  to  $\mathbb{R}$ , is called the dual space of  $X$ .  $X^*$  is a Banach space when equipped with the operator norm, given by

$$\|f\| := \sup_{x \in B_X} \|f(x)\| \quad \text{for all } f \in X^*.$$

- Let  $X$  be a set and  $Y$  a totally ordered set. For any function  $f : X \rightarrow Y$  we define

$$\operatorname{argmax}(f) := \{x \in X : f(y) \leq f(x) \text{ for all } y \in X\},$$

$$\operatorname{argmin}(f) := \{x \in X : f(x) \leq f(y) \text{ for all } y \in X\}.$$

- Let  $A$  be a subset of a vector space  $X$ . Then the convex hull of  $A$ , denoted by  $\operatorname{co}(A)$ , is defined to be the intersection of all convex subsets of  $X$  that contain  $A$ .
- Let  $X$  be a set and let  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  a function. Then

$$\operatorname{Dom}(f) := \{x \in X : f(x) < \infty\}.$$

- If  $X$  is a normed linear space,  $f : X \rightarrow \mathbb{R}$  is a function and  $f$  is Gâteaux (Fréchet) differentiable at  $x_0 \in X$ , then we write  $\nabla f(x_0)$  for the Gâteaux (Fréchet) derivative of  $f$  at  $x_0$ .
- If  $f$  is a convex function defined on a nonempty convex subset  $K$  of a normed linear space  $(X, \|\cdot\|)$  and  $x \in K$ , then we define the subdifferential of  $f$  at  $x$  to be the set  $\partial f(x)$  of all  $x^* \in X^*$  satisfying

$$x^*(y - x) \leq f(y) - f(x) \quad \text{for all } y \in K.$$

- It is assumed that the reader has a basic working knowledge of metric spaces and normed linear spaces. In particular, it is assumed that the reader is familiar with the basics of linear topology (including the Hahn–Banach theorem). In particular, the weak topology on a normed linear space. In this regard, knowledge of the following theorems is assumed.

**THEOREM [8, 14].** *Let  $(X, \|\cdot\|)$  be a normed linear space. Then  $X$  is reflexive if and only if  $B_X$  with the relative weak topology is compact.*

**THEOREM (Eberlein–Šmulian theorem, [26, 30]).** *Every nonempty weakly compact subset  $K$  of a Banach space  $(X, \|\cdot\|)$  is weakly sequentially compact, that is, every sequence in  $K$  possesses a weakly convergent subsequence (with the limit in  $K$ ).*

For a good introduction to the theory of Banach spaces see any of [26, 28, 29, 36, 75].

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